Discussion about Lorenz Equations and Chaos

Richard Cai

Abstract

This article is aimed at analyzing the property of curves of Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = \rho x - y - zx$$

$$\frac{dz}{dt} = xy - \beta z$$

In order to have a basic understanding of chaos.

1. Introduction

1.1 The First Approach of Chaos

In 1963, Edward Lorenz published his paper *Deterministic Nonperiodic Flow* in *Journal of the Atmospheric Sciences,* indicating that there mist be a relationship between the impossibility of exact repetition of

weather and the incapacity of long-term weather forecast. This relationship links the non-cycling and unpredictability, which are two main properties of chaos. In addition, he found that chaos is "extremely sensitive to the initial value", Lorenz's paper, using Lorenz equation to interpret the chaos phenomena, established a new discipline called "Chaos Theory".

Perhaps most of us are unfamiliar with chaos, but we are familiar with the term "Butterfly Effect". The formation of hurricane is determined by whether a butterfly thousands miles away flaps its wings. "Butterfly Effect" shows a "chaotic" way of thinking. Mathematically, chaos is the system that can change dramatically when the initial value changes slightly.

Chaos Theory has a lot of applications in different areas. For examples, chaos theory is widely used in complex systems such as weather or stock market to investigate the reason of unpredictable natural disasters or stock market crashes, which can be ascribe to the slight difference of the initial value that can cause powerful destructive forces to the system. In addition, the use of chaos theory in sociology indicates that a tiny harmful mechanism in the society could lead to the collapse of the whole society. The application of chaos theory in

psychology reveals that a tiny mental irritation could be magnified as the person growing up, which is exactly the phenomenon in the film *Butterfly Effect*.

More specifically, here is a somewhat exaggerated example about the function of chaos. This examples links to the smoking of an American man to the world-wide inflation. Assume an American man is smoking and he throw the cigarette butt to somewhere near his bed before he left home. Unfortunately, the cigarette butt is not quenched and causes a fire. The fire leads to the explosion of the gas cylinder in his home, and consequently the fire gets bigger and becomes overwhelming. The whole building (assume the man lives in an apartment) then becomes burning. Although the fire is finally extinct, panic becomes prevalent among public because people are suspicious that the fire catastrophe is another terrorists attack. The panic makes investor selling their stocks, creating a stock market crash and pull down the American economy, which leads to the deflation of dollars. Because world-wide raw material trade is based on dollars, the deflation of dollars makes the price of raw materials increase and therefore caused the world-wide inflation.

From these examples we can learn that chaos theory is a

comprehensive discipline that can be applied to different areas. Its interpretation of natural mechanism can significantly help us understand the world.
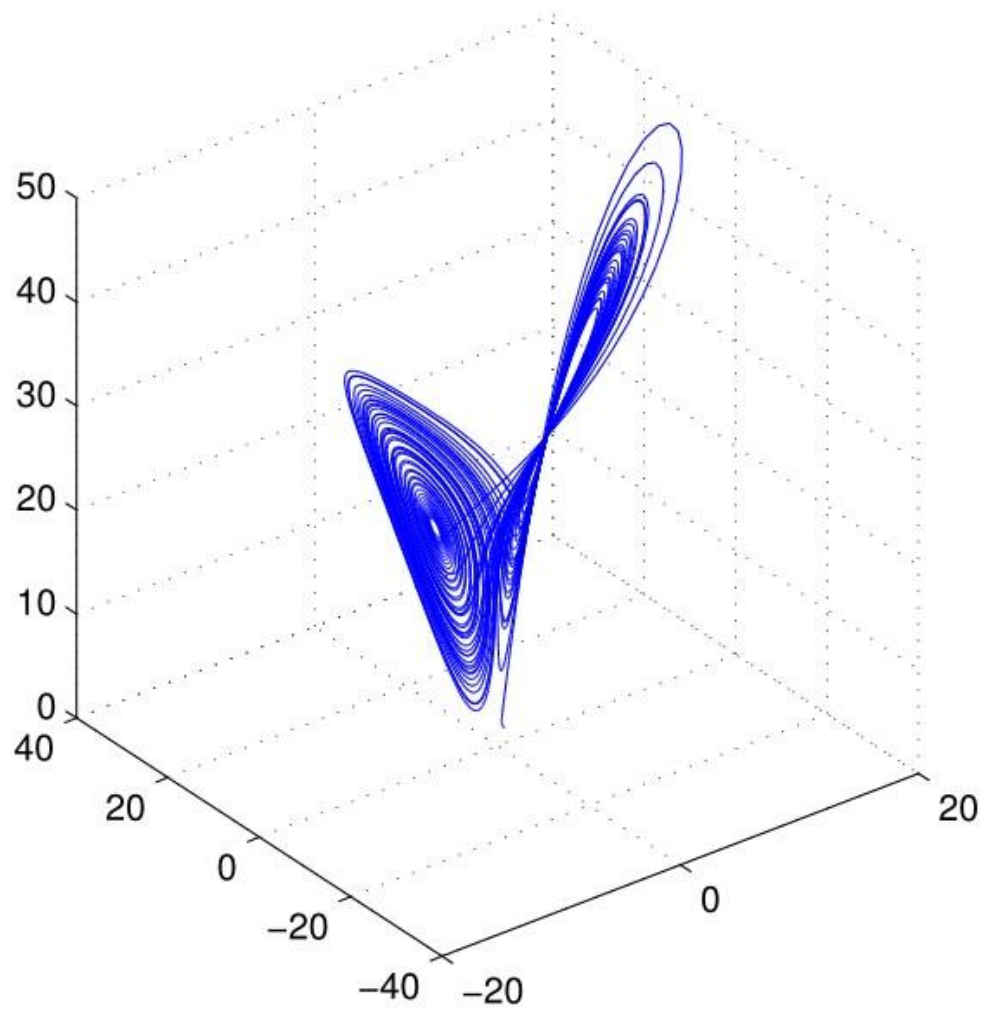
1.2 Lorenz Equation and Chaos

Lorenz equation is a simple 3-dimensional ordinary differential equation:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = \rho x - y - zx$$

$$\frac{dz}{dt} = xy - \beta z$$

This ostensibly simple function is however difficult to analyze.

Here is the 3-dimensional graphic of the Lorenz equation

FIGURE 3D graphic of the equation (N=5000, y(0)=(1,1,1),h=0.01)

In order to the graphic more clearly, we can see its projection on xOy plane:

FIGURE Projection on xOy of the equation (N=15000, y(0)=(1,1,1),h=0.01)
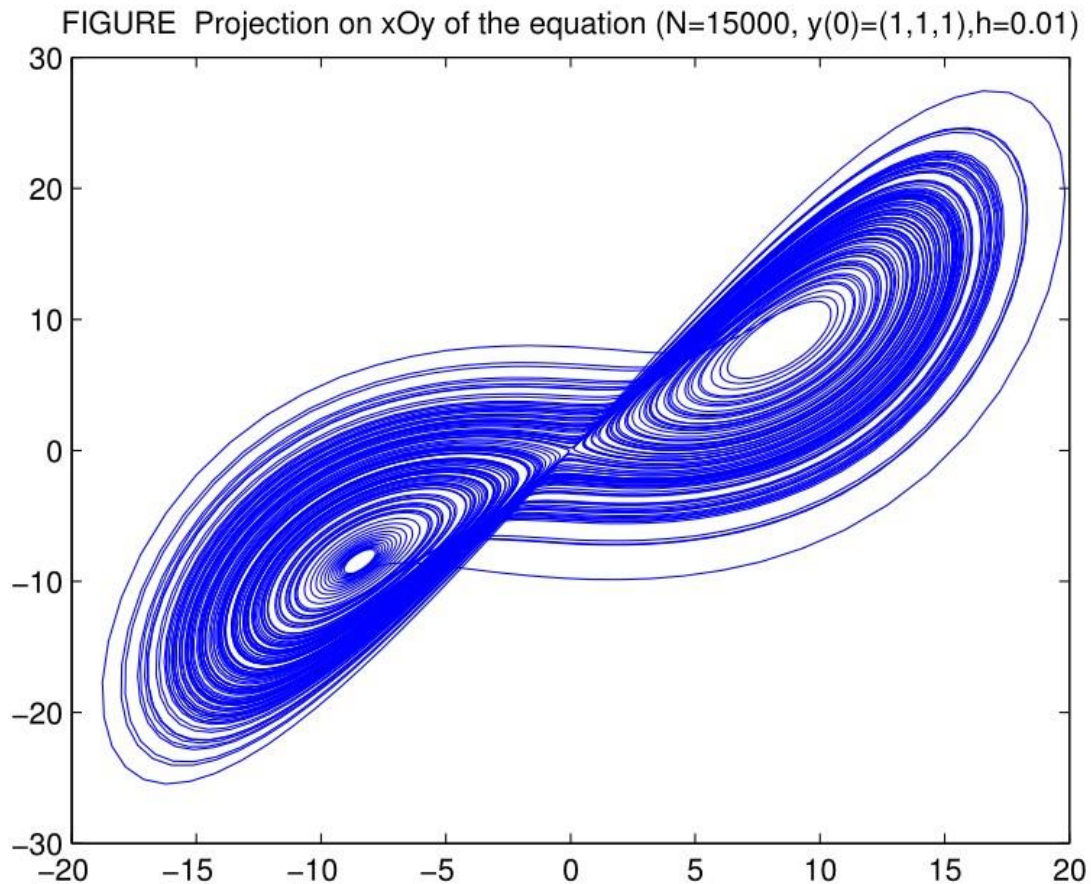
We can apparently see that the projection is consisted of two "ears": the equation consistently winds itself around the center of two "ears", moving approximately periodically.

From the latter analysis, we can learn that the curve of the equation will change dramatically when the initial condition alters slightly, with the constant parameters. This phenomenon indicates that the differential equation system is sensitive, which is so-called "chaos"

Chaos is fairly normal and useful in our daily life. For example, weather

forecast is strongly related to Chaos. Therefore it is necessary to investigate chaos. In here we are only going to give a short introduction to chaos.

2. The Invitation of Initial Value Problem

2.1 Analysis on the image of Lorenz problem

We can clearly see the image of Lorenz equation is limited, and it seems like there exists some pattern of that image, otherwise the image would not be such a beautiful spindle. The following 3 pictures depict the change of Lorenz equation in 3 different axis with time.

FIGURE Change in x–axis with t of the equation (N=10000, y(0)=(1,1,1),h=0.01)
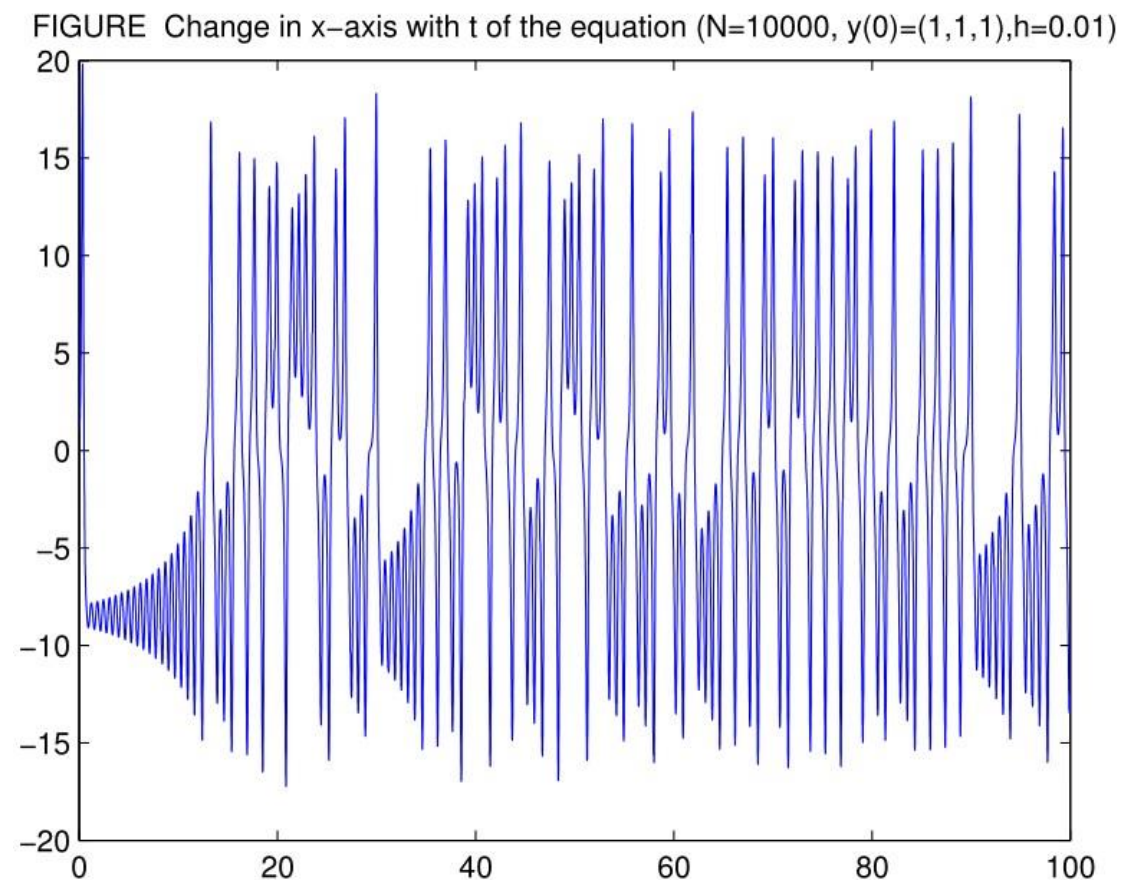
FIGURE Change in y–axis with t of the equation (N=10000, y(0)=(1,1,1),h=0.01)
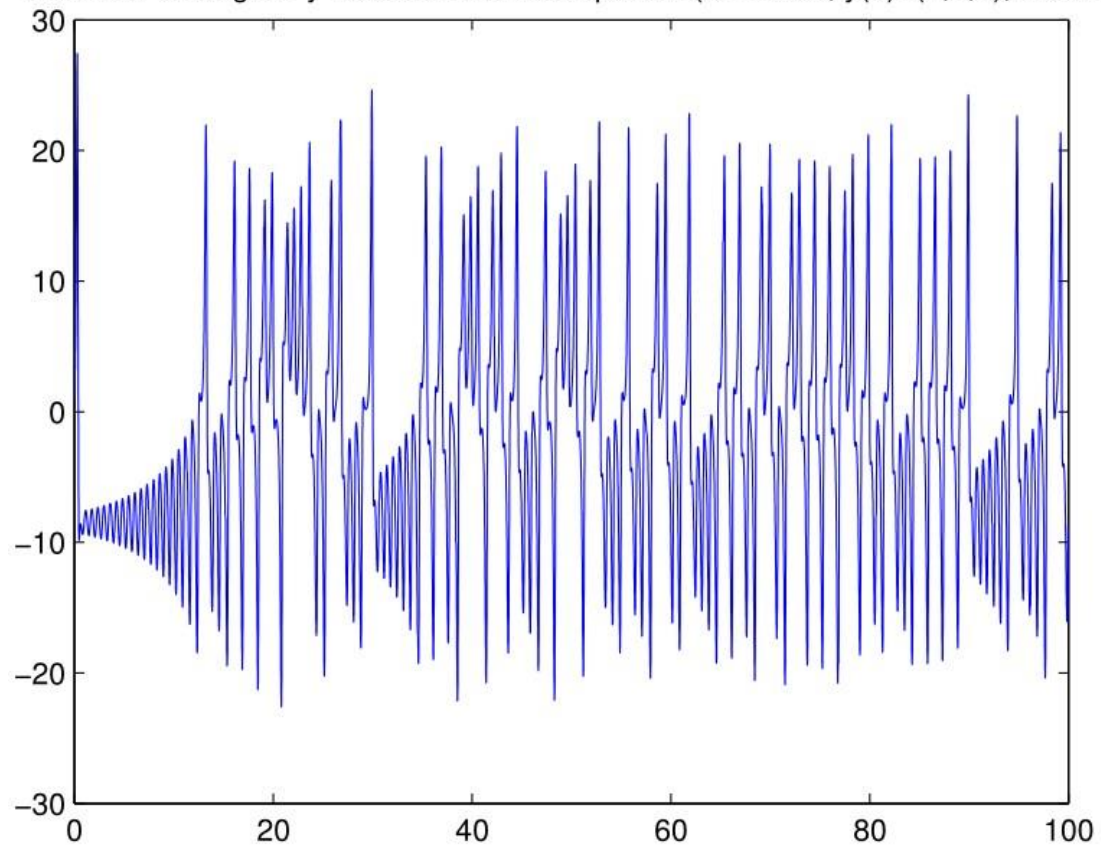
FIGURE Change in z–axis with t of the equation (N=10000, y(0)=(1,1,1),h=0.01)
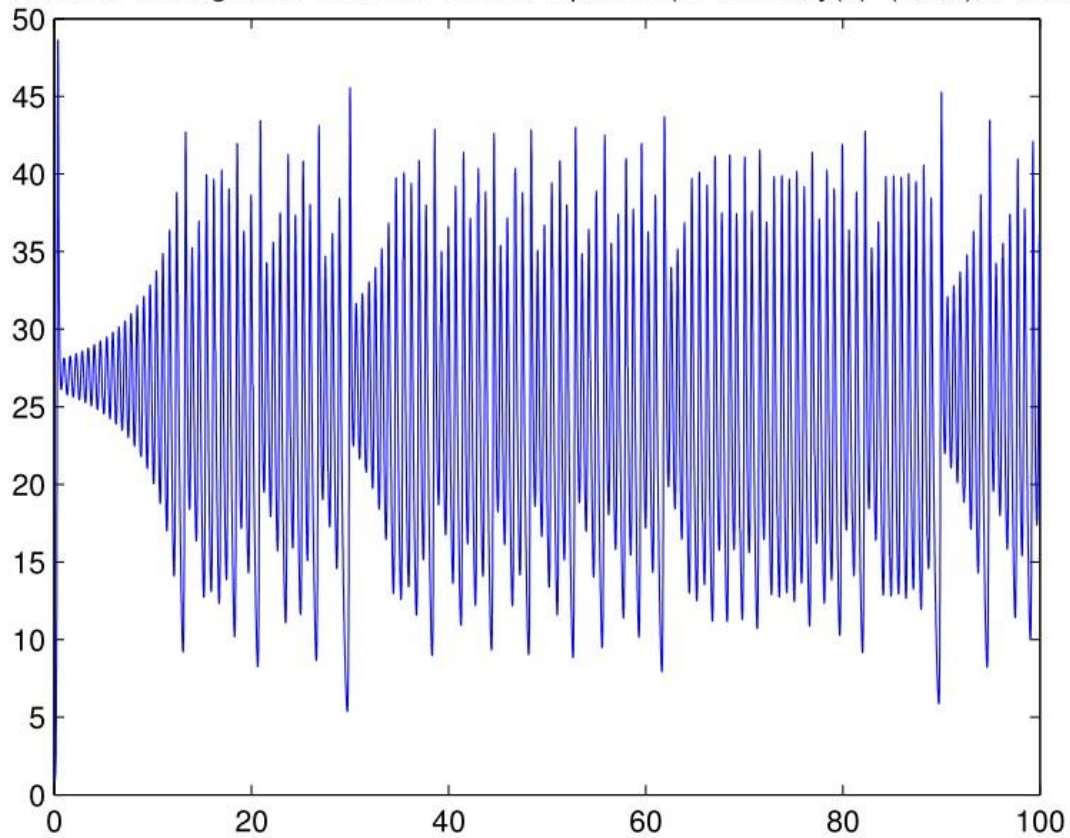
We can see that the projection of Lorenz equation on x-axis is the most featured, including distinctive upper part and lower part. Referring to the xOy projection, we can see that the equation is winding up itself around the "right ear" above the $x=0$ axis, and it is winding up itself around the "left year" below the $x=0$ axis. Therefore we can know when the equation is winding up itself around "right ear" or "left year" by observing the projection of the equation in x-axis, which is hard to see from 3-dimensional graphic of the equation.

2.2 Initial Value Problem

FIGURE 3D graphic of the equation (N=10000, y(0)=(0.001,0.001,0.001),h=0.01)
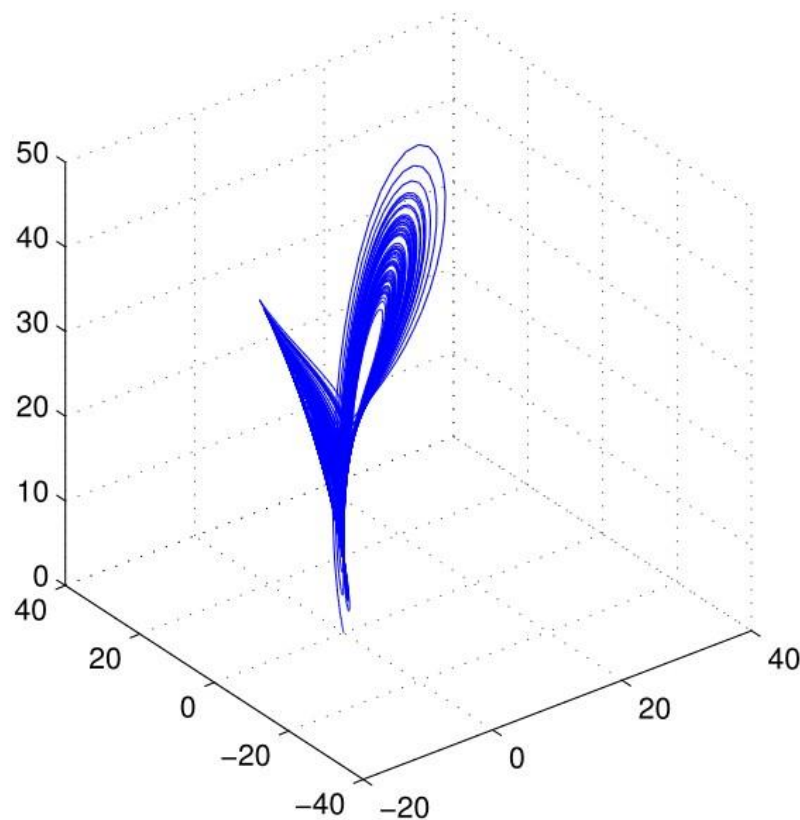
FIGURE 3D graphic of the equation (N=10000, y(0)=(0.002,0.002,0.002),h=0.01)
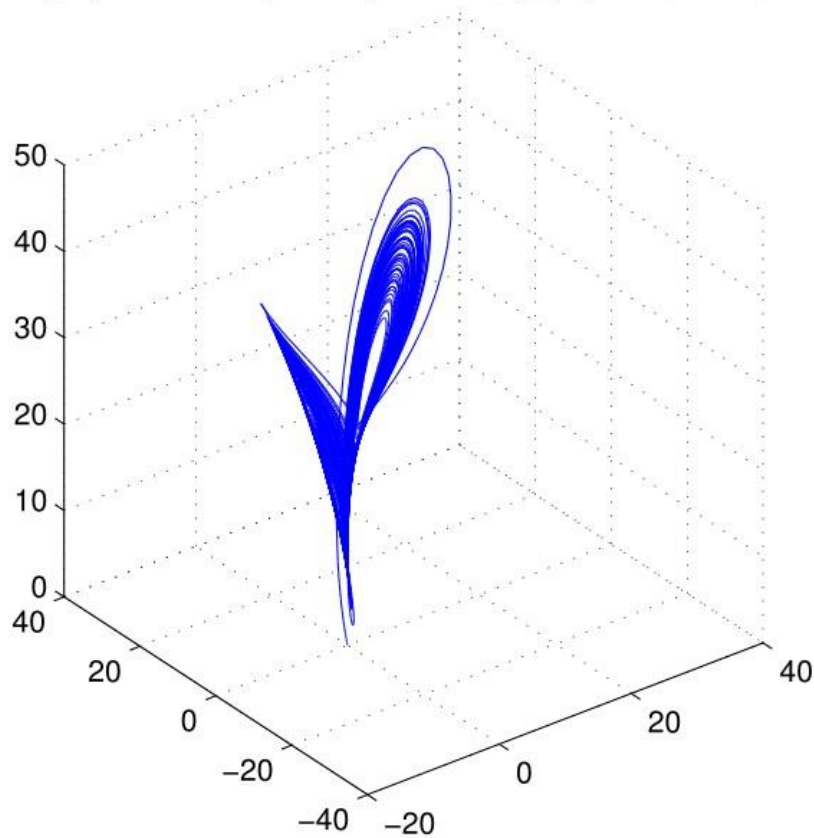
Initial value problem is significant for the discussion of chaos problem, because the formation of chaos system is strongly depended on the instability of the initial value. Now we can observe the graphic of the solution of Lorenz equation when the initial value of x, y and z as (0.001,0.001,0.001) and (0.002,0.002,0.002). The 3-dimensional graphics are shown above.

It is hard to distinguish two solution merely based on the graphics. The range of motion is however very close, which indicates that the large-scale graphic is relatively stable corresponding to the change of initial

value. We can also get the same conclusion from the 2-dimensional

projection on xOy-plane of the equation.

FIGURE Projection on xOy of the equation (N=10000, y(0)=(0.001,0.001,0.001),h=0.01)
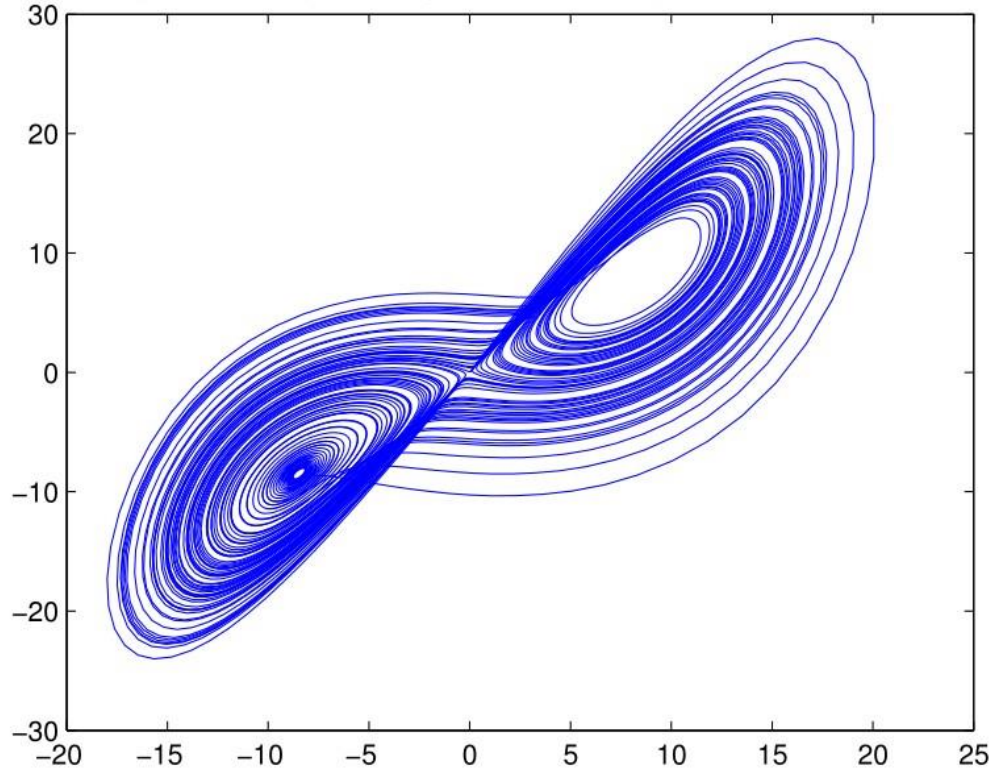
FIGURE Projection on xOy of the equation (N=10000, y(0)=(0.002,0.002,0.002),h=0.01)
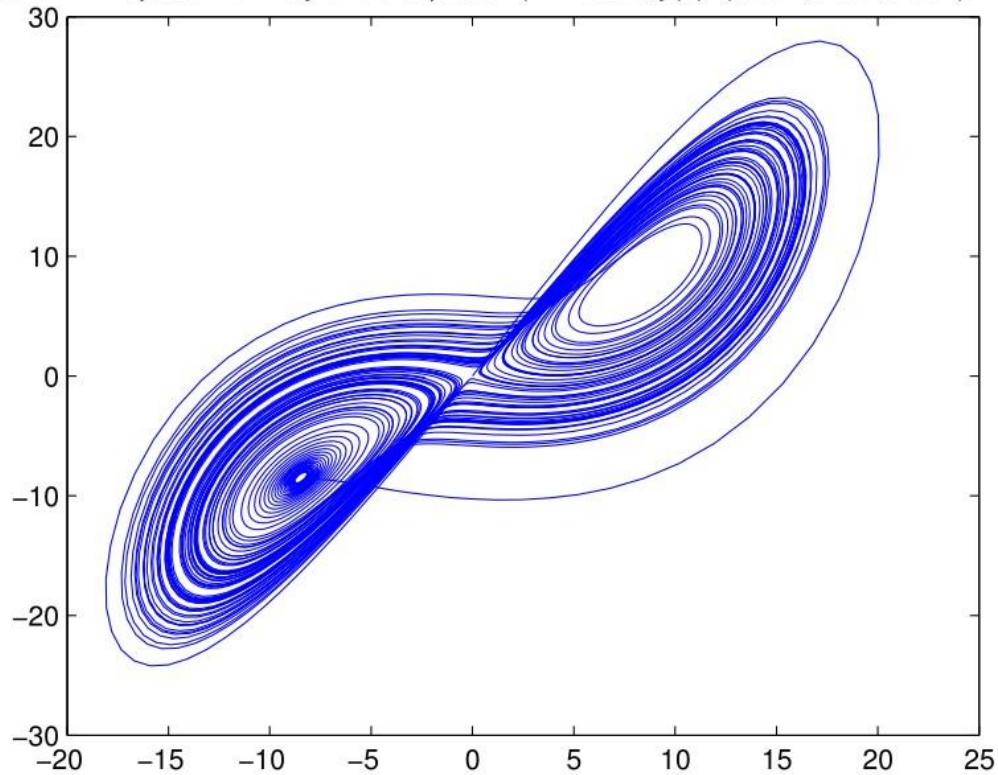
The reason why I give 2-dimensional graphics is that they are more intuitive than 3-dimensional graphics.

One point need to be emphasized is that there are some differences between two graphics, after all the initial value is different. However, we cannot simply ascribe this difference to the instability of the initial value, because the general patterns of two graphics are very close. What is the real difference and how big the difference would be will be discussed later.

3. Lorenz Attractor and Chaos

3.1 The Ways of the Winding

We mentioned above that the large-scale graphic is relatively stable corresponding to the change of the initial value, but it cannot indicate that the microscopic movements of the equations are stable. Furthermore, we have no idea about the ways of winding. If the initial values change slightly, the ways of winding would change dramatically. Therefore microscopically, the graphic is not stable corresponding to the changes of the initial values, which is easy to understand because windings around "left ear" or "right ear" are very different motions.

It is hard to investigate the ways of the winding of the equation merely based on the 3-dimensional or 2-dimensional graphics, but we know how to determine the ways of winding from the projection on x-axis of the equation corresponding to the change of time, based on our former discussion. Therefore, the following discussion is about how to investigate the microscopic motion of the Lorenz equation using the x-axis projection of the equation corresponding to the change of time.

3.2 Lorenz Attractor

Specially, we take the initial value σ=10, ρ=28, β=8/3, and then we are going to discuss the ways of winding of the equations with initial values of (0.001, 0.001, 0.001) and (0.002, 0.002, 0.002). The $x-t$ graphics of two equations based on two initial values are in the following.
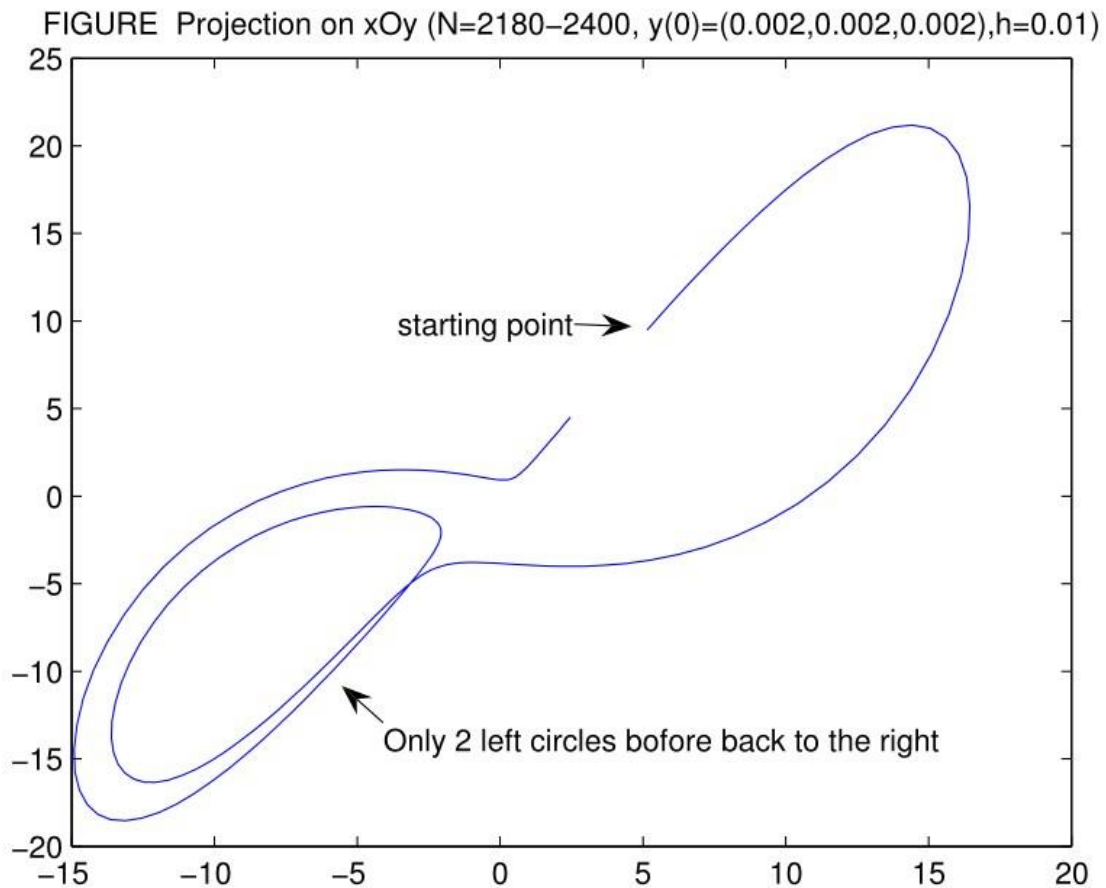
FIGURE Change in x−axis with t (N=3000, y(0)=(0.001,0.001,0.001),h=0.01)

FIGURE Change in x−axis with t (N=3000, y(0)=(0.002,0.002,0.002),h=0.01)
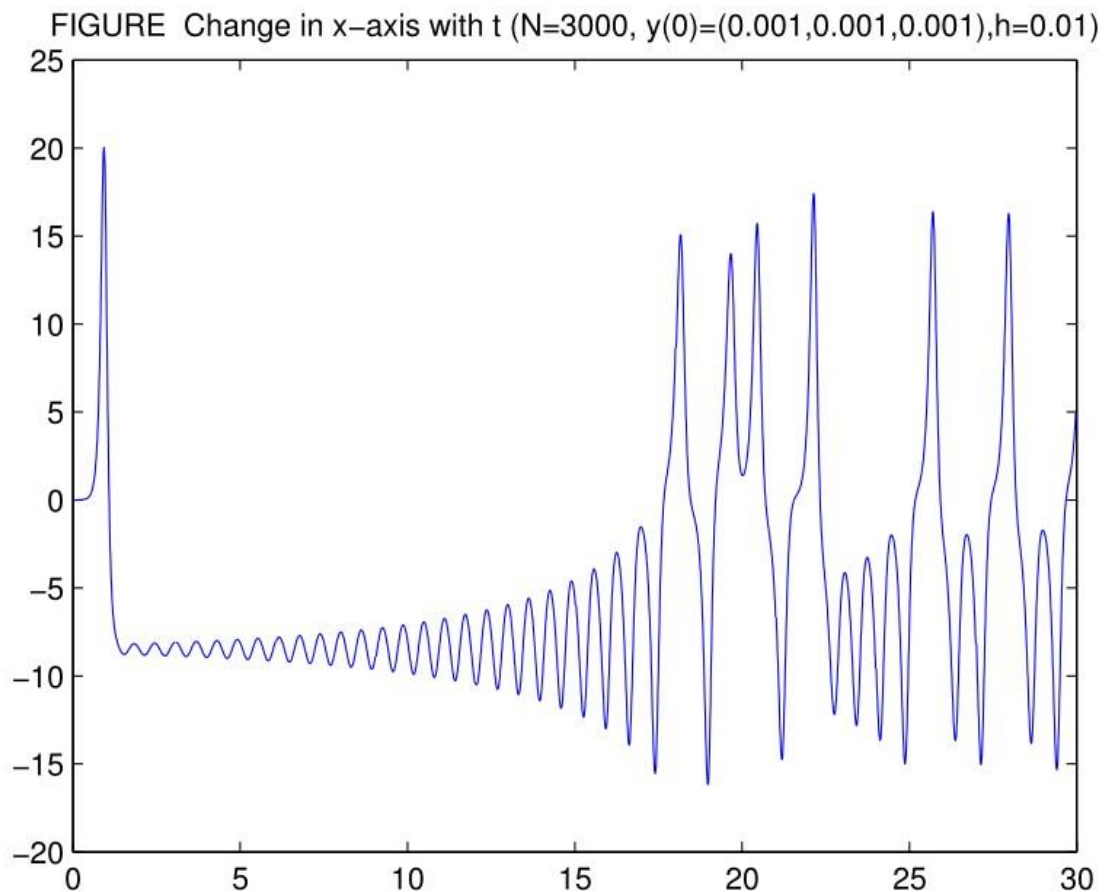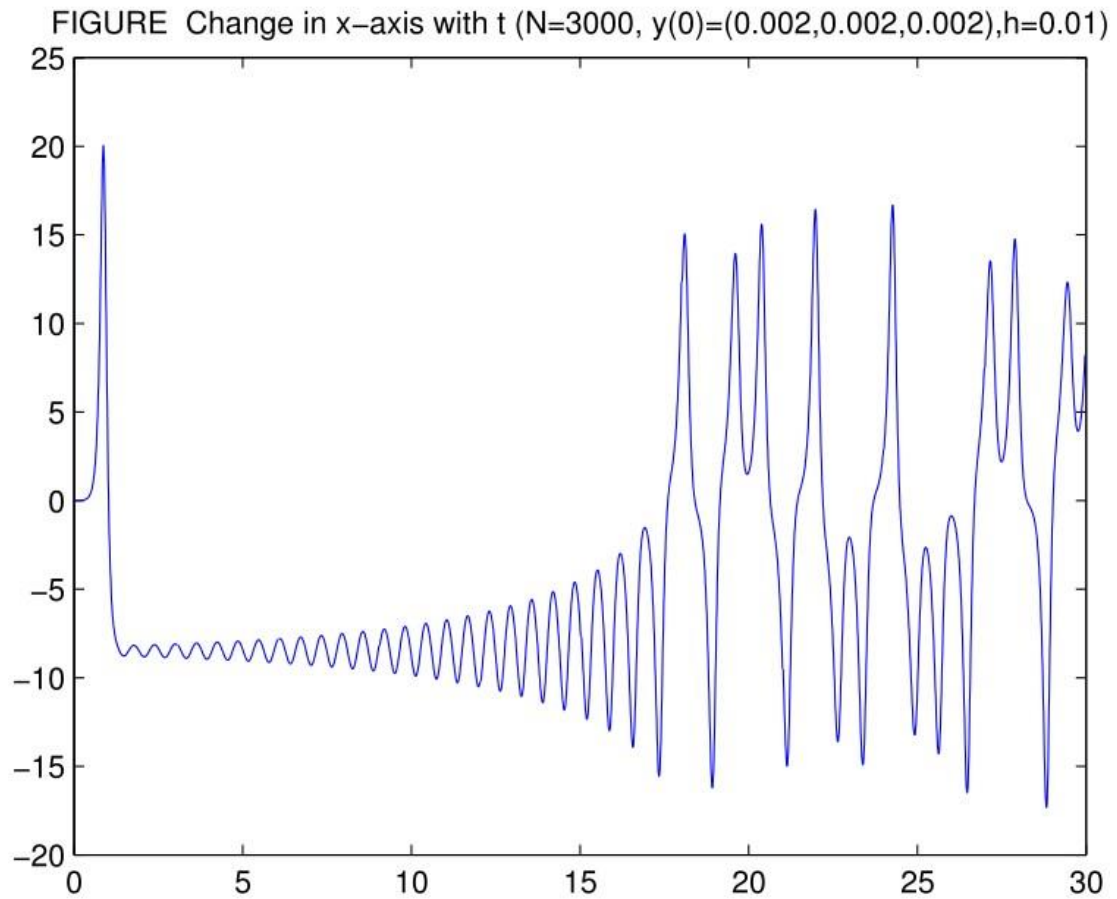
We can conclude a table of the number of windings from the graphics

(L indicates "left ear", R indicates "right ear")

	L	R	L	R	L	R	L	R	L	R	...
(0.001, 0.001, 0.001)	26	1	1	2	1	1	4	1	2	1	...
(0.002, 0.002, 0.002)	26	1	1	2	1	1	2	1	3	2	...

For example, when the initial value of the equation is (0.001,0.001,0.001), the equation winds up itself around "left ear" for 26 times, and then winds up itself around "right ear" for 1 time, then "left year" for 1 time. Based on that description, we can see from both the graphics and the table that the close initial values cause big difference of the curve of the equation. The following graphics are the 2-dimensional projections of the equations, which can intuitively show the difference of the ways of winding.

FIGURE Projection on xOy (N=2200–2600, y(0)=(0.001,0.001,0.001),h=0.01)

FIGURE Projection on xOy (N=2180–2400, y(0)=(0.002,0.002,0.002),h=0.01)
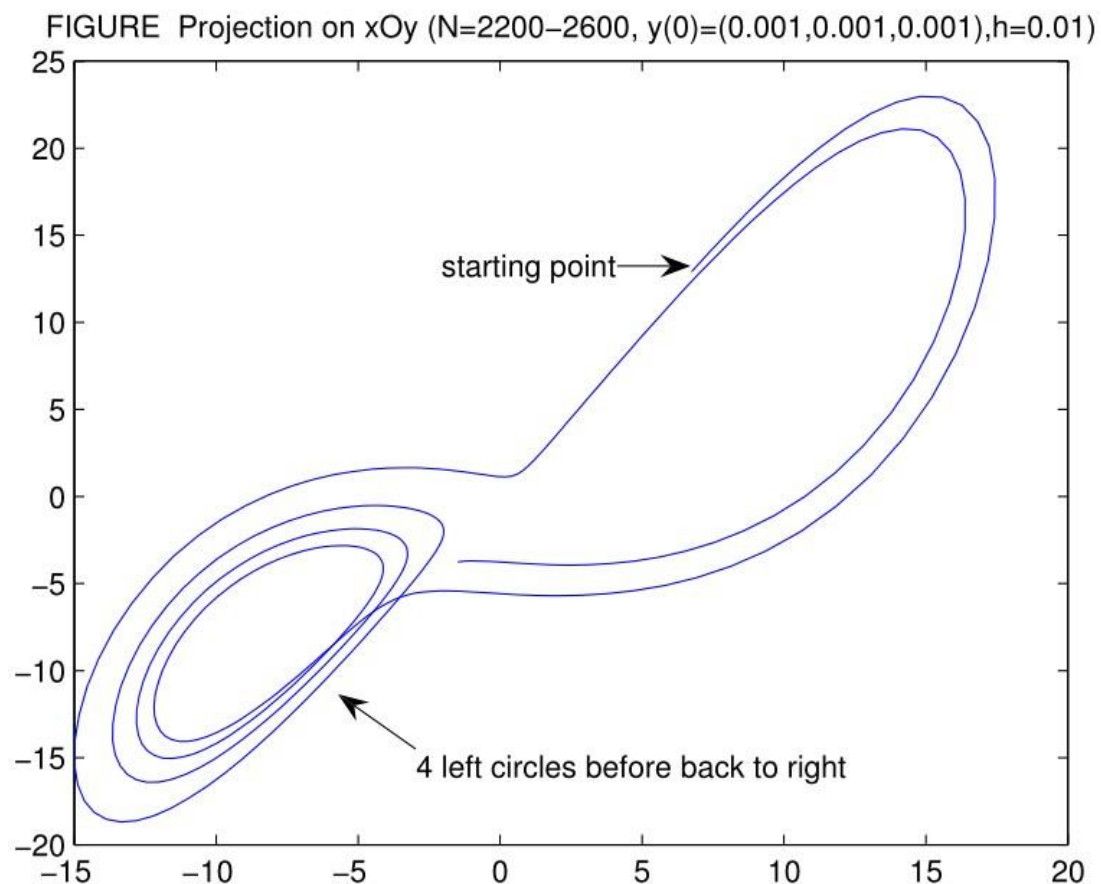
The seemingly normal parameter $\sigma=10$, $\rho=28$, $\beta=8/3$ leads to the instability of the Lorenz equation in microscopic area. The superficially regular and beautiful equation, however, is quite random in details. (We don't know when it winds up itself around "left ear" or "right ear"). We define the equation with these certain parameters as the "strange attractor".

Indeed, strange attractors do not only exist in the Lorenz equation with these certain parameters, many system have the similar phenomena of strange attractors. The reasons why they are strange are that not only

they are not smooth curve or surface with "non-integer" dimensions but also the sensitivity of their motions corresponding to the change of the initial values, which is one significant point we figured out in the former discussion.

4. Discussion about the Parameters

From the analysis above, σ =10, ρ =28, β =8/3 is an extremely mysterious parameter. In order to investigate whether the chaos phenomenon would happen after the change of parameters, we will do an experiment to see the Lorenz equations after the change of parameter.

The Value of ρ	The Graphic of the Function
[0, 1]	The function converges to the origin corresponding to the change of t
(1, 13.4]	The function converges to the center of the "right ear" corresponding to the change of t
(13.4, 21.7]	The function converges to the center of the "right ear" corresponding to the change of t
(21.7, 119)	The function is in chaos
[119, 134]	The function is cycling
(134, 166]	The function is in chaos
(166, 490]	The function is cycling
(490, ∞)	The function converges to a point corresponding to the change of t

Now we will see the result by changing the value of ρ

In here one point need to say is that every following page will provide a 3-dimensional graphic, a xOy projection graphic and an $x-t$ graphic corresponding to a certain value of ρ, all of the results of the experiment are in the table below:
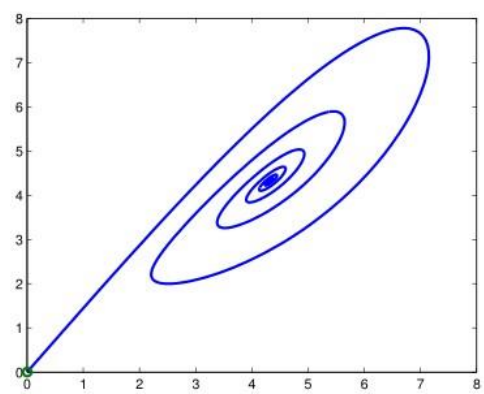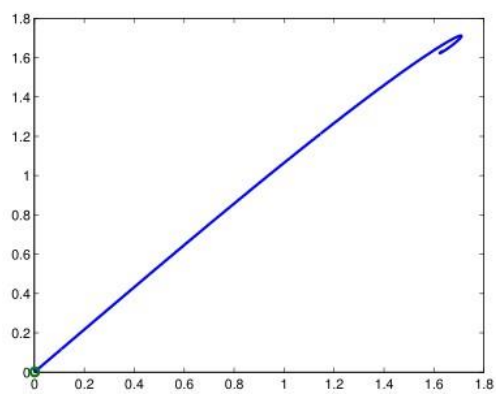
3-dimensional Graphic:

$\rho = 1$ $\rho = 1.01$
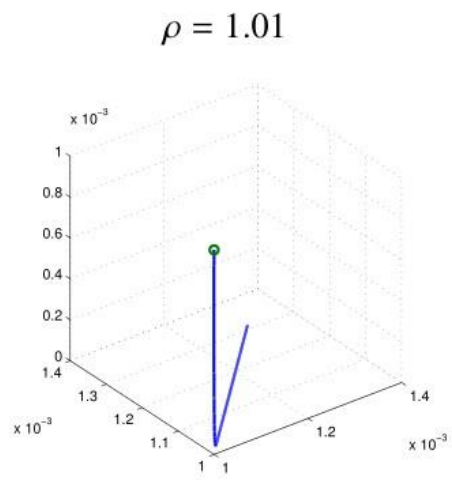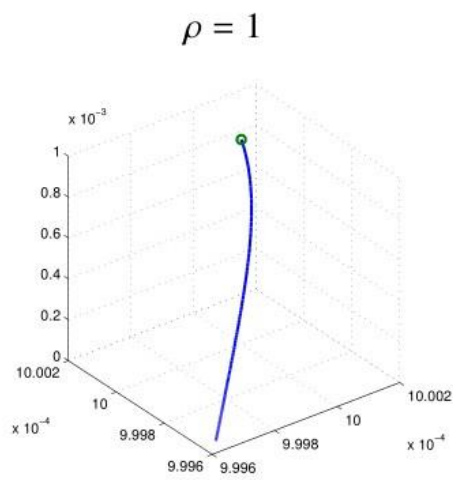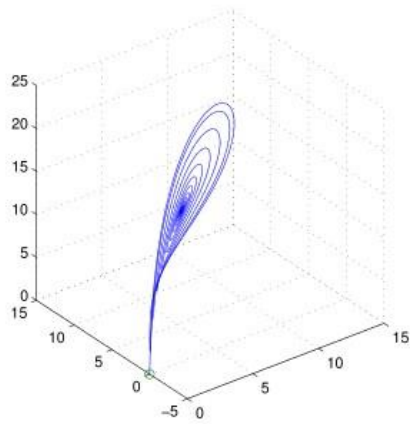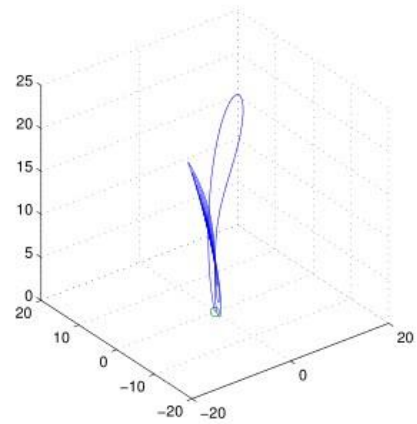

xOy Projection Graphic:
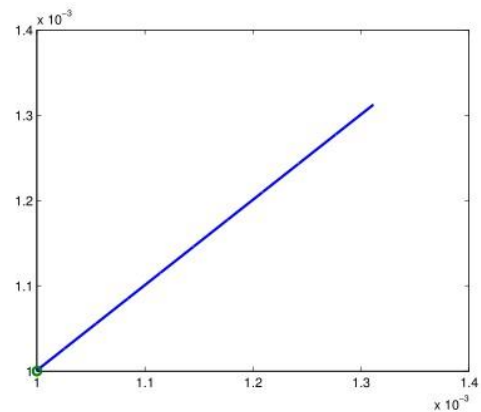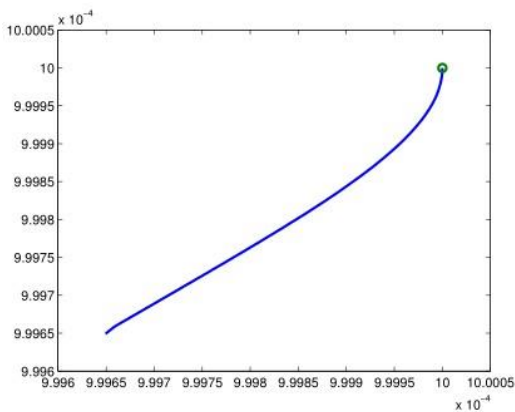
$X - T$ Graphic:
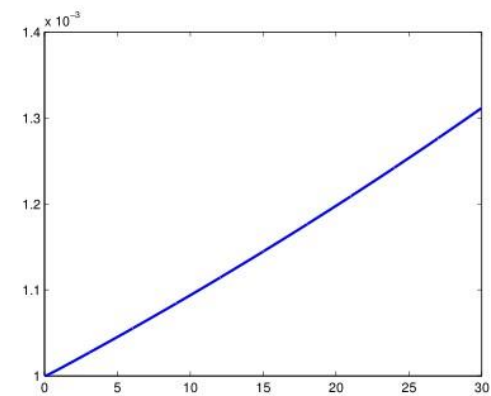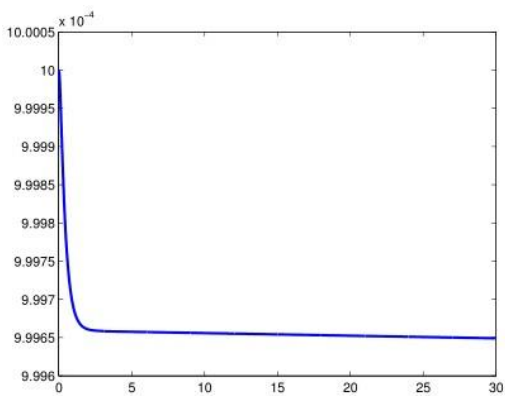
When ρ is not bigger than 1 the function converges to the origin. However, when ρ =1.01, the situation changes and the function does not converge to the origins, converging to the center of "ears", which is the center of the "right ear" in this case, based on the graphics. It is necessary to say that the circled point is the initial point. In order to compare different results conveniently, we set (x, y, z) as (0.001, 0.001, 0.001) We need to focus the difference of xOy projection of the equation with different ρ, which is seemingly similar but completely different. The initial value of the left graphic is located in the upper right corner, and then converges to the origin. The initial value of the right graphic is located in the lower left corner, and then converges to the center of the "right ear".

3-dimensional Graphic:

$$\rho = 2 \qquad\qquad \rho = 8$$
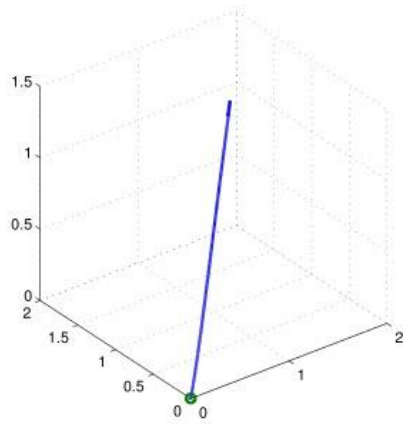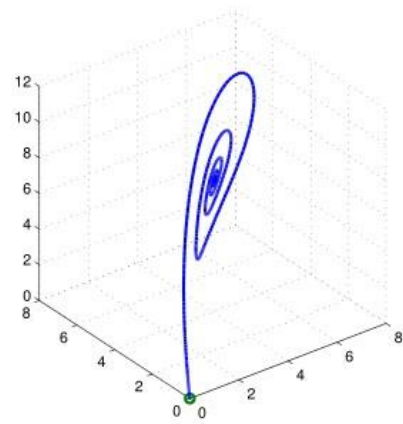
xOy Projection Graphic:
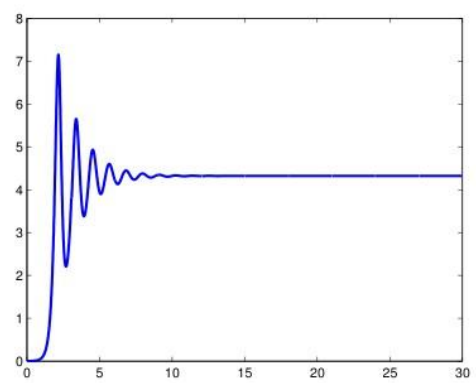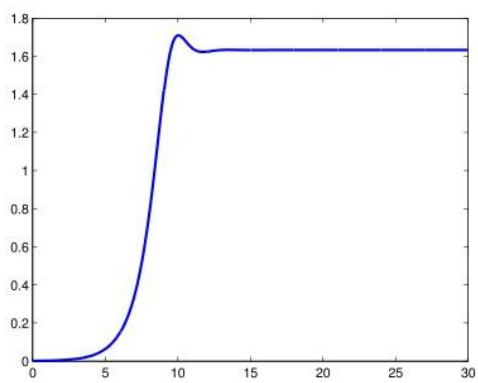
$X - T$ Graphic:

Two graphics in this page shows the convergences towards the center of the "right ear", which is obvious from the $x-t$ graphics. The equation is converging right now, far from being chaos at this time.
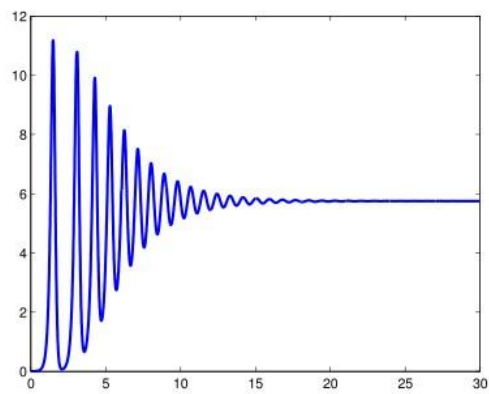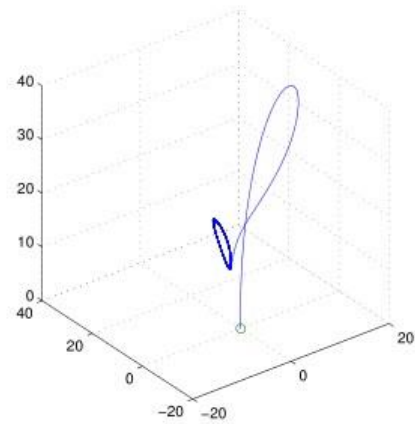
3-dimensional Graphic:

$$\rho = 13.4 \qquad\qquad \rho = 13.5$$

xOy Projection Graphic:
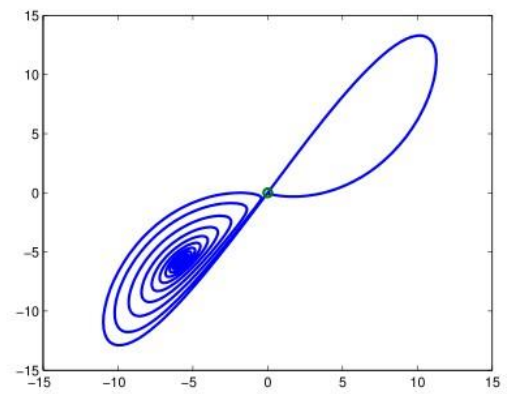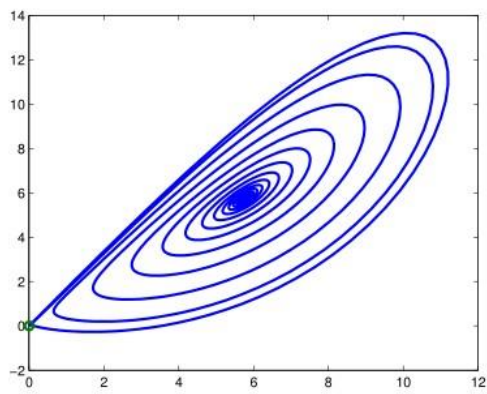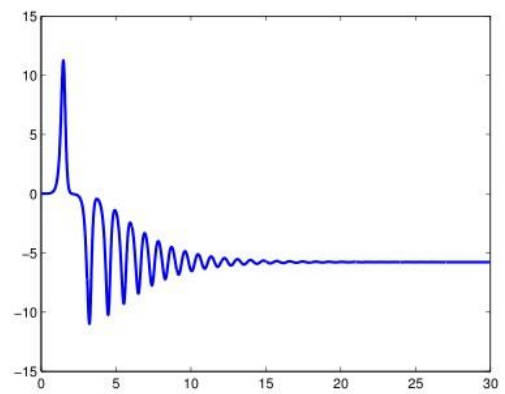
$X - T$ Graphic:

The equation changes from converging to the center of the "right ear" to converging to the center of the "left ear" when ρ changes from 13.4 to 13.5, which is obvious from 2-dimensional graphic. However, the function is still converging, which does not trigger chaos.
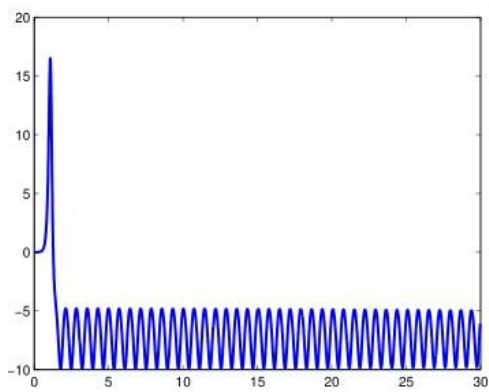
3-dimensional Graphic:

$$\rho = 21.7 \qquad\qquad \rho = 21.8$$
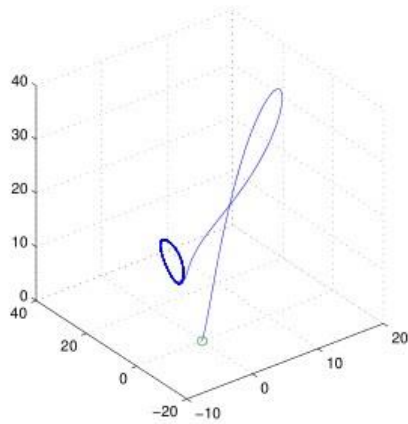

xOy Projection Graphic:
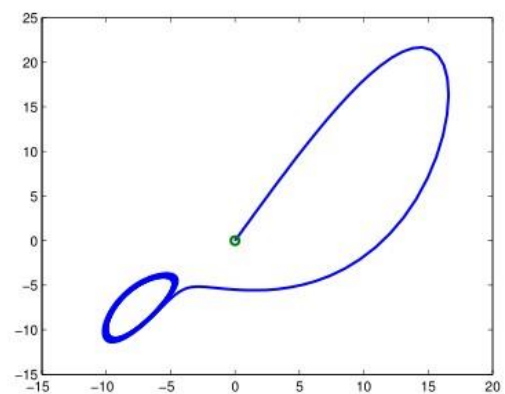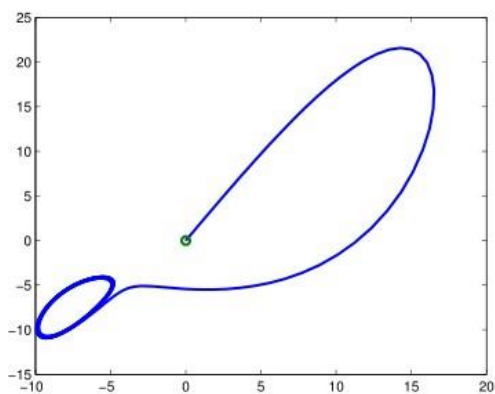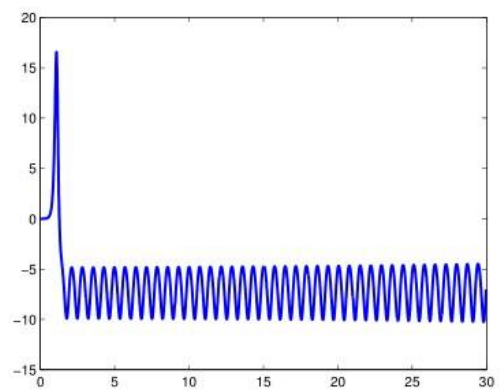
$X - T$ Graphic:

X − T   Graphic with Larger Scale
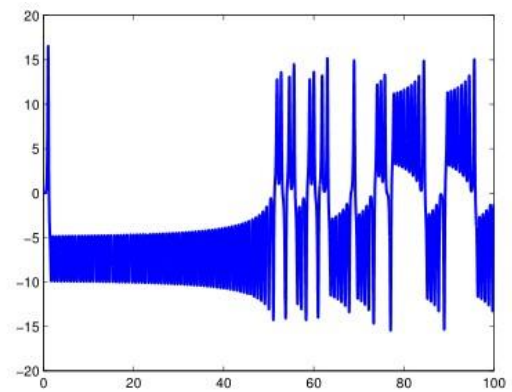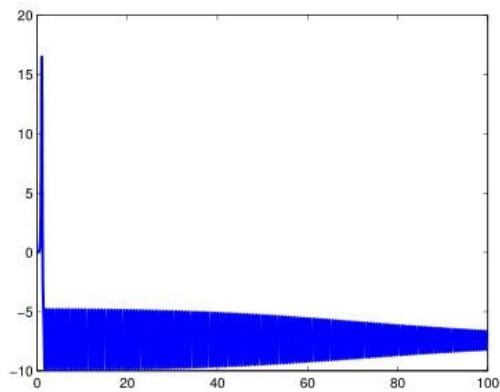
There is a dramatic change between 21.7 and 21.8, which is hard to see from the graph. The function converges to the center of the "left ear" at 21.7, but it diverges at 21.8. Based on our discussion ahead, the equation become a chaos system at 21.8, Therefore the critical point must be between 21.7 and 21.8
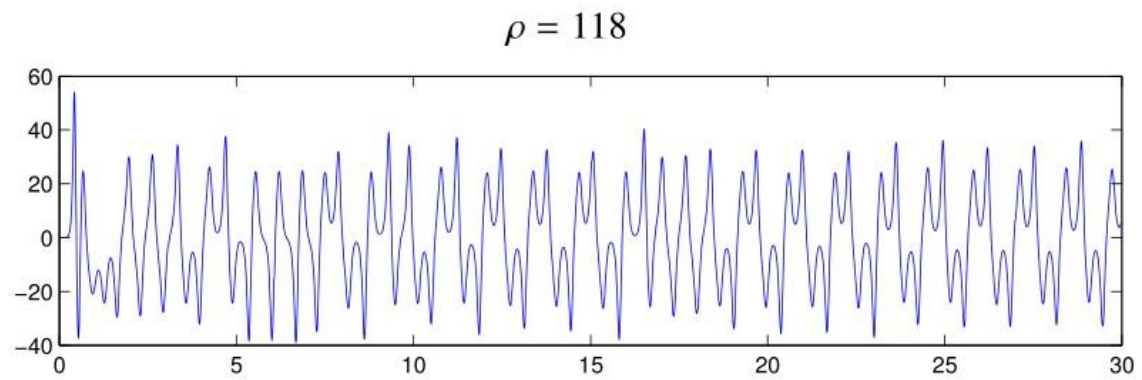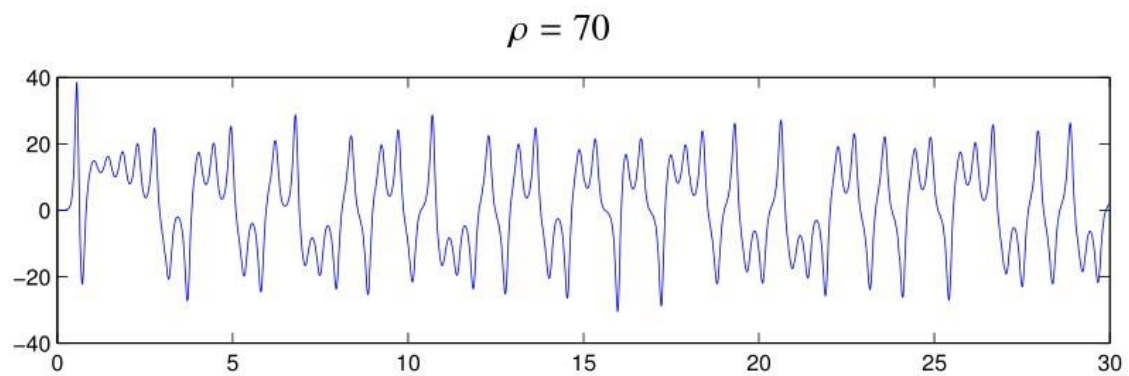
xOy Projection Graphic:
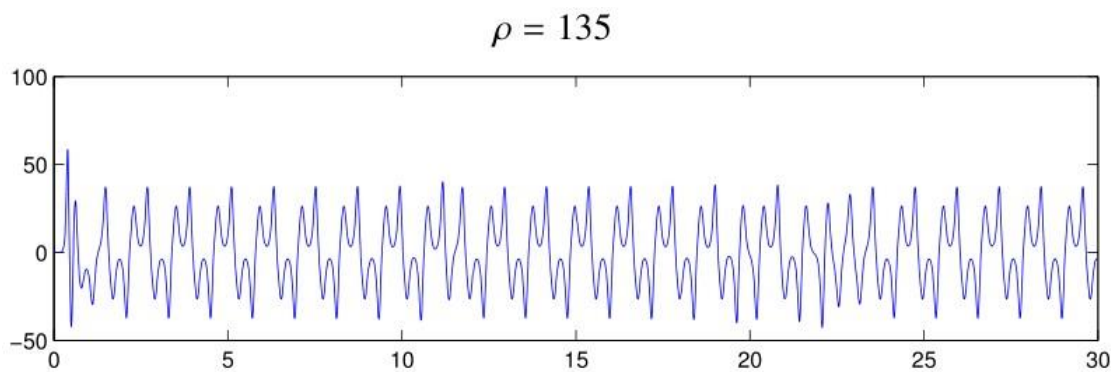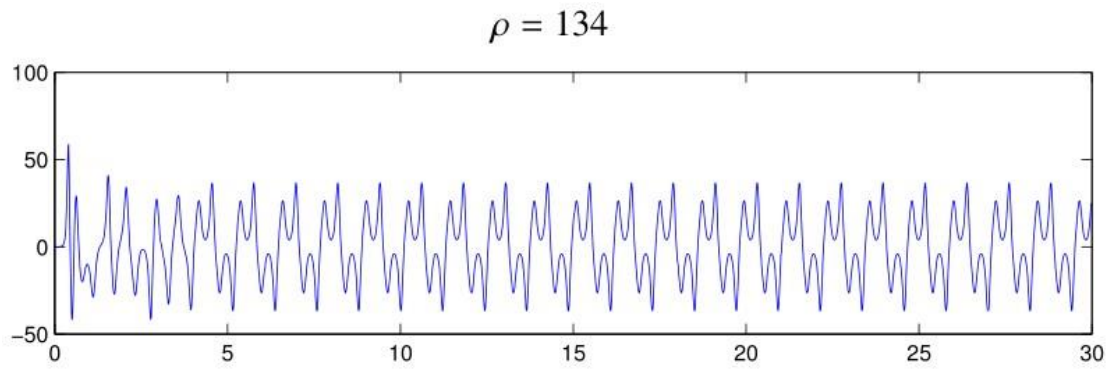

$\rho = 120$ (left) and $\rho = 130$ (right)
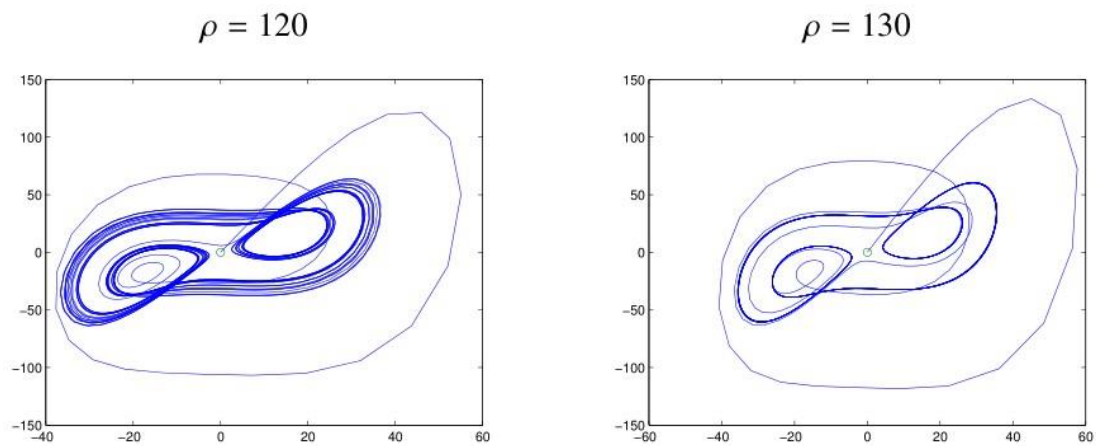
We can see that the graphic is approximately cycling at 120 and 130, indicating that the equation is not chaos when ρ equals to those values because the chaos is randomized, which means two "ears" should not be the same.

Now we can look some $x-t$ graphic to determine the interval of non-chaos
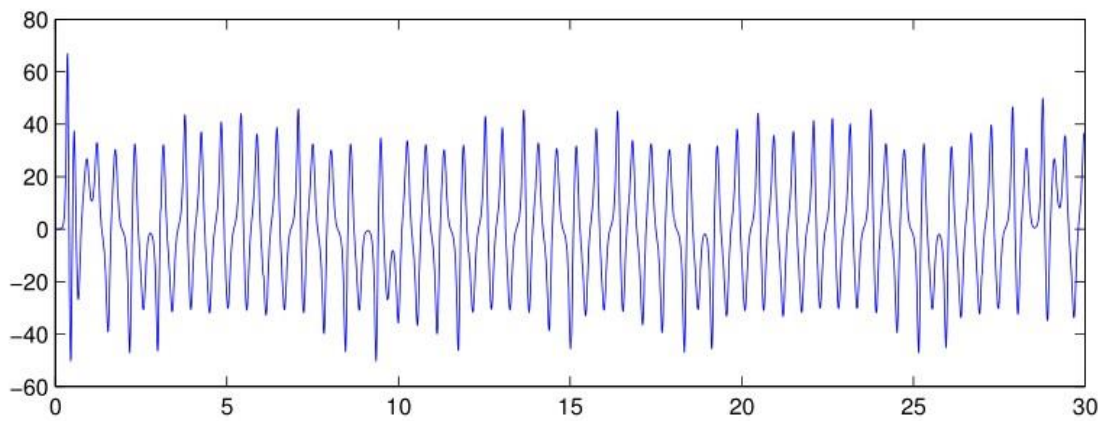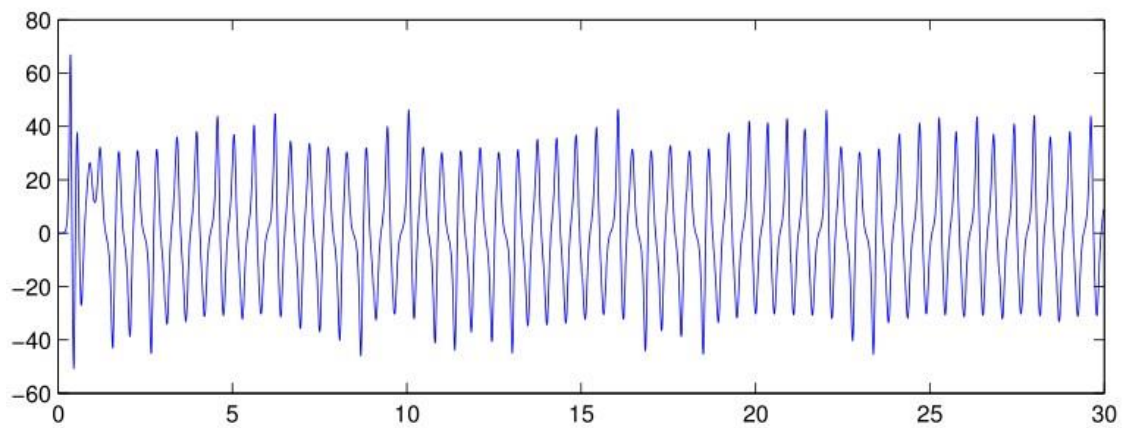
$X-T$ Graphic:

$$\rho = 70$$

$$\rho = 118$$

$$\rho = 119$$

$\rho = 134$

$\rho = 135$
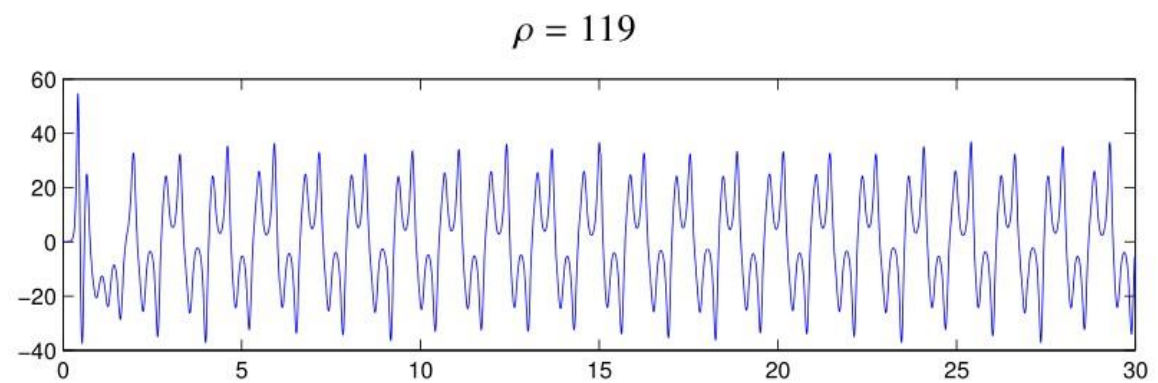
From the graphics above we can see that no pattern of winding with "ears" at 70, but we can see a pattern at 119-134, which indicates the function is not in chaos in this interval. 118 and 135, however, back to the no-pattern situation, indicating that the function goes back to chaos at those time.
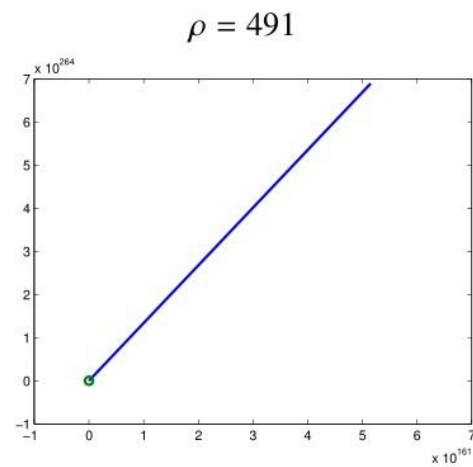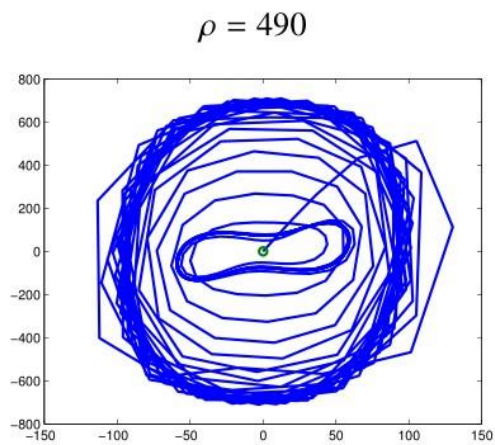
$\rho = 165$

$\rho = 166$

The function is again cycling when $\rho > 166$, indicating that the chaos

phenomenon disappear after 177

xOy Projection Graphic:

Finally, we can see that the function is cycling till 490, but it converges to a point when ρ =491, which is not in our discussion.

In general, the chaos happens when ρ in the interval of (21.7, 119) and [134, 166], including our choice of 28 in 3.2. Therefore we know which parameter can cause the chaos.

5. Summary

Chaos is a very interesting topics. We discussed the stability of the initial value, and talked about the relation between chaos happening and the parameters. We get some interesting results about chaos system, which enhances our understanding of chaos phenomenon.

6. Reference

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