

FINITE ELEMENT APPROXIMATION OF NONLOCAL DYNAMIC FRACTURE MODELS

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ABSTRACT. In this work we estimate the convergence rate for time stepping schemes applied to nonlocal dynamic fracture modeling. Here we use the non-local formulation given by the bond based peridynamic equation of motion. We begin by establishing the existence of H^2 peridynamic solutions over any finite time interval. For this model the gradients can become large and steep slopes appear and localize when the non-locality of the model tends to zero. In this treatment spatial approximation by finite elements are used. We consider the central-difference scheme for time discretization and linear finite elements for discretization in the spatial variable. The fully discrete scheme is shown to converge to the actual H^2 solution in the mean square norm at the rate $C_t \Delta t + C_s h^2 / \epsilon^2$. Here h is the mesh size, ϵ is the length scale of nonlocal interaction and Δt is the time step. The constants C_t and C_s are independent of Δt , and h . In the absence of nonlinearity a CFL like condition for the energy stability of the central difference time discretization scheme is developed. As an example we consider Plexiglass and compute constants in the a-priori error bound.

1. Introduction. In this article we consider non local models for dynamic crack propagation in solids. We focus on the peridynamic bond based formulation introduced in [37]. The basic idea is to redefine the strain in terms of the difference quotients of the displacement field and allow for nonlocal interaction within some finite horizon. The formulation has a natural length scale given by the size of the horizon. The force at any given material point is computed by considering the deformation of all neighboring material points within a radius given by the size of horizon. Here we examine the finite element approximation to the nonlinear non-local model proposed and examined in [27, 28]. One of the important aspects of this model is the that as the size of the horizon goes to zero the model behaves

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as if it is a linear elastic model away from the crack set and has bounded Griffith fracture energy [27, 28]. Therefore in the limit the model not only converges to the local elastic model in regions with small deformation but also has finite Griffith fracture energy associated with a sharp fracture set. The nonlinear potential can be calibrated so that it gives the same fracture toughness as in Linear Elastic Fracture Mechanics models. The force potential used in this model is a smooth version of the prototypical micro elastic brittle bond model introduced in [37]. Further, the slope of the nonlinear force for small strain (as we show in Section 2) is controlled by elastic constant of the material.

Earlier work shows that the linear peridynamic model converges to the linear elastic model, when the nonlocal length scale goes to zero, this is seen in the convergence of the integral operators to the differential operators, see [14], [38], [2]. More fundamentally the convergence of linear peridynamics to local elasticity in the sense of solution operators is shown in [34]. Aspects of crack propagation using the peridynamics model has been considered extensively, see for example [36], [5], [21], [1], [18], [11], [22], [30]. Theoretical analysis of peridynamics models are carried out in [27], [33], [28], [15], [13], [14], [17], [3], [9], [32], [31], [12]. Dynamic phase transformations described by peridynamic theory has been proposed and analyzed in [10]. For thin plates and shells, the peridynamic model is developed and applied in [35, 26, 39].

In this work, we consider the spatial discretization given by the finite element approximation associated with linear conforming elements. Here the potential used to compute the force between points is of double well type. One well corresponds to linear elastic deformation and is zero for zero strain and other corresponds to material rupture and has a well at infinity. To proceed we first show the existence of evolutions described by the twice differentiable time dependent displacement field taking values in $H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d)$, see Theorem 3.2. In Theorem 3.3, we show that the peridynamic evolution can have improved differentiability in time when body forces satisfy differentiability in time. We next address the stability of semi-discrete approximation for the nonlinear model and show that the evolution is energetically stable, see Theorem 4.1. We then consider the linearization of the nonlinear model and provide a stability analysis of the fully discrete approximation. Here we follow [25] and [20] to obtain a stability condition on Δt for the linearized model, see Theorem 5.4. The rationale is that for small strains the material behaves like a linear elastic material. Related work for linear local elastic models establish stability of the general Newmark time discretization [4], [19], [25]. This behavior is shown to persist for nonlocal models in [20], using techniques in [4], [19], [25]. For the nonlinear model we establish a Lax Richtmyer stability, see Lemma 5.3 and 5.1.

The primary contribution of this paper is the approximation of twice differentiable in time peridynamic evolutions taking values in $H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d)$. Here the spatial approximation is given by linear conforming elements. The time stepping approximation with linear finite element approximation in the spatial variables converges to the actual solution in the mean square norm at the rate $C_t \Delta t + C_s h^2 / \epsilon^2$ where h is the mesh size, ϵ is the size of nonlocal interaction and Δt is the time step, see Theorem 5.1. The constant C_t is independent of Δt and h and depends on the L^2 norm of the time derivatives of the exact solution. The constant C_s is also independent of Δt , and h and depends on the H^2 norm of the exact solution. We assess the impact of the constants appearing in Theorem 5.1 for evolution times seen in fracture experiments in Section 6.

Related recent work, [23], addresses the finite difference approximation of Hölder space $C^{0,\gamma}(D; \mathbf{R}^d)$ valued peridynamic evolutions with Hölder exponent $\gamma \in (0, 1]$. A convergence rate of $C_t \Delta t + C_s h^\gamma / \epsilon^2$ is demonstrated. The constant C_t depends on the L^2 norm of the time derivative of the exact solution and the constant C_s depends on the Hölder norm of the exact solution and the Lipschitz constant of peridynamic force. It is clear from these estimates that the rate of convergence for the approximation improves with the differentiability of the solution. Our results show that the convergence rate is an order of magnitude better for finite element approximations (in space) of $H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d)$ valued peridynamic evolutions than finite difference approximations (in space) of Hölder space valued peridynamic evolutions.

The organization of article is as follows: In Section 2, we introduce the class of nonlocal nonlinear potentials used in this article. We establish the existence of $H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d)$ solutions in Section 3. In Section 4 we describe the finite element approximation and establish energy stability for the semi-discrete in time approximation. In Section 5 we consider the central in time discretization and describe the convergence rate of the FEM approximation to the true solution. We establish a CFL like criterion on the time step for stability of the linearized model. We discuss the convergence rate and the associated a-priori error over time scales seen in fracture experiments, see Section 6. We provide concluding remarks addressing the existence of asymptotically compatible schemes in the context of fracture, see Section 7. The proof of claims are given in Appendix A.

2. Class of bond-based nonlinear potentials. In this section, we present the nonlinear nonlocal model. Let $D \subset \mathbf{R}^d$, for $d = 2, 3$ be the material domain with characteristic length-scale of unity. To fix ideas D is assumed to be an open set with C^1 boundary. Every material point $x \in D$ interacts nonlocally with all other material points inside a horizon of length $\epsilon \in (0, 1)$. Let $H_\epsilon(x)$ be the ball of radius ϵ centered at x containing all points y that interact with x . After deformation the material point x assumes position $z = x + u(x)$. The deformation z is given by $z(x) = u(x) + x$ where u is the displacement field. The strain between two points x and y inside D is given by

$$\mathcal{S} = \frac{|z(y) - z(x)| - |y - x|}{|y - x|}. \quad (1)$$

In this treatment we assume infinitesimal displacements $u(x)$ so the deformed configuration is the same as the reference configuration and on linearizing the strain is given by

$$S = S(y, x; u) = \frac{u(y) - u(x)}{|y - x|} \cdot \frac{y - x}{|y - x|}.$$

We let t denote time and the displacement field $u(t, x)$ evolves according to a nonlocal version of Cauchy's equations of motion for a continuum body

$$\rho \partial_{tt}^2 u(t, x) = \mathcal{L}^\epsilon(u(t))(x) + b(t, x) \quad (2)$$

for all $x \in D$. Here the body force applied to the domain D can evolve with time and is denoted by $b(t, x)$. Without loss of generality, we will assume $\rho = 1$. The peridynamic force denoted by $\mathcal{L}^\epsilon(u)(x)$ is given by summing up all forces acting on

x ,

$$\mathcal{L}^\epsilon(u)(x) = \frac{2}{\epsilon^d \omega_d} \int_{H_\epsilon(x)} \partial_S W^\epsilon(S, y-x) \frac{y-x}{|y-x|} dy,$$

where $\partial_S W^\epsilon$ is the force exerted on x by y and is given by the derivative of the nonlocal two point potential $W^\epsilon(S, y-x)$ with respect to the strain and ω_d is volume of unit ball in dimension d .

Let ∂D be the boundary of material domain D . The Dirichlet boundary condition on u is

$$u(t, x) = 0 \quad \forall x \in \partial D, \forall t \in [0, T] \quad (3)$$

and the initial condition is

$$u(0, x) = u_0(x) \quad \text{and} \quad \partial_t u(0, x) = v_0(x), \quad \forall x \in D. \quad (4)$$

The initial data and solution $u(t, x)$ are extended by 0 outside D . We remark that traction boundary conditions can be introduced in the nonlocal context by prescribing a body force along a boundary layer of width ϵ and allowing the displacement to be free there.

Define the energy $\mathcal{E}^\epsilon(u)(t)$ to be the sum of kinetic and potential energy and is given by

$$\mathcal{E}^\epsilon(u)(t) = \frac{1}{2} \|\dot{u}(t)\|_{L^2}^2 + PD^\epsilon(u(t)). \quad (5)$$

where potential energy PD^ϵ is given by

$$PD^\epsilon(u) = \int_D \left[\frac{1}{\epsilon^d \omega_d} \int_{H_\epsilon(x)} |y-x| W^\epsilon(S(u), y-x) dy \right] dx.$$

Using the definition of \mathcal{E}^ϵ in 5, one easily sees that

$$\frac{d}{dt} \mathcal{E}^\epsilon(u)(t) = (\ddot{u}(t), \dot{u}(t)) - (\mathcal{L}^\epsilon(u(t)), \dot{u}(t)). \quad (6)$$

2.1. Nonlocal potential. We now describe the nonlocal potential. We consider potentials W^ϵ of the form

$$W^\epsilon(S, y-x) = \omega(x)\omega(y) \frac{J^\epsilon(|y-x|)}{\epsilon|y-x|} f(|y-x|S^2), \quad (7)$$

where $f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is assumed to be positive, smooth, and concave with following properties

$$\lim_{r \rightarrow 0^+} \frac{f(r)}{r} = f'(0), \quad \lim_{r \rightarrow \infty} f(r) = f_\infty < \infty. \quad (8)$$

The peridynamic force \mathcal{L}^ϵ is written as

$$\mathcal{L}^\epsilon(u)(x) = \frac{4}{\epsilon^{d+1} \omega_d} \int_{H_\epsilon(x)} \omega(x)\omega(y) J^\epsilon(|y-x|) f'(|y-x|S(u)^2) S(u) e_{y-x} dy, \quad (9)$$

where we used the notation $S(u) = S(y, x; u)$ and $e_{y-x} = \frac{y-x}{|y-x|}$.

The function $J^\epsilon(|y-x|)$ models the influence of separation between points y and x . Here $J^\epsilon(|y-x|) = J(|y-x|/\epsilon)$ can be piecewise smooth and we define J to be zero outside the ball $\{\xi : |\xi| < 1\} = H_1(0)$ and $0 \leq J(|\xi|) \leq M$ for all $\xi \in H_1(0)$.

The boundary function $\omega(x)$ is nonnegative and takes the value 1 for points x inside D of distance ϵ away from the boundary ∂D . Inside the boundary layer of width ϵ the function $\omega(x)$ smoothly decreases from 1 to 0 taking the value 0 on ∂D .

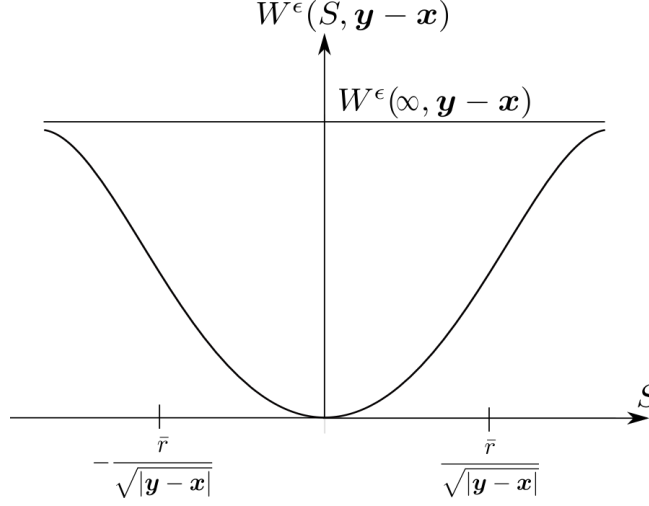


FIGURE 1. Two-point potential $W^\epsilon(S, y-x)$ as a function of strain S for fixed $y-x$.

In the sequel we will set

$$\bar{\omega}_\xi(x) = \omega(x)\omega(x + \epsilon\xi) \quad (10)$$

and we assume

$$|\nabla \bar{\omega}_\xi| \leq C_{\omega_1} < \infty \quad \text{and} \quad |\nabla^2 \bar{\omega}_\xi| \leq C_{\omega_2} < \infty.$$

The potential described in 7 gives the convex-concave dependence, see Figure 1, of $W(S, y-x)$ on the strain S for fixed $y-x$. Initially the deformation is elastic for small strains and then softens as the strain becomes larger. The critical strain where the force between x and y begins to soften is given by $S_c(y, x) := \bar{r}/\sqrt{|y-x|}$ and the force decreases monotonically for

$$|S(y, x; u)| > S_c. \quad (11)$$

Here \bar{r} is the inflection point of $r \mapsto f(r^2)$ and is the root of following equation

$$f'(r^2) + 2r^2 f''(r^2) = 0. \quad (12)$$

In Theorem 5.2 of [28], it is shown that in the limit $\epsilon \rightarrow 0$, the peridynamics solution has bounded linear elastic fracture energy, provided the initial data (u_0, v_0) has bounded linear elastic fracture energy and u_0 is bounded. The elastic constant (Lamé constant λ and μ) and energy release rate of the limiting energy is given by

$$\lambda = \mu = C_d f'(0) M_d, \quad \mathcal{G}_c = \frac{2\omega_{d-1}}{\omega_d} f_\infty M_d$$

where $M_d = \int_0^1 J(r) r^d dr$ and $f_\infty = \lim_{r \rightarrow \infty} f(r)$. $C_d = 2/3, 1/4, 1/5$ for $d = 1, 2, 3$ respectively and $\omega_n = 1, 2, \pi, 4\pi/3$ for $n = 0, 1, 2, 3$. Therefore, $f'(0)$ and f_∞ are determined by the Lamé constant λ and fracture toughness \mathcal{G}_c .

3. Existence of solutions in $H^2 \cap H_0^1$. We consider function space W given by

$$W := H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d) = \{v \in H^2(D; \mathbf{R}^d) : \gamma v = 0, \text{ on } \partial D\}, \quad (13)$$

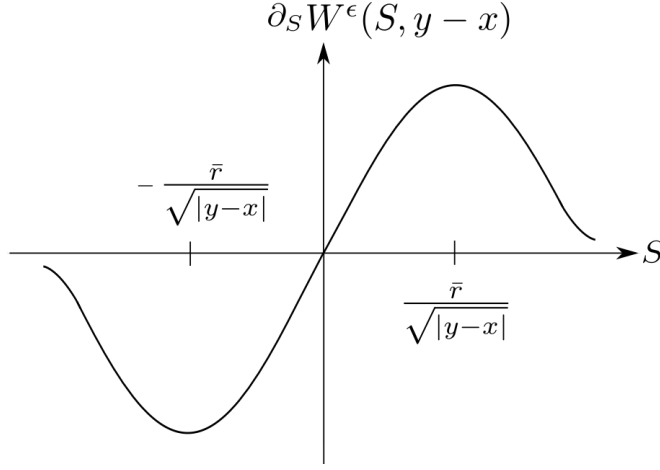


FIGURE 2. Nonlocal force $\partial_S W^\epsilon(S, y - x)$ as a function of strain S for fixed $y - x$. Second derivative of $W^\epsilon(S, y - x)$ is zero at $\pm \bar{r}/\sqrt{|y - x|}$.

where γv is the trace of function v on the boundary ∂D . Norm on W is $H^2(D; \mathbf{R}^d)$ norm. In this section, we show that for suitable initial condition and body force solutions of peridynamic equation exist in W . We will assume that $u \in W$ is extended by zero outside D .

We note the following Sobolev embedding properties of $H^2(D; \mathbf{R}^d)$ when D is a C^1 domain.

- From Theorem 2.72 of [16], there exists a constant C_{e_1} independent of $u \in H^2(D; \mathbf{R}^d)$ such that

$$\|u\|_{L^\infty(D; \mathbf{R}^d)} \leq C_{e_1} \|u\|_{H^2(D; \mathbf{R}^d)}. \quad (14)$$

- Further application of standard embedding theorems (e.g., Theorem 2.72 of [16]), shows there exists a constant C_{e_2} independent of u such that

$$\|\nabla u\|_{L^q(D; \mathbf{R}^{d \times d})} \leq C_{e_2} \|\nabla u\|_{H^1(D; \mathbf{R}^{d \times d})} \leq C_{e_2} \|u\|_{H^2(D; \mathbf{R}^d)}, \quad (15)$$

for any q such that $2 \leq q < \infty$ when $d = 2$ and $2 \leq q \leq 6$ when $d = 3$.

In what follows, we first state the Lipschitz continuity property for $\mathcal{L}^\epsilon(u)$. We then state the existence theorem for solutions over finite time intervals. These are proved in Appendix A. We now write the peridynamic evolution equation as an equivalent first order system with $y_1(t) = u(t)$ and $y_2(t) = v(t)$ with $v(t) = \dot{u}(t)$. Let $y = (y_1, y_2)^T$ where $y_1, y_2 \in W$ and let $F^\epsilon(y, t) = (F_1^\epsilon(y, t), F_2^\epsilon(y, t))^T$ such that

$$F_1^\epsilon(y, t) := y_2, \quad (16)$$

$$F_2^\epsilon(y, t) := \mathcal{L}^\epsilon(y_1) + b(t). \quad (17)$$

The initial boundary value problem is equivalent to the initial boundary value problem for the first order system given by

$$\dot{y}(t) = F^\epsilon(y, t), \quad (18)$$

with initial condition given by $y(0) = (u_0, v_0)^T \in W \times W$. Recall that we denote the norm on $H^2(D; \mathbf{R}^d)$ as $\|\cdot\|_2$.

Theorem 3.1. Lipschitz continuity of the peridynamic force For any $u, v \in W$, we have

$$\begin{aligned} & \|\mathcal{L}^\epsilon(u) - \mathcal{L}^\epsilon(v)\|_2 \\ & \leq \frac{\bar{L}_1 + \bar{L}_2(\|u\|_2 + \|v\|_2) + \bar{L}_3(\|u\|_2 + \|v\|_2)^2}{\epsilon^3} \|u - v\|_2 \end{aligned} \quad (19)$$

where constants $\bar{L}_1, \bar{L}_2, \bar{L}_3$ are independent of ϵ , u , and v , and are defined in 129. Also, for $u \in W$, we have

$$\|\mathcal{L}^\epsilon(u)\|_2 \leq \frac{\bar{L}_4\|u\|_2 + \bar{L}_5\|u\|_2^2}{\epsilon^{5/2}}, \quad (20)$$

where constants are independent of ϵ and u and are defined in 138.

In Theorem 6.1 of [28], the Lipschitz property of the peridynamic force is shown in $L^2(D; \mathbf{R}^d)$ and is given by

$$\|\mathcal{L}^\epsilon(u) - \mathcal{L}^\epsilon(v)\| \leq \frac{L_1}{\epsilon^2} \|u - v\| \quad \forall u, v \in L^2(D; \mathbf{R}^d), \quad (21)$$

with L_1 given by 100.

We state the theorem which shows the existence and uniqueness of solution in any given finite time interval $I_0 = (-T, T)$.

Theorem 3.2. Existence and uniqueness of solutions over finite time intervals For any initial condition $x_0 \in X = W \times W$, time interval $I_0 = (-T, T)$, and right hand side $b(t)$ continuous in time for $t \in I_0$ such that $b(t)$ satisfies $\sup_{t \in I_0} \|b(t)\|_2 < \infty$, there is a unique solution $y(t) \in C^1(I_0; X)$ of

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau) d\tau,$$

or equivalently

$$y'(t) = F^\epsilon(y(t), t), \text{ with } y(0) = x_0,$$

where $y(t)$ and $y'(t)$ are Lipschitz continuous in time for $t \in I_0$.

It is found that the peridynamic evolutions have higher regularity in time for body forces that are differentiable in time. We now state the higher temporal regularity for peridynamic evolutions.

Theorem 3.3. Higher regularity Suppose the initial data and righthand side $b(t)$ satisfy the hypothesis of Theorem 3.2 and suppose further that $\dot{b}(t)$ exists and is continuous in time for $t \in I_0$ and $\sup_{t \in I_0} \|\dot{b}(t)\|_2 < \infty$. Then $u \in C^3(I_0; W)$ and

$$\|\partial_{ttt}^3 u(t, x)\|_2 \leq \frac{C(1 + \sup_{s \in I_0} \|u(s)\|_2 + \sup_{s \in I_0} \|u(s)\|_2^2)}{\epsilon^3} \sup_{s \in I_0} \|\partial_t u(s)\|_2 + \|\dot{b}(t, x)\|_2, \quad (22)$$

where C is a positive constant independent of u .

The proofs of Theorem 3.1, Theorem 3.2, and Theorem 3.3 are given in Appendix A. We now discuss the finite element approximation of the peridynamic evolution.

4. Finite element interpolation. Let V_h be given by linear continuous interpolations over tetrahedral or triangular elements \mathcal{T}_h where h denotes the size of finite element mesh. Here we assume the elements are conforming and the finite element mesh is shape regular and $V_h \subset H_0^1(D; \mathbf{R}^d)$.

For a continuous function u on \bar{D} , $\mathcal{I}_h(u)$ is the continuous piecewise linear interpolant on \mathcal{T}_h . It is given by

$$\mathcal{I}_h(u) \Big|_T = \mathcal{I}_T(u) \quad \forall T \in \mathcal{T}_h, \quad (23)$$

where $\mathcal{I}_T(u)$ is the local interpolant defined over finite element T and is given by

$$\mathcal{I}_T(u) = \sum_{i=1}^n u(x_i) \phi_i. \quad (24)$$

Here n is the number of vertices in an element T , x_i is the position of vertex i , and ϕ_i is the linear interpolant associated to vertex i .

Application of Theorem 4.4.20 and Remark 4.4.27 in [6] gives

$$\|u - \mathcal{I}_h(u)\| \leq ch^2 \|u\|_2, \quad \forall u \in W. \quad (25)$$

Let $r_h(u)$ denote the projection of $u \in W$ on V_h . For the L^2 norm it is defined as

$$\|u - r_h(u)\| = \inf_{\tilde{u} \in V_h} \|u - \tilde{u}\|. \quad (26)$$

and satisfies

$$(r_h(u), \tilde{u}) = (u, \tilde{u}), \quad \forall \tilde{u} \in V_h. \quad (27)$$

Since $\mathcal{I}_h(u) \in V_h$, and 25 we see that

$$\|u - r_h(u)\| \leq ch^2 \|u\|_2, \quad \forall u \in W. \quad (28)$$

4.1. Semi-discrete approximation. Let $u_h(t) \in V_h$ be the approximation of $u(t)$ which satisfies the following

$$(\ddot{u}_h(t), \tilde{u}) = (\mathcal{L}^\epsilon(u_h(t)), \tilde{u})(b(t), \tilde{u}), \quad \forall \tilde{u} \in V_h. \quad (29)$$

We now show that the semi-discrete approximation is stable, i.e. the energy at time t is bounded by the initial energy and work done by the body force.

Theorem 4.1. Energy stability of the semi-discrete approximation *The semi-discrete scheme is energetically stable and the energy $\mathcal{E}^\epsilon(u_h)(t)$, defined in 5, satisfies the following bound*

$$\mathcal{E}^\epsilon(u_h)(t) \leq \left[\sqrt{\mathcal{E}^\epsilon(u_h)(0)} + \int_0^t \|b(\tau)\| d\tau \right]^2.$$

Proof. Letting $\tilde{u} = \dot{u}_h(t)$ in 29 and applying the identity 6, we get

$$\frac{d}{dt} \mathcal{E}^\epsilon(u_h)(t) = (b(t), \dot{u}_h(t)) \leq \|b(t)\| \|\dot{u}_h(t)\|.$$

We also have

$$\|\dot{u}_h(t)\| \leq 2\sqrt{\frac{1}{2}\|\dot{u}_h\|^2 + PD^\epsilon(u_h(t))} = 2\sqrt{\mathcal{E}^\epsilon(u_h)(t)}$$

where we used the fact that $PD^\epsilon(u)(t)$ is nonnegative and

$$\frac{d}{dt}\mathcal{E}^\epsilon(u_h)(t) \leq 2\sqrt{\mathcal{E}^\epsilon(u_h)(t)} \|b(t)\|.$$

Fix $\delta > 0$ and let $A(t) = \mathcal{E}^\epsilon(u_h(t)) + \delta$. Then from the equation above we easily see that

$$\frac{d}{dt}A(t) \leq 2\sqrt{A(t)} \|b(t)\| \Rightarrow \frac{1}{2} \frac{\frac{d}{dt}A(t)}{\sqrt{A(t)}} \leq \|b(t)\|.$$

Noting that $\frac{1}{\sqrt{a(t)}} \frac{da(t)}{dt} = 2 \frac{d}{dt} \sqrt{a(t)}$, integrating from $t = 0$ to τ and relabeling τ as t , we get

$$\sqrt{A(t)} \leq \sqrt{A(0)} + \int_0^t \|b(s)\| ds.$$

Letting $\delta \rightarrow 0$ and taking the square of both side proves the claim. \square

5. Central difference in time discretization paired with the finite element spatial discretization. We consider the central difference approximation scheme in time paired with the finite element approximation in space. We present the convergence rate for this fully discrete approximation to this nonlinear and nonlocal problem. We conclude with a discussion of the linearized peridynamic evolution and a demonstration of CFL like conditions for stability of the fully discrete scheme.

Let Δt be the time step. The exact solution at $t^k = k\Delta t$ (or time step k) is denoted as (u^k, v^k) , with $v^k = \partial u^k / \partial t$, and the projection onto V_h at t^k is given by $(r_h(u^k), r_h(v^k))$. The solution of the discrete problem at time step k is denoted as (u_h^k, v_h^k) .

We first describe the finite element approximation. The initial data for displacement u_0 and velocity v_0 are approximated by their projections $r_h(u_0)$ and $r_h(v_0)$. Let $u_h^0 = r_h(u_0)$ and $v_h^0 = r_h(v_0)$. Let b_h denotes the projection of body force $b(t^k)$. The finite element approximation of the peridynamic evolution is given as follows. For $k \geq 1$, (u_h^k, v_h^k) satisfies, for all $\tilde{u} \in V_h$,

$$\begin{aligned} \left(\frac{u_h^{k+1} - u_h^k}{\Delta t}, \tilde{u} \right) &= (v_h^{k+1}, \tilde{u}), \\ \left(\frac{v_h^{k+1} - v_h^k}{\Delta t}, \tilde{u} \right) &= (\mathcal{L}^\epsilon(u_h^k), \tilde{u}) + (b_h^k, \tilde{u}). \end{aligned} \quad (30)$$

Combining the two equations delivers the central difference equation for u_h^k . We have

$$\left(\frac{u_h^{k+1} - 2u_h^k + u_h^{k-1}}{\Delta t^2}, \tilde{u} \right) = (\mathcal{L}^\epsilon(u_h^k), \tilde{u}) + (b_h^k, \tilde{u}), \quad \forall \tilde{u} \in V_h. \quad (31)$$

For $k = 0$, we have $\forall \tilde{u} \in V_h$

$$\left(\frac{u_h^1 - u_h^0}{\Delta t^2}, \tilde{u} \right) = \frac{1}{2}(\mathcal{L}^\epsilon(u_h^0), \tilde{u}) + \frac{1}{\Delta t}(v_h^0, \tilde{u}) + \frac{1}{2}(b_h^0, \tilde{u}). \quad (32)$$

We now study the convergence of FE approximation stated in [30](#).

5.1. Convergence of approximation. We show how to establish a uniform bound on the L^2 norm of the discretization error and prove that approximate solution converges to the exact solution at the rate $C_t \Delta t + C_s h^2 / \epsilon^2$ for fixed $\epsilon > 0$. We first compare the exact solution with its projection in V_h and then compare the projection with approximate solution. We further divide the calculation of error between projection and approximate solution in two parts, namely consistency analysis and error analysis.

Error E^k is given by

$$E^k := \|u_h^k - u(t^k)\| + \|v_h^k - v(t^k)\|.$$

We split the error as follows

$$E^k \leq (\|u^k - r_h(u^k)\| + \|v^k - r_h(v^k)\|) + (\|u_h^k - r_h(u^k)\| + \|v_h^k - r_h(v^k)\|),$$

where first term is error between exact solution and projections, and second term is error between projections and approximate solution. Let

$$e_h^k(u) := u_h^k - r_h(u^k) \quad \text{and} \quad e_h^k(v) := v_h^k - r_h(v^k) \quad (33)$$

and

$$e^k := \|e_h^k(u)\| + \|e_h^k(v)\|. \quad (34)$$

Using 28, we have

$$E^k \leq C_p h^2 + e^k, \quad (35)$$

where

$$C_p := c \left[\sup_t \|u(t)\|_2 + \sup_t \left\| \frac{\partial u(t)}{\partial t} \right\|_2 \right]. \quad (36)$$

We have following main result

Theorem 5.1. Convergence of the fully discrete approximation with respect to the L^2 norm. *Let (u, v) be the exact solution of the peridynamics equation in 2. For the k^{th} time step (u_h^k, v_h^k) are the FE approximate solution of 31 and 32. If $u, v \in C^2([0, T]; W)$, then the scheme is consistent and the error E^k satisfies following bound*

$$\begin{aligned} & \sup_{k \leq T/\Delta t} E^k \\ &= C_p h^2 + \exp[T(1 + L_1/\epsilon^2)(\frac{1}{1 - \Delta t})] \left[e^0 + \left(\frac{T}{1 - \Delta t} \right) \left(C_t \Delta t + C_s \frac{h^2}{\epsilon^2} \right) \right] \end{aligned} \quad (37)$$

where the constants C_p , C_t , and C_s are given by 36 and 47. Here the constant L_1/ϵ^2 is the Lipschitz constant of $\mathcal{L}^\epsilon(u)$ in L^2 , see 21 and 100. If the error in initial data is zero then E^k is of the order of $C_t \Delta t + C_s h^2 / \epsilon^2$.

Theorem 3.3 shows that $u, v \in C^2([0, T]; W)$ for right hand side $b \in C^1([0, T]; W)$. In Section 6 we discuss the behavior of the exponential constant appearing in Theorem 5.1 for evolution times seen in fracture experiments. Since we are approximating the solution of an ODE on a Banach space the proof of Theorem 5.1 will follow from the Lipschitz continuity of the force $\mathcal{L}^\epsilon(u)$ with respect to the L^2 norm. The proof is given in the following two sections.

5.1.1. *Truncation error analysis and consistency.* We derive the equation for evolution of $e_h^k(u)$ as follows

$$\begin{aligned}
& \left(\frac{u_h^{k+1} - u_h^k}{\Delta t} - \frac{r_h(u^{k+1}) - r_h(u^k)}{\Delta t}, \tilde{u} \right) \\
&= (v_h^{k+1}, \tilde{u}) - \left(\frac{r_h(u^{k+1}) - r_h(u^k)}{\Delta t}, \tilde{u} \right) \\
&= (v_h^{k+1}, \tilde{u}) - (r_h(v^{k+1}), \tilde{u}) + (r_h(v^{k+1}), \tilde{u}) - (v^{k+1}, \tilde{u}) \\
&\quad + (v^{k+1}, \tilde{u}) - \left(\frac{\partial u^{k+1}}{\partial t}, \tilde{u} \right) \\
&\quad + \left(\frac{\partial u^{k+1}}{\partial t}, \tilde{u} \right) - \left(\frac{u^{k+1} - u^k}{\Delta t}, \tilde{u} \right) \\
&\quad + \left(\frac{u^{k+1} - u^k}{\Delta t}, \tilde{u} \right) - \left(\frac{r_h(u^{k+1}) - r_h(u^k)}{\Delta t}, \tilde{u} \right).
\end{aligned}$$

Using property $(r_h(u), \tilde{u}) = (u, \tilde{u})$ for $\tilde{u} \in V_h$ and the fact that $\frac{\partial u(t^{k+1})}{\partial t} = v^{k+1}$ where u is the exact solution, we get

$$\left(\frac{e_h^{k+1}(u) - e_h^k(u)}{\Delta t}, \tilde{u} \right) = (e_h^{k+1}(v), \tilde{u}) + \left(\frac{\partial u^{k+1}}{\partial t}, \tilde{u} \right) - \left(\frac{u^{k+1} - u^k}{\Delta t}, \tilde{u} \right). \quad (38)$$

Let $(\tau_h^k(u), \tau_h^k(v))$ be the truncation error in the time discretization given by

$$\tau_h^k(u) := \frac{\partial u^{k+1}}{\partial t} - \frac{u^{k+1} - u^k}{\Delta t}, \quad (39)$$

$$\tau_h^k(v) := \frac{\partial v^k}{\partial t} - \frac{v^{k+1} - v^k}{\Delta t}. \quad (40)$$

With the above notation, we have

$$(e_h^{k+1}(u), \tilde{u}) = (e_h^k(u), \tilde{u}) + \Delta t (e_h^{k+1}(v), \tilde{u}) + \Delta t (\tau_h^k(u), \tilde{u}). \quad (41)$$

We now derive the equation for $e_h^k(v)$ as follows

$$\begin{aligned}
& \left(\frac{v_h^{k+1} - v_h^k}{\Delta t} - \frac{r_h(v^{k+1}) - r_h(v^k)}{\Delta t}, \tilde{u} \right) \\
&= (\mathcal{L}^\epsilon(u_h^k), \tilde{u}) + (b_h^k, \tilde{u}) - \left(\frac{r_h(v^{k+1}) - r_h(v^k)}{\Delta t}, \tilde{u} \right) \\
&= (\mathcal{L}^\epsilon(u_h^k), \tilde{u}) + (b^k, \tilde{u}) - \left(\frac{\partial v^k}{\partial t}, \tilde{u} \right) \\
&\quad + \left(\frac{\partial v^k}{\partial t}, \tilde{u} \right) - \left(\frac{v^{k+1} - v^k}{\Delta t}, \tilde{u} \right) \\
&\quad + \left(\frac{v^{k+1} - v^k}{\Delta t}, \tilde{u} \right) - \left(\frac{r_h(v^{k+1}) - r_h(v^k)}{\Delta t}, \tilde{u} \right) \\
&= (\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(u^k), \tilde{u}) + (b_h^k - b(t^k), \tilde{u})
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\partial v^k}{\partial t}, \tilde{u} \right) - \left(\frac{v^{k+1} - v^k}{\Delta t}, \tilde{u} \right) + \left(\frac{v^{k+1} - v^k}{\Delta t}, \tilde{u} \right) - \left(\frac{r_h(v^{k+1}) - r_h(v^k)}{\Delta t}, \tilde{u} \right) \\
& = (\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(u^k), \tilde{u}) + \left(\frac{\partial v^k}{\partial t} - \frac{v^{k+1} - v^k}{\Delta t}, \tilde{u} \right)
\end{aligned}$$

where we used the property of $r_h(u)$ and the fact that

$$(\mathcal{L}^\epsilon(u^k), \tilde{u}) + (b^k, \tilde{u}) - \left(\frac{\partial v^k}{\partial t}, \tilde{u} \right) = 0, \quad \forall \tilde{u} \in V_h.$$

We further divide the error in the peridynamics force as follows

$$\begin{aligned}
& (\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(u^k), \tilde{u}) \\
& = (\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(r_h(u^k)), \tilde{u}) + (\mathcal{L}^\epsilon(r_h(u^k)) - \mathcal{L}^\epsilon(u^k), \tilde{u}).
\end{aligned}$$

We will see in next section that second term is related to the truncation error in the spatial discretization. Therefore, we define another truncation error term $\sigma_{per,h}^k(u)$ as follows

$$\sigma_{per,h}^k(u) := \mathcal{L}^\epsilon(r_h(u^k)) - \mathcal{L}^\epsilon(u^k). \quad (42)$$

After substituting the notations related to truncation errors, we get

$$\begin{aligned}
(e_h^{k+1}(v), \tilde{u}) & = (e_h^k(v), \tilde{u}) + \Delta t (\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(r_h(u^k)), \tilde{u}) \\
& \quad + \Delta t (\tau_h^k(v), \tilde{u}) + \Delta t (\sigma_{per,h}^k(u), \tilde{u}).
\end{aligned} \quad (43)$$

When u, v are C^2 in time, we easily see

$$\|\tau_h^k(u)\| \leq \Delta t \sup_t \left\| \frac{\partial^2 u}{\partial t^2} \right\| \quad \text{and} \quad \|\tau_h^k(v)\| \leq \Delta t \sup_t \left\| \frac{\partial^2 v}{\partial t^2} \right\|.$$

Here u and v are C^2 in time for differentiable in time body forces as stated in Theorem 3.3 and Theorem A.3.

To estimate $\sigma_{per,h}^k(u)$, we note the Lipschitz property of the peridynamics force in L^2 norm, see 21. This leads us to

$$\|\sigma_{per,h}^k(u)\| \leq \frac{L_1}{\epsilon^2} \|u^k - r_h(u^k)\| \leq \frac{L_1 c}{\epsilon^2} h^2 \sup_t \|u(t)\|_2. \quad (44)$$

We now state the consistency of this approach.

Lemma 5.2. Consistency *Let τ be given by*

$$\tau := \sup_k (\|\tau_h^k(u)\| + \|\tau_h^k(v)\| + \|\sigma_{per,h}^k(u)\|), \quad (45)$$

then the approach is consistent in that

$$\tau \leq C_t \Delta t + C_s \frac{h^2}{\epsilon^2}. \quad (46)$$

where

$$C_t := \sup_t \left\| \frac{\partial^2 u}{\partial t^2} \right\| + \sup_t \left\| \frac{\partial^2 v}{\partial t^2} \right\| \quad \text{and} \quad C_s := L_1 c \sup_t \|u(t)\|_2. \quad (47)$$

5.1.2. *Stability analysis.* In equation for $e_h^k(u)$, see 41, we take $\tilde{u} = e_h^{k+1}(u)$. Note that $e_h^{k+1}(u) = u_h^k - r_h(u^k) \in V_h$. We have

$$\|e_h^{k+1}(u)\|^2 = (e_h^k(u), e_h^{k+1}(u)) + \Delta t(e_h^{k+1}(v), e_h^{k+1}(u)) + \Delta t(\tau_h^k(u), e_h^{k+1}(u)),$$

and we get

$$\|e_h^{k+1}(u)\|^2 \leq \|e_h^k(u)\| \|e_h^{k+1}(u)\| + \Delta t \|e_h^{k+1}(v)\| \|e_h^{k+1}(u)\| + \Delta t \|\tau_h^k(u)\| \|e_h^{k+1}(u)\|.$$

Canceling $\|e_h^{k+1}(u)\|$ from both sides gives

$$\|e_h^{k+1}(u)\| \leq \|e_h^k(u)\| + \Delta t \|e_h^{k+1}(v)\| + \Delta t \|\tau_h^k(u)\|. \quad (48)$$

Similarly, if we choose $\tilde{u} = e_h^{k+1}(v)$ in 43, and use the steps similar to above, we get

$$\begin{aligned} \|e_h^{k+1}(v)\| &\leq \|e_h^k(v)\| + \Delta t \|\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(r_h(u^k))\| \\ &\quad + \Delta t (\|\tau_h^k(v)\| + \|\sigma_{per,h}^k(u)\|). \end{aligned} \quad (49)$$

Using the Lipschitz property of the peridynamics force in L^2 , we have

$$\|\mathcal{L}^\epsilon(u_h^k) - \mathcal{L}^\epsilon(r_h(u^k))\| \leq \frac{L_1}{\epsilon^2} \|u_h^k - r_h(u^k)\| = \frac{L_1}{\epsilon^2} \|e_h^k(u)\|. \quad (50)$$

After adding 48 and 49, and substituting 50, we get

$$\|e_h^{k+1}(u)\| + \|e_h^{k+1}(v)\| \leq \|e_h^k(u)\| + \|e_h^k(v)\| + \Delta t \|e_h^{k+1}(v)\| + \frac{L_1}{\epsilon^2} \Delta t \|e_h^k(u)\| + \Delta t \tau$$

where τ is defined in 46.

Let $e^k := \|e_h^k(u)\| + \|e_h^k(v)\|$. Assuming $L_1/\epsilon^2 \geq 1$, we get

$$\begin{aligned} e^{k+1} &\leq e^k + \Delta t e^{k+1} + \Delta t \frac{L_1}{\epsilon^2} e^k + \Delta t \tau \\ \Rightarrow e^{k+1} &\leq \frac{1 + \Delta t L_1/\epsilon^2}{1 - \Delta t} e^k + \frac{\Delta t}{1 - \Delta t} \tau. \end{aligned}$$

Substituting for e^k recursively in the equation above, we get

$$e^{k+1} \leq \left(\frac{1 + \Delta t L_1/\epsilon^2}{1 - \Delta t} \right)^{k+1} e^0 + \frac{\Delta t}{1 - \Delta t} \tau \sum_{j=0}^k \left(\frac{1 + \Delta t L_1/\epsilon^2}{1 - \Delta t} \right)^{k-j}.$$

Noting $(1 + a\Delta t)^k \leq \exp[ka\Delta t] \leq \exp[Ta]$ for $a > 0$ and

$$\frac{1 + \Delta t L_1/\epsilon^2}{1 - \Delta t} = 1 + \frac{(1 + L_1/\epsilon^2)}{1 - \Delta t} \Delta t$$

we get

$$\left(\frac{1 + \Delta t L_1/\epsilon^2}{1 - \Delta t} \right)^k \leq \exp\left[\frac{T(1 + L_1/\epsilon^2)}{1 - \Delta t}\right].$$

Substituting above estimates, we can easily show that

$$\begin{aligned} e^{k+1} &\leq \exp\left[\frac{T(1 + L_1/\epsilon^2)}{1 - \Delta t}\right] \left[e^0 + \frac{\Delta t}{1 - \Delta t} \tau \sum_{j=0}^k 1 \right] \\ &\leq \exp\left[\frac{T(1 + L_1/\epsilon^2)}{1 - \Delta t}\right] \left[e^0 + \frac{k\Delta t}{1 - \Delta t} \tau \right]. \end{aligned}$$

Finally, we substitute above into 35 to conclude

Lemma 5.3. Stability

$$E^k \leq C_p h^2 + \exp\left[\frac{T(1 + L_1/\epsilon^2)}{1 - \Delta t}\right] \left[e^0 + \frac{k\Delta t}{1 - \Delta t}\tau\right]. \quad (51)$$

After taking sup over $k \leq T/\Delta t$ and substituting the bound on τ from Lemma 5.2, we get the desired result and proof of Theorem 5.1 is complete.

We now consider a stronger notion of stability for the linearized peridynamics model.

5.2. Linearized peridynamics and energy stability. In this section, we linearize the peridynamics model and obtain a CFL like stability condition. For problems where strains are small, the stability condition for the linearized model is expected to apply to the nonlinear model. The slope of peridynamics potential f is constant for sufficiently small strain and therefore for small strain the nonlinear model behaves like a linear model. When displacement field is smooth, the difference between the linearized peridynamics force and the nonlinear peridynamics force is of the order of ϵ . See Proposition 4 of [24].

For strain far below the critical strain, ie., $|S(u)| < S_c$ we expand the integrand of 9 in a Taylor series about zero to obtain the linearized peridynamic force given by

$$\mathcal{L}_l^\epsilon(u)(x) = \frac{4}{\epsilon^{d+1}\omega_d} \int_{H_\epsilon(x)} \omega(x)\omega(y)J^\epsilon(|y-x|)f'(0)S(u)e_{y-x}dy. \quad (52)$$

The corresponding bilinear form is denoted as a_l^ϵ and is given by

$$a_l^\epsilon(u, v) = \frac{2}{\epsilon^{d+1}\omega_d} \int_D \int_{H_\epsilon(x)} \omega(x)\omega(y)J^\epsilon(|y-x|)f'(0)|y-x|S(u)S(v)dydx. \quad (53)$$

We have

$$(\mathcal{L}_l^\epsilon(u), v) = -a_l^\epsilon(u, v).$$

We now discuss the stability of the FEM approximation to the linearized problem. We replace \mathcal{L}^ϵ by its linearization denoted by \mathcal{L}_l^ϵ in 31 and 32. The corresponding approximate solution in V_h is denoted by $u_{l,h}^k$ where

$$\left(\frac{u_{l,h}^{k+1} - 2u_{l,h}^k + u_{l,h}^{k-1}}{\Delta t^2}, \tilde{u}\right) = (\mathcal{L}_l^\epsilon(u_{l,h}^k), \tilde{u}) + (b_h^k, \tilde{u}), \quad \forall \tilde{u} \in V_h \quad (54)$$

and

$$\left(\frac{u_{l,h}^1 - u_{l,h}^0}{\Delta t^2}, \tilde{u}\right) = \frac{1}{2}(\mathcal{L}_l^\epsilon(u_{l,h}^0), \tilde{u}) + \frac{1}{\Delta t}(v_{l,h}^0, \tilde{u}) + \frac{1}{2}(b_h^0, \tilde{u}), \quad \forall \tilde{u} \in V_h. \quad (55)$$

We will adopt following notations

$$\begin{aligned} \bar{u}_h^{k+1} &:= \frac{u_h^{k+1} + u_h^k}{2}, \quad \bar{u}_h^k := \frac{u_h^k + u_h^{k-1}}{2}, \\ \bar{\partial}_t^+ u_h^k &:= \frac{u_h^{k+1} - u_h^{k-1}}{2\Delta t}, \quad \bar{\partial}_t^- u_h^k := \frac{u_h^{k+1} - u_h^k}{\Delta t}, \quad \bar{\partial}_t^- u_h^k := \frac{u_h^k - u_h^{k-1}}{\Delta t}. \end{aligned} \quad (56)$$

With above notations, we have

$$\bar{\partial}_t u_h^k = \frac{\bar{\partial}_t^+ u_h^k + \bar{\partial}_t^- u_h^k}{2} = \frac{\bar{u}_h^{k+1} - \bar{u}_h^k}{\Delta t}.$$

We also define

$$\bar{\partial}_{tt}u_h^k := \frac{u_h^{k+1} - 2u_h^k + u_h^{k-1}}{\Delta t^2} = \frac{\bar{\partial}_t^+ u_h^k - \bar{\partial}_t^- u_h^k}{\Delta t}.$$

We introduce the discrete energy associated with $u_{l,h}^k$ at time step k as defined by

$$\mathcal{E}(u_{l,h}^k) := \frac{1}{2} \left[\|\bar{\partial}_t^+ u_{l,h}^k\|^2 - \frac{\Delta t^2}{4} a_l^\epsilon(\bar{\partial}_t^+ u_{l,h}^k, \bar{\partial}_t^+ u_{l,h}^k) + a_l^\epsilon(\bar{u}_{l,h}^{k+1}, \bar{u}_{l,h}^{k+1}) \right]$$

Following [Theorem 4.1, [25]], we have

Theorem 5.4. Energy Stability of the Central difference approximation of linearized peridynamics Let $u_{l,h}^k$ be the approximate solution of 54 and 55 with respect to linearized peridynamics. In the absence of body force $b(t) = 0$ for all t , if Δt satisfies the CFL like condition

$$\frac{\Delta t^2}{4} \sup_{u \in V_h \setminus \{0\}} \frac{a_l^\epsilon(u, u)}{(u, u)} \leq 1, \quad (57)$$

The discrete energy is positive and we have the stability

$$\mathcal{E}(u_{l,h}^k) = \mathcal{E}(u_{l,h}^{k-1}). \quad (58)$$

Proof. Set $b(t) = 0$. Noting that a_l^ϵ is bilinear, after adding and subtracting term $(\Delta t^2/4)a_l^\epsilon(\bar{\partial}_{tt}u_{l,h}^k, \tilde{u})$ to 54, and noting following

$$u_{l,h}^k + \frac{\Delta t^2}{4} \bar{\partial}_{tt}u_{l,h}^k = \frac{\bar{u}_{l,h}^{k+1}}{2} + \frac{\bar{u}_{l,h}^k}{2}$$

we get

$$(\bar{\partial}_{tt}u_{l,h}^k, \tilde{u}) - \frac{\Delta t^2}{4} a_l^\epsilon(\bar{\partial}_{tt}u_{l,h}^k, \tilde{u}) + \frac{1}{2} a_l^\epsilon(\bar{u}_{l,h}^{k+1} + \bar{u}_{l,h}^k, \tilde{u}) = 0.$$

We let $\tilde{u} = \bar{\partial}_t u_{l,h}^k$, to write

$$(\bar{\partial}_{tt}u_{l,h}^k, \bar{\partial}_t u_{l,h}^k) - \frac{\Delta t^2}{4} a_l^\epsilon(\bar{\partial}_{tt}u_{l,h}^k, \bar{\partial}_t u_{l,h}^k) + \frac{1}{2} a_l^\epsilon(\bar{u}_{l,h}^{k+1} + \bar{u}_{l,h}^k, \bar{\partial}_t u_{l,h}^k) = 0.$$

It is easily shown that

$$(\bar{\partial}_{tt}u_{l,h}^k, \bar{\partial}_t u_{l,h}^k) = \left(\frac{\bar{\partial}_t^+ u_{l,h}^k - \bar{\partial}_t^- u_{l,h}^k}{\Delta t}, \frac{\bar{\partial}_t^+ u_{l,h}^k + \bar{\partial}_t^- u_{l,h}^k}{2} \right) = \frac{1}{2\Delta t} \left[\|\bar{\partial}_t^+ u_{l,h}^k\|^2 - \|\bar{\partial}_t^- u_{l,h}^k\|^2 \right]$$

and

$$a_l^\epsilon(\bar{\partial}_{tt}u_{l,h}^k, \bar{\partial}_t u_{l,h}^k) = \frac{1}{2\Delta t} \left[a_l^\epsilon(\bar{\partial}_t^+ u_{l,h}^k, \bar{\partial}_t^+ u_{l,h}^k) - a_l^\epsilon(\bar{\partial}_t^- u_{l,h}^k, \bar{\partial}_t^- u_{l,h}^k) \right].$$

Noting that $\bar{\partial}_t u_{l,h}^k = (\bar{u}_{l,h}^{k+1} - \bar{u}_{l,h}^k)/\Delta t$, we get

$$\begin{aligned} & \frac{1}{2\Delta t} a_l^\epsilon(\bar{u}_{l,h}^{k+1} + \bar{u}_{l,h}^k, \bar{u}_{l,h}^{k+1} - \bar{u}_{l,h}^k) \\ &= \frac{1}{2\Delta t} \left[a_l^\epsilon(\bar{u}_{l,h}^{k+1}, \bar{u}_{l,h}^{k+1}) - a_l^\epsilon(\bar{u}_{l,h}^k, \bar{u}_{l,h}^k) \right]. \end{aligned}$$

After combining the above equations, we get

$$\begin{aligned} & \frac{1}{\Delta t} \left[\left(\frac{1}{2} \|\bar{\partial}_t^+ u_{l,h}^k\|^2 - \frac{\Delta t^2}{8} a_l^\epsilon(\bar{\partial}_t^+ u_{l,h}^k, \bar{\partial}_t^+ u_{l,h}^k) + \frac{1}{2} a_l^\epsilon(\bar{u}_{l,h}^{k+1}, \bar{u}_{l,h}^{k+1}) \right) \right. \\ & \quad \left. - \left(\frac{1}{2} \|\bar{\partial}_t^- u_{l,h}^k\|^2 - \frac{\Delta t^2}{8} a_l^\epsilon(\bar{\partial}_t^- u_{l,h}^k, \bar{\partial}_t^- u_{l,h}^k) + \frac{1}{2} a_l^\epsilon(\bar{u}_{l,h}^k, \bar{u}_{l,h}^k) \right) \right] = 0. \quad (59) \end{aligned}$$

We recognize the first term in bracket as $\mathcal{E}(u_{l,h}^k)$. We next prove that the second term is $\mathcal{E}(u_{l,h}^{k-1})$. We substitute $k = k - 1$ in the definition of $\mathcal{E}(u_{l,h}^k)$, to get

$$\mathcal{E}(u_{l,h}^{k-1}) = \frac{1}{2} \left[\|\bar{\partial}_t^+ u_{l,h}^{k-1}\|^2 - \frac{\Delta t^2}{4} a_l^\epsilon(\bar{\partial}_t^+ u_{l,h}^{k-1}, \bar{\partial}_t^+ u_{l,h}^{k-1}) + a_l^\epsilon(\bar{u}_{l,h}^k, \bar{u}_{l,h}^k) \right].$$

We clearly have $\bar{\partial}_t^+ u_{l,h}^{k-1} = \frac{u_{l,h}^{k-1+1} - u_{l,h}^{k-1}}{\Delta t} = \bar{\partial}_t^- u_{l,h}^k$ and this implies that the second term in 59 is $\mathcal{E}(u_{l,h}^{k-1})$. It now follows from 59, that $\mathcal{E}(u_{l,h}^k) = \mathcal{E}(u_{l,h}^{k-1})$.

We now establish the positivity of the discrete energy $\mathcal{E}(u_{l,h}^k)$. To do this we derive a condition on the time step that insures the sum of the first two terms is positive and the positivity of $\mathcal{E}(u_{l,h}^k)$ will follow. Let $v = \bar{\partial}_t^+ u_{l,h}^k \in V_h$, then we require

$$\|v\|^2 - \frac{\Delta t^2}{4} a_l^\epsilon(v, v) \geq 0 \quad \Rightarrow \quad \frac{\Delta t^2}{4} \frac{a_l^\epsilon(v, v)}{\|v\|^2} \leq 1 \quad (60)$$

Clearly if Δt satisfies

$$\frac{\Delta t^2}{4} \sup_{v \in V_h \setminus \{0\}} \frac{a_l^\epsilon(v, v)}{\|v\|^2} \leq 1 \quad (61)$$

then 60 is also satisfied and the discrete energy is positive. Iteration gives $\mathcal{E}(u_{l,h}^k) = \mathcal{E}(u_{l,h}^0)$ and the theorem is proved. \square

6. Estimates on error accumulation in numerical simulations. The peridynamic equation analyzed so far is assumed to be nondimensional. In this section we show how to apply the a-priori error bound obtained for nondimensional peridynamic equation to peridynamic equation for the material constants characterizing Plexiglass. In this section we will assume a two dimensional problem to fix ideas. Let \bar{D} is the material domain with characteristic length scale L_0 and let $x \in \bar{D}$ are coordinates with dimensions of length. The final simulation time is \bar{T} is expressed in units of time and $\bar{t} \in [0, \bar{T}]$. Let $\bar{\epsilon}$ denote the size of horizon with units of length. The displacement field is $\bar{u}(\bar{x}, \bar{t})$ and has units of length. The influence function $\bar{J}(\bar{\xi}) = a(1 - \bar{\xi})$ is a non dimensional function of $\bar{\xi} = |\bar{x} - \bar{y}|/\bar{\epsilon}$ with specified constant $a > 0$. We set the boundary function $\omega = 1$ and body force $b = 0$.

The nonlinear peridynamic force is given in terms of potential function \bar{f} . We let $\bar{f}(\bar{r}) = \bar{C}(1 - \exp[-\bar{\beta}\bar{r}])$ where \bar{r} has units of length, \bar{C} has units of force/length, and $\bar{\beta}$ has units of 1/length. Let the bulk modulus K , density $\bar{\rho}$, and critical energy release rate G correspond to Plexiglass at room temperature. Following equations (5.7) and (5.8) of [28], the parameters $\bar{C}, \bar{\beta}$ are given in 2-d by

$$\bar{C} = \frac{G}{2(\omega_1/\omega_2)M}, \quad \bar{\beta} = \frac{\lambda}{(1/4)\bar{C}M}, \quad M = \int_0^1 \bar{J}(s)s^2 ds, \quad (62)$$

where $\omega_1 = 2, \omega_2 = \pi$. Here the Lamé parameter is related to K by $\lambda = 3K/5$. For $\bar{J}(s) = a(1 - s)$, $M = a/12$. Substituting, we have

$$\bar{C} = \frac{3\pi G}{a}, \quad \bar{\beta} = \frac{48K}{5\pi G} \quad (63)$$

and also

$$\bar{C}\bar{\beta} = \frac{144}{5a}K. \quad (64)$$

Displacement field \bar{u} satisfies

$$\bar{\rho} \partial_{tt}^2 \bar{u}(\bar{t}, \bar{x}) = \bar{\mathcal{L}}^\epsilon(\bar{u}(\bar{t}))(\bar{x}), \quad \forall (\bar{x}, \bar{t}) \in \bar{D} \times [0, \bar{T}]. \quad (65)$$

The solution \bar{u} takes the boundary condition $\bar{u}(t) = 0$ for all $\bar{x} \in \partial \bar{D}$ and the initial condition $\bar{u}(0) = \bar{u}_0, \dot{\bar{u}}(0) = \bar{v}_0$.

6.1. Nondimensionalization of peridynamic equation. Now we associate a local wave speed for the peridynamic material and an associated local time scale given by

$$v_0 = \sqrt{\frac{\bar{C}\bar{\beta}}{\bar{\rho}}}, \quad T_0 = \frac{L_0}{v_0}. \quad (66)$$

The change to non-dimensional variables is given by

$$x = \frac{\bar{x}}{L_0}, \quad t = \frac{\bar{t}}{T_0}, \quad \epsilon = \frac{\bar{\epsilon}}{L_0}, \quad u(x, t) = \frac{\bar{u}(\bar{x}, \bar{t})}{L_0}. \quad (67)$$

From above it is easy to see that $\bar{S}(\bar{x}, \bar{y}, \bar{t}) = \frac{\bar{u}(\bar{y}, \bar{t}) - \bar{u}(\bar{x}, \bar{t})}{|\bar{y} - \bar{x}|} \cdot \frac{\bar{y} - \bar{x}}{|\bar{y} - \bar{x}|} = S(x, y, t)$. We write

$$\bar{r} = |\bar{x} - \bar{y}| \bar{S}^2 = L_0 |x - y| S = L_0 r, \quad (68)$$

where $r = |x - y| S^2$. The non-dimensional potential function f is related to \bar{f} by

$$f(r) = \frac{\bar{f}(L_0 r)}{L_0 \bar{\rho} v_0^2} = \frac{1}{L_0 \bar{\rho} v_0^2} \bar{C} (1 - \exp[-L_0 \bar{\beta} r]). \quad (69)$$

It is now clear that the dimension of \bar{f} is the same as $L_0 \bar{\rho} v_0^2$ and therefore f is non-dimensional. We have,

$$f'(r) = \frac{\bar{f}'(L_0 r)}{\bar{\rho} v_0^2} = \frac{\bar{C} \bar{\beta}}{\bar{\rho} v_0^2} \exp[-L_0 \bar{\beta} r]. \quad (70)$$

Collecting results we now see that the peridynamic equation 65 is equivalent to the non-dimensional equation of motion 2 with density $\rho = 1$, i.e.,

$$\left(\frac{\bar{\rho} v_0^2}{L_0} \right) \partial_{tt}^2 u = \bar{\rho} \partial_{tt}^2 \bar{u} = \bar{\mathcal{L}}^\epsilon(\bar{u}(\bar{t}))(\bar{x}) = \left(\frac{\bar{\rho} v_0^2}{L_0} \right) (\mathcal{L}^\epsilon(u(t)))(x), \quad (71)$$

so

$$\partial_{tt}^2 u = \mathcal{L}^\epsilon(u)(x). \quad (72)$$

6.2. Bound on error. The exact solution is in $u \in H^2(D; \mathbf{R}^2) \cap H_0^1(D; \mathbf{R}^2)$, and the bound on the spatial discretization error is given by, see 37,

$$\begin{aligned} \sup_k E^k &\leq \exp \left[\frac{T(1 + L_1/\epsilon^2)}{1 - \Delta t} \right] \frac{T}{1 - \Delta t} (C_s/\epsilon^2) h^2 \\ &\leq \exp [T(1 + L_1/\epsilon^2)] T (C_s/\epsilon^2) h^2, \end{aligned} \quad (73)$$

where

$$L_1 = 4C_2 \bar{J}_1, \quad C_2 = \sup_r |\partial_{rr}^2 f(r^2)|, \quad \bar{J}_1 = \frac{1}{\omega_2} \int_{H_1(0)} J(|\xi|)/|\xi| d\xi, \quad \omega_2 = |H_1(0)| = \pi$$

and

$$C_s = L_1 c \sup_t \|u(t)\|_2.$$

Constant c depends only on the triangulation \mathcal{T}_h .

For $f(r) = \frac{1}{L_0 \bar{\rho} v_0^2} \bar{C}(1 - \exp[-L_0 \bar{\beta} r])$ and $J(r) = a(1 - \xi)$, it can be seen that

$$C_2 = \frac{2\bar{C}\bar{\beta}}{\bar{\rho}v_0^2}, \quad \bar{J}_1 = a. \quad (74)$$

We have $\bar{C}\bar{\beta} = \bar{\rho}v_0^2$ from 66. So

$$L_1 = \frac{8a}{\epsilon^2}. \quad (75)$$

The upper bound on error is given by

$$\sup_k E^k \leq \exp[(1 + \frac{8a}{\epsilon^2})T] T \frac{8a}{\epsilon^2} c h^2 \sup_t \|u(t)\|_2,$$

and the a-priori upper bound on the relative error is denoted by α where

$$\alpha = \exp[(1 + \frac{8a}{\epsilon^2})T] \frac{8acTh^2}{\epsilon^2}. \quad (76)$$

Numerical value of α : We set $L_0 = 1, \epsilon = 1/10, h = 1/100$ and we fix $a = 0.001$ and $v_0 = \sqrt{\frac{\bar{C}\bar{\beta}}{\bar{\rho}}}$. We also assume $c = 1$. The material properties of Plexiglass at room temperature are given by the density $\bar{\rho} = 1200 \text{ kg/m}^3$, the bulk modulus $K = 25 \text{ GPa}$, and the critical energy release rate $G = 500 \text{ J/m}^2$. We then have

$$\alpha = \exp[1.8T] 80 \times 10^{-6} T. \quad (77)$$

Here the relative error upper bound $\alpha < 1/10$ when the non-dimensional time $T \leq \frac{5.94}{1.8} = 3.3$. Therefore the actual time in seconds of the simulation can be $\bar{T} = T_0 \times T \leq (L_0/v_0) \times 3.3 = 4.26 \mu\text{s}$.

6.3. Discussion on error accumulation. Fracture in notched Plexiglass samples can last up to several hundred microseconds. From the previous subsection we see that error increases by factor 1/10 every $4.26 \mu\text{s}$ for nonlinear peridynamic material. This gives us about $20 \mu\text{s}$ of simulation time till the a-priori bound on the relative error is about 1/2.

The simulations are expected to be stable for much larger time because the region where nonlinearity is strong is restricted to a very small region, with area $L_0 \times 2\bar{\epsilon}$ in 2-d for a single crack see [29, 28]. For points in the region away from the crack the deformation is smooth. We can argue that in this region the material behaves like a linear elastic material up to a small error of the order of $O(\bar{\epsilon})$. This has been shown for this model when the solution is sufficiently smooth and using [Proposition 6, 24]] we write

$$\bar{\mathcal{L}}^{\bar{\epsilon}}(\bar{u})(\bar{x}) = \nabla \cdot \mathbb{C} \mathcal{E} \bar{u}(\bar{x}) + O(\bar{\epsilon}), \quad (78)$$

where

$$\mathcal{E} \bar{u}(\bar{x}) = \frac{1}{2} (\nabla \bar{u}(\bar{x}) + \nabla \bar{u}(\bar{x})^T), \quad (79)$$

$$\mathbb{C}_{ijkl} = 2\mu \frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2} + \lambda \delta_{ij}\delta_{kl}, \quad (80)$$

$$\lambda = \mu = \frac{\bar{f}'(0)}{4} \int_0^1 J(\xi) \xi^2 d\xi = \bar{C}\bar{\beta} \frac{a}{48}, \quad (81)$$

where last equation is for $d = 2$ and for $J(\xi) = a(1 - \xi)$. We now observe that for the non-dimensional function $f(r) = \frac{1}{L_0 \bar{\rho} \tilde{v}_0^2} \bar{C}(1 - \exp[-L_0 \bar{\beta} r])$, $f'(0) = 1$. Using this we can write

$$\bar{\mathcal{L}}^{\bar{\epsilon}}(\bar{u})(\bar{x}) = \frac{\bar{C} \bar{\beta}}{L_0} \frac{a}{48} \nabla \cdot \hat{\mathbb{C}} \mathcal{E} u(x) + O(\bar{\epsilon}), \quad (82)$$

where $\hat{\mathbb{C}}$ is given by 80 for the choice $\lambda = \mu = 1$.

Substituting 78 into 72 we get

$$\bar{\mathcal{L}}^{\bar{\epsilon}}(u)(x) = \left(\frac{\bar{\rho} \tilde{v}_0^2}{L_0} \right) \nabla \cdot \hat{\mathbb{C}} \mathcal{E} u(x) + O(\bar{\epsilon}), \quad (83)$$

with

$$\tilde{v}_0 = \sqrt{\frac{\bar{C} \bar{\beta}}{48 \bar{\rho}}} = \sqrt{\frac{\lambda}{\bar{\rho}}}. \quad (84)$$

where we have used the relation 64 and $\lambda = \mu$ and \tilde{v}_0 is the s -wave speed in Plexiglass.

It follows from 83, that for regions where nonlinearity is negligible then the solution should be an approximation to the solution of the linear elastic wave equation. This is shown for smooth solutions in [Theorem 5, [24]] so the total error accumulated at each time step is far less than in the nonlinear region. The error due to the truly nonlinear peridynamic interaction is restricted to a region of small area $2L_0 \bar{\epsilon}$.

7. Conclusions. We have considered a canonical nonlinear peridynamic model and have shown the existence of a unique $H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d)$ solution for any finite time interval. It has been demonstrated that finite element approximation converges to the exact solution at the rate $C_t \Delta t + C_s \frac{h^2}{\epsilon^2}$ for fixed ϵ . The constants C_t and C_s are independent of time step Δt and mesh size h . The constant C_t depends on the L^2 norm of the first and second time derivatives of the solution. The constant C_s depends on the H^2 norm of the solution. A stability condition for the length of time step has been obtained for the linearized peridynamic model. It is expected that this stability condition is also in force for the nonlinear model in regions where the deformation is sufficiently small.

We have described the connection between the non-dimensionalized dynamics used in the a-priori convergence analysis and the simulated dynamics using dimensional quantities. The numerics are carried out for Plexiglass. The a-priori estimates predict a simulation time of a few microseconds before the relative error grows too large. However, due to the fact that the nonlinearity is isolated on a set of small area related to the crack set, the simulation is expected to be stable for much longer time. Away from the crack set the evolution is linearly elastic and characterized by the shear wave speed of Plexiglass. This observation motivates future work that will address posteriori error estimation and mesh adaptivity.

We remark that there is a large amount of work regarding asymptotically compatible schemes, in which one can pass to the limit $\epsilon \rightarrow 0$ and retain convergence of the numerical method, see [40, 8]. Such a scheme may be contemplated only when the convergence rate to the solution of the limit problem with respect to ϵ is known. Unfortunately an asymptotically compatible scheme is not yet possible for the nonlinear nonlocal evolutions treated here because the convergence rate of solutions with respect to $\epsilon > 0$ is not known. One fundamental barrier to obtaining a rate is that the complete characterization of the $\epsilon = 0$ evolution is not yet

known. What is known so far is the characterization developed in the earlier work [27, 28]. Here the evolution u^ϵ for the nonlinear nonlocal model is shown, on passage to subsequences, to converge in the $C([0, T]; L^2(D; \mathbf{R}^3))$ norm as $\epsilon \rightarrow 0$ to a limit evolution $u(t)$ the space of SBD^2 functions. The fracture set at time t is given by the jump set $J_{u(t)}$ of $u(t)$. ($J_{u(t)}$ is the countable union of components contained in smooth manifolds and has finite Hausdorff $d-1$ measure.) At each time the associated energies $PD^\epsilon(u^\epsilon)$ Γ -converge to the energy of linear elastic fracture mechanics evaluated at the limit evolution $u(t)$. This energy is found to be bounded. Away from the fracture set the limit evolution has been shown to evolve according to the linear elastic wave equation [27, 28]. What remains missing is the dynamics of the fracture set $J_{u(t)}$. Once this is known a convergence rate may be sought and an asymptotically compatible scheme may be contemplated.

As shown in this paper the nonlinear nonlocal model is well posed in H^2 for all $\epsilon > 0$. However the H^2 norm of the solution gets progressively larger as $\epsilon \rightarrow 0$ if gradients steepen due to forces acting on the body. On the other hand if it is known that the solution is bounded in a C^p norm uniformly for $\epsilon > 0$ and if p large enough then one can devise a finite difference scheme with truncation error that goes to zero independent of the peridynamic horizon [Proposition 5, [24]]. For example if $p \geq 4$ then the peridynamic evolutions converge to the elastodynamics evolution independently of horizon and an asymptotically compatible scheme can be developed for the linearized peridynamic force, [Proposition 5, [24]]. We note that the nonlinear and linearized kernels treated here and in [24] are different than those treated in [40] where asymptotically compatible schemes are first proposed.

A. Proof of claims. In this section we establish the Lipschitz continuity in the space $W = H^2(D; \mathbf{R}^d) \cap H_0^1(D; \mathbf{R}^d)$ and the existence of a differentiable in time solution to the peridynamic evolution belonging to W . We outline the proof of Lipschitz continuity of $Q(v; u)$, see 152, required to show the higher regularity of solutions in time.

A.1. Lipschitz continuity in $H^2 \cap H_0^1$. We now prove the Lipschitz continuity given by Theorem 3.1.

To simplify the presentation, we write the peridynamics force $\mathcal{L}^\epsilon(u)$ as $P(u)$. We need to analyze $\|P(u) - P(v)\|_2$.

We first introduce the following convenient notations

$$s_\xi = \epsilon|\xi|, \quad e_\xi = \frac{\xi}{|\xi|}, \quad \bar{J}_\alpha = \frac{1}{\omega_d} \int_{H_1(0)} J(|\xi|) \frac{1}{|\xi|^\alpha} d\xi, \quad (85)$$

$$S_\xi(u) = \frac{u(x + \epsilon\xi) - u(x)}{s_\xi} \cdot e_\xi, \quad (86)$$

$$S_\xi(\nabla u) = \nabla S_\xi(u) = \frac{\nabla u^T(x + \epsilon\xi) - \nabla u^T(x)}{s_\xi} e_\xi, \quad (87)$$

$$S_\xi(\nabla^2 u) = \nabla S_\xi(\nabla u) = \nabla \left[\frac{\nabla u^T(x + \epsilon\xi) - \nabla u^T(x)}{s_\xi} e_\xi \right]. \quad (88)$$

In indicial notation, we have

$$S_\xi(\nabla u)_i = \frac{u_{k,i}(x + \epsilon\xi) - u_{k,i}(x)}{s_\xi} (e_\xi)_k,$$

$$S_\xi(\nabla^2 u)_{ij} = \left[\frac{u_{k,i}(x + \epsilon\xi) - u_{k,i}(x)}{s_\xi} (e_\xi)_k \right]_{,j} = \frac{u_{k,ij}(x + \epsilon\xi) - u_{k,ij}(x)}{s_\xi} (e_\xi)_k \quad (89)$$

and

$$[e_\xi \otimes S_\xi(\nabla^2 u)]_{ijk} = (e_\xi)_i S_\xi(\nabla^2 u)_{jk}, \quad (90)$$

where $u_{i,j} = (\nabla u)_{ij}$, $u_{k,ij} = (\nabla^2 u)_{kij}$, and $(e_\xi)_k = \xi_k/|\xi|$.

We now examine the Lipschitz properties of the peridynamic force. Let $F_1(r) := f(r^2)$ where f is described in the Section 2. We have $F_1'(r) = f'(r^2)2r$. Thus, $2Sf'(\epsilon|\xi|S^2) = F_1'(\sqrt{\epsilon|\xi|}S)/\sqrt{\epsilon|\xi|}$. We define following constants related to nonlinear potential

$$C_1 := \sup_r |F_1'(r)|, \quad C_2 := \sup_r |F_1''(r)|, \quad C_3 := \sup_r |F_1'''(r)|, \quad C_4 := \sup_r |F_1''''(r)|. \quad (91)$$

The potential function f as chosen here satisfies $C_1, C_2, C_3, C_4 < \infty$. Let

$$\bar{\omega}_\xi(x) = \omega(x)\omega(x + \epsilon\xi), \quad (92)$$

and we choose ω such that

$$|\nabla \bar{\omega}_\xi| \leq C_{\omega_1} < \infty \quad \text{and} \quad |\nabla^2 \bar{\omega}_\xi| \leq C_{\omega_2} < \infty. \quad (93)$$

With notations described so far, we write peridynamics force $P(u)$ as

$$P(u)(x) = \frac{2}{\epsilon\omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) \frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} e_\xi d\xi. \quad (94)$$

The gradient of $P(u)(x)$ is given by

$$\begin{aligned} \nabla P(u)(x) &= \frac{2}{\epsilon\omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes \nabla S_\xi(u) d\xi \\ &\quad + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} e_\xi \otimes \nabla \bar{\omega}_\xi(x) d\xi \\ &= g_1(u)(x) + g_2(u)(x), \end{aligned} \quad (95)$$

where we denote first and second term as $g_1(u)(x)$ and $g_2(u)(x)$ respectively. We also have

$$\begin{aligned} \nabla^2 P(u)(x) &= \frac{2}{\epsilon\omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) F_1'''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes S_\xi(\nabla^2 u) d\xi \\ &\quad + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) \sqrt{s_\xi} F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes S_\xi(\nabla u) \otimes S_\xi(\nabla u) d\xi \\ &\quad + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes S_\xi(\nabla u) \otimes \nabla \bar{\omega}_\xi(x) d\xi \\ &\quad + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \frac{F_1'(\sqrt{s_\xi} S_\xi(u))}{\sqrt{s_\xi}} e_\xi \otimes \nabla^2 \bar{\omega}_\xi(x) d\xi \\ &\quad + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes \nabla \bar{\omega}_\xi(x) \otimes S_\xi(\nabla u) d\xi \\ &= h_1(u)(x) + h_2(u)(x) + h_3(u)(x) + h_4(u)(x) + h_5(u)(x) \end{aligned} \quad (96)$$

where we denote first, second, third, fourth, and fifth terms as h_1, h_2, h_3, h_4, h_5 respectively.

Estimating $\|P(u) - P(v)\|$. From 94, we have

$$\begin{aligned}
& |P(u)(x) - P(v)(x)| \\
& \leq \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \frac{1}{\sqrt{s_\xi}} |F'_1(\sqrt{s_\xi}S_\xi(u)) - F'_1(\sqrt{s_\xi}S_\xi(v))| d\xi \\
& \leq \frac{2}{\epsilon\omega_d} \left(\sup_r |F'_1(r)| \right) \int_{H_1(0)} J(|\xi|) \frac{1}{\sqrt{s_\xi}} |\sqrt{s_\xi}S_\xi(u) - \sqrt{s_\xi}S_\xi(v)| d\xi \\
& = \frac{2C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(u) - S_\xi(v)| d\xi,
\end{aligned} \tag{97}$$

where we used the fact that $|\bar{\omega}_\xi(x)| \leq 1$ and $|F'_1(r_1) - F'_1(r_2)| \leq C_2|r_1 - r_2|$.

From 97, we have

$$\begin{aligned}
& \|P(u) - P(v)\|^2 \\
& \leq \int_D \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \frac{J(|\eta|)}{|\eta|} |\xi| |S_\xi(u) - S_\xi(v)| |\eta| |S_\eta(u) - S_\eta(v)| d\xi d\eta dx.
\end{aligned}$$

Using the identities $|a||b| \leq |a|^2/2 + |b|^2/2$ and $(a+b)^2 \leq 2a^2 + 2b^2$ we get

$$\begin{aligned}
& \|P(u) - P(v)\|^2 \\
& \leq \int_D \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \frac{J(|\eta|)}{|\eta|} \frac{|\xi|^2 |S_\xi(u) - S_\xi(v)|^2 + |\eta|^2 |S_\eta(u) - S_\eta(v)|^2}{2} d\xi d\eta dx \\
& = 2 \int_D \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \frac{J(|\eta|)}{|\eta|} \frac{|\xi|^2 |S_\xi(u) - S_\xi(v)|^2}{2} d\xi d\eta dx \\
& = \int_D \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} |\xi|^2 \frac{2|u(x + \epsilon\xi) - v(x + \epsilon\xi)|^2 + 2|u(x) - v(x)|^2}{\epsilon^2 |\xi|^2} d\xi dx \\
& = \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \frac{1}{\epsilon^2} \left[2 \int_D (|u(x + \epsilon\xi) - v(x + \epsilon\xi)|^2 + |u(x) - v(x)|^2) dx \right] d\xi \\
& \leq \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \frac{1}{\epsilon^2} [4\|u - v\|^2] d\xi,
\end{aligned} \tag{98}$$

where we used the symmetry with respect to ξ and η in second equation. This gives

$$\|P(u) - P(v)\| \leq \frac{L_1}{\epsilon^2} \|u - v\| \leq \frac{L_1}{\epsilon^2} \|u - v\|_2, \tag{99}$$

where

$$L_1 := 4C_2 \bar{J}_1. \tag{100}$$

Estimating $\|\nabla P(u) - \nabla P(v)\|$. From 95, we have

$$\|\nabla P(u) - \nabla P(v)\| \leq \|g_1(u) - g_1(v)\| + \|g_2(u) - g_2(v)\|.$$

Using $|\bar{\omega}_\xi(x)| \leq 1$, we get

$$\begin{aligned}
& |g_1(u)(x) - g_1(v)(x)| \\
& \leq \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |F''_1(\sqrt{s_\xi}S_\xi(u)) \nabla S_\xi(u) - F''_1(\sqrt{s_\xi}S_\xi(v)) \nabla S_\xi(v)| d\xi \\
& \leq \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |F''_1(\sqrt{s_\xi}S_\xi(u)) - F''_1(\sqrt{s_\xi}S_\xi(v))| |\nabla S_\xi(u)| d\xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |F_1''(\sqrt{s_\xi} S_\xi(v))| |\nabla S_\xi(u) - \nabla S_\xi(v)| d\xi \\
& \leq \frac{2C_3}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |S_\xi(u) - S_\xi(v)| |\nabla S_\xi(u)| d\xi \\
& + \frac{2C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |\nabla S_\xi(u) - \nabla S_\xi(v)| d\xi \\
& = I_1(x) + I_2(x)
\end{aligned} \tag{101}$$

where we denote first and second term as $I_1(x)$ and $I_2(x)$. Proceeding similar to [98](#), we can show

$$\begin{aligned}
||I_1||^2 & = \int_D \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} \frac{J(|\eta|)}{|\eta|^{3/2}} |\xi|^{3/2} |\eta|^{3/2} \sqrt{s_\xi} \sqrt{s_\eta} \\
& \quad \times |S_\xi(u) - S_\xi(v)| |\nabla S_\xi(u)| |S_\eta(u) - S_\eta(v)| |\nabla S_\eta(u)| d\xi d\eta dx \\
& \leq \int_D \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_{3/2} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} |\xi|^3 s_\xi |S_\xi(u) - S_\xi(v)|^2 |\nabla S_\xi(u)|^2 d\xi dx.
\end{aligned} \tag{102}$$

Now

$$\begin{aligned}
& \int_D |S_\xi(u) - S_\xi(v)|^2 |\nabla S_\xi(u)|^2 dx \\
& \leq \frac{4||u - v||_\infty^2}{\epsilon^2 |\xi|^2} \frac{1}{\epsilon^2 |\xi|^2} \int_D 2(|\nabla u(x + \epsilon\xi)|^2 + |\nabla u(x)|^2) dx \\
& \leq \frac{16||\nabla u||^2 ||u - v||_\infty^2}{\epsilon^4 |\xi|^4}.
\end{aligned}$$

By Sobolev embedding property, see [14](#), we have $||u - v||_\infty \leq C_{e_1} ||u - v||_2$. Thus, we get

$$\int_D |S_\xi(u) - S_\xi(v)|^2 |\nabla S_\xi(u)|^2 dx \leq \frac{16C_{e_1}^2 ||\nabla u||^2 ||u - v||_2^2}{\epsilon^4 |\xi|^4}.$$

Substituting above in [102](#) to get

$$\begin{aligned}
||I_1||^2 & \leq \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_{3/2} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} |\xi|^3 \epsilon |\xi| \frac{16C_{e_1}^2 ||u||_2^2}{\epsilon^4 |\xi|^4} ||u - v||_2^2 d\xi \\
& = \left(\frac{8C_3 C_{e_1} \bar{J}_{3/2} ||u||_2}{\epsilon^{5/2}} \right)^2 ||u - v||_2^2.
\end{aligned}$$

Let $L_2 = 8C_3 C_{e_1} \bar{J}_{3/2}$ to write

$$||I_1|| \leq \frac{L_2 (||u||_2 + ||v||_2)}{\epsilon^{5/2}} ||u - v||_2. \tag{103}$$

Similarly

$$\begin{aligned}
||I_2||^2 & = \int_D \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \frac{J(|\eta|)}{|\eta|} |\xi| |\eta| \\
& \quad \times |\nabla S_\xi(u) - \nabla S_\xi(v)| |\nabla S_\eta(u) - \nabla S_\eta(v)| d\xi d\eta dx \\
& \leq \left(\frac{2C_2}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} |\xi|^2 \left[\int_D |\nabla S_\xi(u) - \nabla S_\xi(v)|^2 dx \right] d\xi.
\end{aligned}$$

This gives

$$\|I_2\| \leq \frac{4C_2\bar{J}_1}{\epsilon^2} \|u - v\|_2 = \frac{L_1}{\epsilon^2} \|u - v\|_2. \quad (104)$$

Thus

$$\|g_1(u) - g_1(v)\| \leq \frac{\sqrt{\epsilon}L_1 + L_2(\|u\|_2 + \|v\|_2)}{\epsilon^{5/2}} \|u - v\|_2. \quad (105)$$

We now work on $|g_2(u)(x) - g_2(v)(x)|$, see [95](#). Noting the bound on $\nabla\bar{\omega}_\xi$, we get

$$\begin{aligned} & |g_2(u)(x) - g_2(v)(x)| \\ &= \left| \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \left[\frac{F'_1(\sqrt{s_\xi}S_\xi(u))}{\sqrt{s_\xi}} - \frac{F'_1(\sqrt{s_\xi}S_\xi(v))}{\sqrt{s_\xi}} \right] e_\xi \otimes \nabla\bar{\omega}_\xi(s) d\xi \right| \\ &\leq \frac{2C_{\omega_1}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \left| \frac{F'_1(\sqrt{s_\xi}S_\xi(u))}{\sqrt{s_\xi}} - \frac{F'_1(\sqrt{s_\xi}S_\xi(v))}{\sqrt{s_\xi}} \right| d\xi \\ &\leq \frac{2C_{\omega_1}C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(u) - S_\xi(v)| d\xi. \end{aligned} \quad (106)$$

Note that the above inequality is similar to [97](#) and therefore we get

$$\|g_2(u) - g_2(v)\| \leq \frac{4C_{\omega_1}C_2\bar{J}_1}{\epsilon^2} \|u - v\|_2 = \frac{C_{\omega_1}L_1}{\epsilon^2} \|u - v\|_2. \quad (107)$$

Combining [105](#) and [107](#) to write

$$\|\nabla P(u) - \nabla P(v)\| \leq \frac{\sqrt{\epsilon}(1 + C_{\omega_1})L_1 + L_2(\|u\|_2 + \|v\|_2)}{\epsilon^{5/2}} \|u - v\|_2. \quad (108)$$

Estimating $\|\nabla^2 P(u) - \nabla^2 P(v)\|$. From [96](#), we have

$$\begin{aligned} & \|\nabla^2 P(u) - \nabla^2 P(v)\| \\ &\leq \|h_1(u) - h_1(v)\| + \|h_2(u) - h_2(v)\| + \|h_3(u) - h_3(v)\| \\ &\quad + \|h_4(u) - h_4(v)\| + \|h_5(u) - h_5(v)\|. \end{aligned} \quad (109)$$

We can show, using the fact $|\bar{\omega}_\xi(x)| \leq 1$ and $|F''_1(r_1) - F''_1(r_2)| \leq C_3|r_1 - r_2|$, that

$$\begin{aligned} |h_1(u)(x) - h_1(v)(x)| &\leq \frac{2C_3}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |S_\xi(u) - S_\xi(v)| |S_\xi(\nabla^2 u)| d\xi \\ &\quad + \frac{2C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(\nabla^2 u) - S_\xi(\nabla^2 v)| d\xi \\ &= I_3(x) + I_4(x). \end{aligned} \quad (110)$$

Following similar steps used above we can show

$$\|I_3\| \leq \frac{8C_3C_{e_1}\bar{J}_{3/2}\|u\|_2}{\epsilon^{5/2}} \|u - v\|_2 \leq \frac{L_2(\|u\|_2 + \|v\|_2)}{\epsilon^{5/2}} \|u - v\|_2 \quad (111)$$

and

$$\|I_4\| \leq \frac{4C_2\bar{J}_1}{\epsilon^2} \|u - v\|_2 = \frac{L_1}{\epsilon^2} \|u - v\|_2, \quad (112)$$

where $L_1 = 4C_2\bar{J}_1$, $L_2 = 8C_3C_{e_1}\bar{J}_{3/2}$.

Next we focus on $|h_2(u)(x) - h_2(v)(x)|$. We have

$$\begin{aligned}
& |h_2(u)(x) - h_2(v)(x)| \\
& \leq \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |F_1'''(\sqrt{s_\xi} S_\xi(u)) - F_1'''(\sqrt{s_\xi} S_\xi(v))| |S_\xi(\nabla u)|^2 d\xi \\
& + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |F_1'''(\sqrt{s_\xi} S_\xi(v))| |S_\xi(\nabla u) \otimes S_\xi(\nabla u) - S_\xi(\nabla v) \otimes S_\xi(\nabla v)| d\xi \\
& \leq \frac{2C_4}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) s_\xi |S_\xi(u) - S_\xi(v)| |S_\xi(\nabla u)|^2 d\xi \\
& + \frac{2C_3}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |S_\xi(\nabla u) \otimes S_\xi(\nabla u) - S_\xi(\nabla v) \otimes S_\xi(\nabla v)| d\xi \\
& = I_5(x) + I_6(x). \tag{113}
\end{aligned}$$

Proceeding as below for $\|I_5\|^2$

$$\begin{aligned}
& \|I_5\|^2 \\
& \leq \int_D \left(\frac{2C_4}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2} \frac{J(|\eta|)}{|\eta|^2} |\xi|^2 s_\xi |\eta|^2 s_\eta \\
& \quad \times |S_\xi(u) - S_\xi(v)| |S_\xi(\nabla u)|^2 |S_\eta(u) - S_\eta(v)| |S_\eta(\nabla u)|^2 d\xi d\eta dx \\
& \leq \int_D \left(\frac{2C_4}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_2 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2} |\xi|^4 s_\xi^2 |S_\xi(u) - S_\xi(v)|^2 |S_\xi(\nabla u)|^4 d\xi dx \\
& \leq \left(\frac{2C_4}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_2 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2} |\xi|^4 s_\xi^2 \frac{4\|u - v\|_\infty^2}{\epsilon^2 |\xi|^2} \left[\int_D |S_\xi(\nabla u)|^4 dx \right] d\xi. \tag{114}
\end{aligned}$$

We estimate the term in square bracket. Using the identity $(|a| + |b|)^4 \leq (2|a|^2 + 2|b|^2)^2 \leq 8|a|^4 + 8|b|^4$, we have

$$\begin{aligned}
\int_D |S_\xi(\nabla u)|^4 dx & \leq \frac{8}{\epsilon^4 |\xi|^4} \int_D (|\nabla u(x + \epsilon\xi)|^4 + |\nabla u(x)|^4) dx \\
& \leq \frac{16}{\epsilon^4 |\xi|^4} \|\nabla u\|_{L^4(D; \mathbf{R}^{d \times d})}^4. \tag{115}
\end{aligned}$$

where $\|u\|_{L^4(D; \mathbf{R}^d)} = [\int_D |u|^4 dx]^{1/4}$. Using the Sobolev embedding property of $u \in H^2(D; \mathbf{R}^d)$ as mentioned in [15](#), we get

$$\int_D |S_\xi(\nabla u)|^4 dx \leq \frac{16}{\epsilon^4 |\xi|^4} C_{e_2}^4 \|\nabla u\|_{H^1(D; \mathbf{R}^{d \times d})}^4 \leq \frac{16C_{e_2}^4}{\epsilon^4 |\xi|^4} \|u\|_2^4. \tag{116}$$

Using $\|u - v\|_\infty \leq C_{e_1} \|u - v\|_2$ and above estimate in [114](#) to have

$$\|I_5\|^2 \leq \left(\frac{2C_4}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_2 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^2} |\xi|^4 s_\xi^2 \frac{4C_{e_1}^2 \|u - v\|_2^2}{\epsilon^2 |\xi|^2} \frac{16C_{e_2}^4}{\epsilon^4 |\xi|^4} \|u\|_2^4 d\xi,$$

and we obtain

$$\|I_5\| \leq \frac{16C_4 C_{e_1} C_{e_2}^2 \bar{J}_2 \|u\|_2^2}{\epsilon^3} \|u - v\|_2 \leq \frac{L_3 (\|u\|_2 + \|v\|_2)^2}{\epsilon^3} \|u - v\|_2 \tag{117}$$

where we let $L_3 = 16C_4 C_{e_1} C_{e_2}^2 \bar{J}_2$.

Next we use

$$|S_\xi(\nabla u) \otimes S_\xi(\nabla u) - S_\xi(\nabla v) \otimes S_\xi(\nabla v)| \leq (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|) |S_\xi(\nabla u) - S_\xi(\nabla v)|$$

to estimate $\|I_6\|$ as follows

$$\begin{aligned}
& \|I_6\|^2 \\
& \leq \int_D \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \int_{H_1(0)} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} \frac{J(|\eta|)}{|\eta|^{3/2}} |\xi|^{3/2} |\eta|^{3/2} \sqrt{s_\xi s_\eta} \\
& \quad \times (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|) |S_\xi(\nabla u) - S_\xi(\nabla v)| \\
& \quad \times (|S_\eta(\nabla u)| + |S_\eta(\nabla v)|) |S_\eta(\nabla u) - S_\eta(\nabla v)| d\xi d\eta dx \\
& \leq \int_D \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_{3/2} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} |\xi|^3 \epsilon |\xi| (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^2 |S_\xi(\nabla u) - S_\xi(\nabla v)|^2 d\xi dx \\
& = \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_{3/2} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} |\xi|^3 \epsilon |\xi| \left[\int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^2 |S_\xi(\nabla u) - S_\xi(\nabla v)|^2 dx \right] d\xi.
\end{aligned} \tag{118}$$

We focus on the term in square bracket. Using the Hölder inequality, we have

$$\begin{aligned}
& \int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^2 |S_\xi(\nabla u) - S_\xi(\nabla v)|^2 dx \\
& \leq \left(\int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^4 dx \right)^{1/2} \left(\int_D |S_\xi(\nabla u) - S_\xi(\nabla v)|^4 dx \right)^{1/2}.
\end{aligned} \tag{119}$$

Using $(|a| + |b|)^4 \leq 8|a|^4 + 8|b|^4$, we get

$$\begin{aligned}
& \int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^4 dx \\
& \leq 8 \left[\int_D |S_\xi(\nabla u)|^4 dx + \int_D |S_\xi(\nabla v)|^4 dx \right] \\
& \leq 8 \left[\frac{8}{\epsilon^4 |\xi|^4} \int_D (|\nabla u(x + \epsilon\xi)|^4 + |\nabla u(x)|^4) dx + \frac{8}{\epsilon^4 |\xi|^4} \int_D (|\nabla v(x + \epsilon\xi)|^4 + |\nabla v(x)|^4) dx \right] \\
& \leq \frac{128}{\epsilon^4 |\xi|^4} (\|\nabla u\|_{L^4(D; \mathbf{R}^{d \times d})}^4 + \|\nabla v\|_{L^4(D; \mathbf{R}^{d \times d})}^4) \\
& \leq \frac{128C_{e_2}^4}{\epsilon^4 |\xi|^4} (\|\nabla u\|_{H^1(D; \mathbf{R}^{d \times d})}^4 + \|\nabla v\|_{H^1(D; \mathbf{R}^{d \times d})}^4) \\
& \leq \frac{128C_{e_2}^4}{\epsilon^4 |\xi|^4} (\|u\|_2^4 + \|v\|_2^4) \\
& \leq \frac{128C_{e_2}^4}{\epsilon^4 |\xi|^4} (\|u\|_2 + \|v\|_2)^4.
\end{aligned} \tag{120}$$

where we used Sobolev embedding property 15 in third last step. Proceeding similarly to get

$$\begin{aligned}
& \int_D |S_\xi(\nabla u) - S_\xi(\nabla v)|^4 dx \\
& \leq \frac{8}{\epsilon^4 |\xi|^4} \left[\int_D |\nabla(u - v)(x + \epsilon\xi)|^4 dx + \int_D |\nabla(u - v)(x)|^4 dx \right] \\
& \leq \frac{16}{\epsilon^4 |\xi|^4} \|\nabla(u - v)\|_{L^4(D; \mathbf{R}^{d \times d})}^4 \\
& \leq \frac{16C_{e_2}^4}{\epsilon^4 |\xi|^4} \|u - v\|_2^4.
\end{aligned} \tag{121}$$

Substituting 120 and 121 into 119 to get

$$\begin{aligned}
& \int_D (|S_\xi(\nabla u)| + |S_\xi(\nabla v)|)^2 |S_\xi(\nabla u) - S_\xi(\nabla v)|^2 dx \\
& \leq \left(\frac{128C_{e_2}^4}{\epsilon^4 |\xi|^4} (||u||_2 + ||v||_2)^4 \right)^{1/2} \left(\frac{16C_{e_2}^4}{\epsilon^4 |\xi|^4} ||u - v||_2^4 \right)^{1/2} \\
& = \frac{32\sqrt{2}C_{e_2}^4}{\epsilon^4 |\xi|^4} (||u||_2 + ||v||_2)^2 ||u - v||_2^2 \\
& \leq \frac{64C_{e_2}^4}{\epsilon^4 |\xi|^4} (||u||_2 + ||v||_2)^2 ||u - v||_2^2.
\end{aligned}$$

Substituting above in 118 to get

$$\begin{aligned}
& ||I_6||^2 \\
& \leq \left(\frac{2C_3}{\epsilon\omega_d} \right)^2 \omega_d \bar{J}_{3/2} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} |\xi|^3 \epsilon |\xi| \left[\frac{64C_{e_2}^4}{\epsilon^4 |\xi|^4} (||u||_2 + ||v||_2)^2 ||u - v||_2^2 \right] d\xi.
\end{aligned}$$

From above we have

$$||I_6|| \leq \frac{16C_3 C_{e_2}^2 \bar{J}_{3/2} (||u||_2 + ||v||_2)}{\epsilon^{5/2}} ||u - v||_2 = \frac{L_4 (||u||_2 + ||v||_2)}{\epsilon^{5/2}} ||u - v||_2, \quad (122)$$

where we let $L_4 = 16C_3 C_{e_2}^2 \bar{J}_{3/2}$.

From the expression of $h_3(u)(x)$ and $h_5(u)(x)$ we find that it is similar to term $g_1(u)(x)$ from the point of view of L^2 norm. Also, $h_4(u)(x)$ is similar to $P(u)(x)$. We easily have

$$|h_4(u)(x) - h_4(v)(x)| \leq \frac{2C_2 C_{\omega_2}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(u) - S_\xi(v)| d\xi,$$

where we used the fact that $|\nabla^2 \bar{\omega}_\xi(x)| \leq C_{\omega_2}$. Above is similar to the bound on $|P(u)(x) - P(v)(x)|$, see 97, therefore we have

$$||h_4(u) - h_4(v)|| \leq \frac{L_1 C_{\omega_2}}{\epsilon^2} ||u - v||_2. \quad (123)$$

Similarly, we have

$$\begin{aligned}
& |h_3(u)(x) - h_3(v)(x)| \\
& \leq \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |F_1''(\sqrt{s_\xi} S_\xi(u)) - F_1''(\sqrt{s_\xi} S_\xi(v))| |\nabla \bar{\omega}_\xi(x)| |S_\xi(\nabla u)| d\xi \\
& + \frac{2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |F_1''(\sqrt{s_\xi} S_\xi(v))| |e_\xi \otimes \nabla \bar{\omega}_\xi(x) \otimes S_\xi(\nabla u) - e_\xi \otimes \nabla \bar{\omega}_\xi(x) \otimes S_\xi(\nabla v)| d\xi \\
& \leq \frac{2C_3 C_{\omega_1}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) \sqrt{s_\xi} |S_\xi(u) - S_\xi(v)| |S_\xi(\nabla u)| d\xi \\
& + \frac{2C_2 C_{\omega_1}}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |S_\xi(\nabla u) - S_\xi(\nabla v)| d\xi \\
& = C_{\omega_1} (I_1(x) + I_2(x)), \quad (124)
\end{aligned}$$

where $I_1(x)$ and $I_2(x)$ are given by 101. From 103 and 104, have

$$\begin{aligned}
||h_3(u) - h_3(v)|| & \leq C_{\omega_1} (||I_1|| + ||I_2||) \\
& \leq \frac{\sqrt{\epsilon} C_{\omega_1} L_1 + C_{\omega_1} L_2 (||u||_2 + ||v||_2)}{\epsilon^{5/2}} ||u - v||_2. \quad (125)
\end{aligned}$$

Expression of $h_3(u)$ and $h_5(u)$ is similar and hence we have

$$\begin{aligned} \|h_5(u) - h_5(v)\| &\leq C_{\omega_1}(\|I_1\| + \|I_2\|) \\ &\leq \frac{\sqrt{\epsilon}C_{\omega_1}L_1 + C_{\omega_1}L_2(\|u\|_2 + \|v\|_2)}{\epsilon^{5/2}}\|u - v\|_2. \end{aligned} \quad (126)$$

We collect results to get

$$\begin{aligned} &\|\nabla^2 P(u) - \nabla^2 P(v)\| \\ &\leq \left[\frac{\epsilon L_1 + \sqrt{\epsilon}L_2(\|u\|_2 + \|v\|_2) + L_3(\|u\|_2 + \|v\|_2)^2 + \sqrt{\epsilon}L_4(\|u\|_2 + \|v\|_2)}{\epsilon^3} \right. \\ &\quad \left. + \frac{\epsilon C_{\omega_2}L_1 + 2\epsilon C_{\omega_1}L_1 + 2\sqrt{\epsilon}C_{\omega_1}L_2(\|u\|_2 + \|v\|_2)}{\epsilon^3} \right] \|u - v\|_2 \\ &\leq \left[\frac{\epsilon(1 + 2C_{\omega_1} + C_{\omega_2})L_1 + \sqrt{\epsilon}(L_2 + 2C_{\omega_1}L_2 + L_4)(\|u\|_2 + \|v\|_2)}{\epsilon^3} \right. \\ &\quad \left. + \frac{L_3(\|u\|_2 + \|v\|_2)^2}{\epsilon^3} \right] \|u - v\|_2. \end{aligned} \quad (127)$$

We now combine [99](#), [108](#), and [127](#), to get

$$\begin{aligned} &\|P(u) - P(v)\|_2 \\ &\leq \left[\frac{2\epsilon L_1 + \epsilon(1 + C_{\omega_1})L_1 + \sqrt{\epsilon}(\|u\|_2 + \|v\|_2)}{\epsilon^3} \right. \\ &\quad + \frac{\epsilon(1 + 2C_{\omega_1} + C_{\omega_2})L_1 + \sqrt{\epsilon}(L_2 + 2C_{\omega_1}L_2 + L_4)(\|u\|_2 + \|v\|_2)}{\epsilon^3} \\ &\quad \left. + \frac{L_3(\|u\|_2 + \|v\|_2)^2}{\epsilon^3} \right] \|u - v\|_2. \end{aligned} \quad (128)$$

We let

$$\bar{L}_1 := (4 + 3C_{\omega_1} + C_{\omega_2})L_1, \quad \bar{L}_2 := (1 + 2C_{\omega_1})L_2 + L_4, \quad \bar{L}_3 := L_3 \quad (129)$$

to write

$$\begin{aligned} &\|P(u) - P(v)\|_2 \\ &\leq \frac{\bar{L}_1 + \bar{L}_2(\|u\|_2 + \|v\|_2) + \bar{L}_3(\|u\|_2 + \|v\|_2)^2}{\epsilon^3} \|u - v\|_2 \end{aligned} \quad (130)$$

and this completes the proof of [19](#).

We now bound the peridynamic force. Note that $F'_1(0) = 0$, and $S_\xi(v) = 0$ if $v = 0$. Substituting $v = 0$ in [99](#) to get

$$\|P(u)\| \leq \frac{L_1}{\epsilon^2} \|u\|_2. \quad (131)$$

For $\|g_1(u)\|$ and $\|g_2(u)\|$ we proceed differently. For $\|g_2(u)\|$, we substitute $v = 0$ in [107](#) to get

$$\|g_2(u)\| \leq \frac{C_{\omega_1}L_1}{\epsilon^2} \|u\|_2. \quad (132)$$

For $\|g_1(u)\|$, we proceed as follows

$$\begin{aligned} |g_1(u)(x)| &\leq \frac{2C_2}{\epsilon\omega_d} \int_{H_1(0)} J(|\xi|) |\nabla S_\xi(u)| d\xi \\ &\leq \frac{2C_2}{\epsilon^2\omega_d} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} (|\nabla u(x + \epsilon\xi)| + |\nabla u(x)|) d\xi, \end{aligned} \quad (133)$$

and we have

$$\begin{aligned} \|g_1(u)\|^2 &\leq \left(\frac{2C_2}{\epsilon^2\omega_d} \right)^2 \omega_d \bar{J}_1 \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|} \left[\int_D (|\nabla u(x + \epsilon\xi)| + |\nabla u(x)|)^2 dx \right] d\xi \\ &\leq \left(\frac{4C_2\bar{J}_1}{\epsilon^2} \right)^2 \|\nabla u\|^2, \end{aligned} \quad (134)$$

i.e.

$$\|g_1(u)\| \leq \frac{L_1}{\epsilon^2} \|u\|_2. \quad (135)$$

Combining 132 and 135 gives

$$\|\nabla P(u)\| \leq \frac{(1 + C_{\omega_1})L_1}{\epsilon^2} \|u\|_2. \quad (136)$$

We now estimate $\|\nabla^2 P(u)\|$ from above. We have from 96

$$\|\nabla^2 P(u)\| \leq \|h_1(u)\| + \|h_2(u)\| + \|h_3(u)\| + \|h_4(u)\| + \|h_5(u)\|.$$

From expression of $h_1(u)$ we find that

$$\|h_1(u)\| \leq \frac{4C_2\bar{J}_1}{\epsilon^2} \|u\|_2 = \frac{L_1}{\epsilon^2} \|u\|_2$$

Bound on $\|h_2(u)\|$ is similar to I_6 , see 113, and we have

$$\|h_2(u)\| \leq \frac{8C_3C_{e_2}^2\bar{J}_{3/2}}{\epsilon^{5/2}} \|u\|_2^2 \leq \frac{L_4}{\epsilon^{5/2}} \|u\|_2^2,$$

where $L_4 = 16C_3C_{e_2}^2\bar{J}_{3/2}$. Case of $\|h_3(u)\|$ and $\|h_5(u)\|$ is similar to $\|g_1(u)\|$, and case of $\|h_4(u)\|$ is similar to $\|P(u)\|$. We thus have

$$\|h_4(u)\| \leq \frac{C_{\omega_2}L_1}{\epsilon^2} \|u\|_2$$

and

$$\|h_3(u)\| \leq \frac{C_{\omega_1}L_1}{\epsilon^2} \|u\|_2 \quad \text{and} \quad \|h_5(u)\| \leq \frac{C_{\omega_1}L_1}{\epsilon^2} \|u\|_2.$$

Combining above to get

$$\|\nabla^2 P(u)\| \leq \frac{\sqrt{\epsilon}(1 + C_{\omega_2} + 2C_{\omega_1})L_1 + L_4\|u\|_2}{\epsilon^{5/2}} \|u\|_2. \quad (137)$$

Finally, after combining 131, 136, and 137, we get

$$\|P(u)\|_2 \leq \frac{\sqrt{\epsilon}(4 + 3C_{\omega_1} + C_{\omega_2})L_1 + L_4\|u\|_2}{\epsilon^{5/2}} \|u\|_2.$$

We let

$$\bar{L}_4 := \bar{L}_1 \quad \text{and} \quad \bar{L}_5 := L_4 \quad (138)$$

to write

$$\|P(u)\|_2 \leq \frac{\bar{L}_4\|u\|_2 + \bar{L}_5\|u\|_2^2}{\epsilon^{5/2}}. \quad (139)$$

This completes the proof of 20.

A.2. Local and global existence of solution in $H^2 \cap H_0^1$ space. In this section, we prove Theorem 3.2. We first prove local existence for a finite time interval. We find that we can choose this time interval independent of the initial data. We repeat the local existence theorem to uniquely continue the local solution over any finite time interval. The existence and uniqueness of local solutions is stated in the following theorem.

Theorem A.1. Local existence and uniqueness *Given $X = W \times W$, $b(t) \in W$, and initial data $x_0 = (u_0, v_0) \in X$. We suppose that $b(t)$ is continuous in time over some time interval $I_0 = (-T, T)$ and satisfies $\sup_{t \in I_0} \|b(t)\|_2 < \infty$. Then, there exists a time interval $I' = (-T', T') \subset I_0$ and unique solution $y = (y^1, y^2)$ such that $y \in C^1(I'; X)$ and*

$$y(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau) d\tau, \text{ for } t \in I' \quad (140)$$

or equivalently

$$y'(t) = F^\epsilon(y(t), t), \text{ with } y(0) = x_0, \text{ for } t \in I'$$

where $y(t)$ and $y'(t)$ are Lipschitz continuous in time for $t \in I' \subset I_0$.

Proof. To prove Theorem A.1, we proceed as follows. Write $y(t) = (y^1(t), y^2(t))$ with $\|y\|_X = \|y^1(t)\|_2 + \|y^2(t)\|_2$. Let us consider $R > \|x_0\|_X$ and define the ball $B(0, R) = \{y \in X : \|y\|_X < R\}$. Let $r < \min\{1, R - \|x_0\|_X\}$. We clearly have $r^2 < (R - \|x_0\|_X)^2$ as well as $r^2 < r < R - \|x_0\|_X$. Consider the ball $B(x_0, r^2)$ defined by

$$B(x_0, r^2) = \{y \in X : \|y - x_0\|_X < r^2\}. \quad (141)$$

Then we have $B(x_0, r^2) \subset B(x_0, r) \subset B(0, R)$, see Fig 3.

To recover the existence and uniqueness we introduce the transformation

$$S_{x_0}(y)(t) = x_0 + \int_0^t F^\epsilon(y(\tau), \tau) d\tau.$$

Introduce $0 < T' < T$ and the associated set $Y(T')$ of functions in W taking values in $B(x_0, r^2)$ for $I' = (-T', T') \subset I_0 = (-T, T)$. The goal is to find appropriate interval $I' = (-T', T')$ for which S_{x_0} maps into the corresponding set $Y(T')$. Writing out the transformation with $y(t) \in Y(T')$ gives

$$S_{x_0}^1(y)(t) = x_0^1 + \int_0^t y^2(\tau) d\tau \quad (142)$$

$$S_{x_0}^2(y)(t) = x_0^2 + \int_0^t (\mathcal{L}^\epsilon(y^1(\tau)) + b(\tau)) d\tau. \quad (143)$$

We have from 142

$$\|S_{x_0}^1(y)(t) - x_0^1\|_2 \leq \sup_{t \in (-T', T')} \|y^2(t)\|_2 T'. \quad (144)$$

Using bound on \mathcal{L}^ϵ in Theorem 3.1, we have from 143

$$\|S_{x_0}^2(y)(t) - x_0^2\|_2 \leq \int_0^t \left[\frac{\bar{L}_4}{\epsilon^{5/2}} \|y^1(\tau)\|_2 + \frac{\bar{L}_5}{\epsilon^{5/2}} \|y^1(\tau)\|_2^2 + \|b(\tau)\|_2 \right] d\tau. \quad (145)$$

Let $\bar{b} = \sup_{t \in I_0} \|b(t)\|_2$. Noting that transformation S_{x_0} is defined for $t \in I' = (-T', T')$ and $y(\tau) = (y^1(\tau), y^2(\tau)) \in B(x_0, r^2) \subset B(0, R)$ as $y \in Y(T')$, we have from 145 and 144

$$\begin{aligned} \|S_{x_0}^1(y)(t) - x_0^1\|_2 &\leq RT', \\ \|S_{x_0}^2(y)(t) - x_0^2\|_2 &\leq \left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + \bar{b} \right] T'. \end{aligned}$$

Adding these inequalities delivers

$$\|S_{x_0}(y)(t) - x_0\|_X \leq \left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right] T'. \quad (146)$$

Choosing T' as below

$$T' < \frac{r^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right]} \quad (147)$$

will result in $S_{x_0}(y) \in Y(T')$ for all $y \in Y(T')$ as

$$\|S_{x_0}(y)(t) - x_0\|_X < r^2. \quad (148)$$

Since $r^2 < (R - \|x_0\|_X)^2$, we have

$$T' < \frac{r^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right]} < \frac{(R - \|x_0\|_X)^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right]}.$$

Let $\theta(R)$ be given by

$$\theta(R) := \frac{(R - \|x_0\|_X)^2}{\left[\frac{\bar{L}_4 R + \bar{L}_5 R^2}{\epsilon^{5/2}} + R + \bar{b} \right]}. \quad (149)$$

$\theta(R)$ is increasing with $R > 0$ and satisfies

$$\theta_\infty := \lim_{R \rightarrow \infty} \theta(R) = \frac{\epsilon^{5/2}}{\bar{L}_5}. \quad (150)$$

So given R and $\|x_0\|_X$ we choose T' according to

$$\frac{\theta(R)}{2} < T' < \theta(R), \quad (151)$$

and set $I' = (-T', T')$. This way we have shown that for time domain I' the transformation $S_{x_0}(y)(t)$ as defined in 140 maps $Y(T')$ into itself. The required Lipschitz continuity follows from 19 and existence and uniqueness of solution can be established using an obvious modification of [Theorem VII.3, [7]]. \square

We now prove Theorem 3.2. From the proof of Theorem A.1 above, we have a unique local solution over a time domain $(-T', T')$ with $\frac{\theta(R)}{2} < T'$. Since $\theta(R) \nearrow \epsilon^{5/2}/\bar{L}_5$ as $R \nearrow \infty$ we can fix a tolerance $\eta > 0$ so that $[(\epsilon^{5/2}/2\bar{L}_5) - \eta] > 0$. Then for any initial condition in W and $b = \sup_{t \in [-T, T]} \|b(t)\|_2$ we can choose R sufficiently large so that $\|x_0\|_X < R$ and $0 < (\epsilon^{5/2}/2\bar{L}_5) - \eta < T'$. Since choice of T' is independent of initial condition and R , we can always find local solutions for time intervals $(-T', T')$ for T' larger than $[(\epsilon^{5/2}/2\bar{L}_5) - \eta] > 0$. Therefore we apply the local existence and uniqueness result to uniquely continue local solutions up to an arbitrary time interval $(-T, T)$.

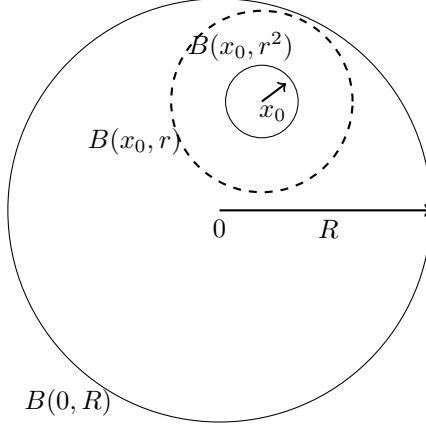


FIGURE 3. Geometry.

A.3. Proof of the higher regularity with respect to time. In this section we prove that the peridynamic evolutions have higher regularity in time for body forces that are differentiable in time. To see this we take the time derivative of [2](#) to get a second order differential equation in time for $v = \dot{u}$ given by

$$\rho \partial_{tt}^2 v(t, x) = Q(v(t); u(t))(x) + \dot{b}(t, x), \quad (152)$$

where $Q(v; u)$ is an operator that depends on the solution u of [2](#) and acts on v . It is given by, $\forall x \in D$,

$$Q(v; u)(x) := \frac{2}{\epsilon^d \omega_d} \int_{H_\epsilon(x)} \partial_S^2 W^\epsilon(S(y, x; u), y - x) S(y, x; v) \frac{y - x}{|y - x|} dy. \quad (153)$$

Clearly, for given u , $Q(v; u)$ acts linearly on v which implies that the equation for v [152](#) is a linear nonlocal equation. The linearity of $Q(v; u)$ implies Lipschitz continuity for $v \in W$ as stated below.

Theorem A.2. Lipschitz continuity of Q Let $u \in W$ be any given field. Then for all $v, w \in W$, we have

$$\|Q(v; u) - Q(w; u)\|_2 \leq \frac{L_6(1 + \|u\|_2 + \|u\|_2^2)}{\epsilon^3} \|v - w\|_2 \quad (154)$$

where constant L_6 does not depend on u, v, w . This gives for all $v \in W$ the upper bound,

$$\|Q(v; u)\|_2 \leq \frac{L_6(1 + \|u\|_2 + \|u\|_2^2)}{\epsilon^3} \|v\|_2. \quad (155)$$

We postpone the proof of Theorem [A.2](#) and provide it in the following subsection. From the theorem, we see that if u is a solution of [2](#) and $u \in C^2(I_0; W)$ then we have for all $t \in I_0$ the inequality

$$\|Q(v; u(t))\|_2 \leq \frac{L_6(1 + \sup_{s \in I_0} \|u(s)\|_2 + \sup_{s \in I_0} \|u(s)\|_2^2)}{\epsilon^3} \|v\|_2. \quad (156)$$

Next we remark that the Lipschitz continuity of $y'(t)$ stated in Theorem [3.2](#) implies $\lim_{t \rightarrow 0^\pm} \partial_{tt}^2 u(t, x) = \partial_{tt}^2 u(0, x)$. We now show that $v(t, x) = \partial_t u(t, x)$ is the unique solution of the following initial boundary value problem.

Theorem A.3. Initial value problem for $v(t, x)$ Suppose the initial data and righthand side $b(t)$ satisfy the hypothesis of Theorem 3.2 and we suppose further that $\dot{b}(t)$ exists and is continuous in time for $t \in I_0$ and $\sup_{t \in I_0} \|\dot{b}(t)\|_2 < \infty$. Then $v(t, x)$ is the unique solution to the initial value problem $v(0, x) = v_0(x)$, $\partial_t v(0, x) = \partial_{tt}^2 u(0, x)$,

$$\rho \partial_{tt}^2 v(t, x) = Q(v(t); u(t))(x) + \dot{b}(t, x), \quad t \in I_0, \quad (157)$$

$v \in C^2(I_0; W)$ and

$$\|\partial_{ttt}^3 v(t, x)\|_2 \leq \|Q(v(t); u(t))(x)\|_2 + \|\dot{b}(t, x)\|_2. \quad (158)$$

Theorem 3.3 now follows immediately from Theorem A.3 noting that $\partial_t u(t, x) = v(t, x)$ together with 156 and 158. The proof of Theorem A.3 follows from the Lipschitz continuity 156 and the Banach fixed point theorem as in [7].

A.4. Lipschitz continuity of $Q(v; u)$ in $H^2 \cap H_0^1$. We conclude by explicitly establishing the Lipschitz continuity of $Q(v; u)$. Recall that $Q(v; u)$ is given by

$$Q(v; u)(x) = \frac{2}{\epsilon^d \omega_d} \int_{H_\epsilon(x)} \partial_S^2 W^\epsilon(S(y, x; u), y - x) S(y, x; v) \frac{y - x}{|y - x|} dy.$$

From expression of W^ϵ in 7 and using the notation $F_1(r) = f(r^2)$ we have

$$\begin{aligned} \partial_S^2 W^\epsilon(S, y - x) &= \partial_S^2 \left(\omega(x) \omega(y) \frac{J^\epsilon(|y - x|)}{\epsilon |y - x|} F_1(\sqrt{|y - x|} S) \right) \\ &= \omega(x) \omega(y) \frac{J^\epsilon(|y - x|)}{\epsilon} F_1''(\sqrt{|y - x|} S). \end{aligned}$$

Substituting above, using the change of variable $y = x + \epsilon \xi$, $\xi \in H_1(0)$ and using the notation of previous subsections, we get

$$Q(v; u)(x) = \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) S_\xi(v) e_\xi d\xi. \quad (159)$$

We study $\|Q(v; u) - Q(w; u)\|_2$ where $u \in W$ is a given field and v, w are any two fields in W . Following the same steps used in the estimation of $\|P(u) - P(v)\|$ together with the bounds on the derivatives of F_1 , a straight forward calculation shows that

$$\|Q(v; u) - Q(w; u)\| \leq \frac{L_1}{\epsilon^2} \|v - w\| \leq \frac{L_1}{\epsilon^2} \|v - w\|_2, \quad (160)$$

where $L_1 = 4C_2 \bar{J}_1$.

We now examine $\|\nabla Q(v; u) - \nabla Q(w; u)\|$. Taking gradient of 159 we get

$$\begin{aligned} \nabla Q(v; u)(x) &= \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) e_\xi \otimes \nabla S_\xi(v) d\xi \\ &\quad + \frac{2}{\epsilon \omega_d} \int_{H_1(0)} J(|\xi|) F_1''(\sqrt{s_\xi} S_\xi(u)) S_\xi(v) e_\xi \otimes \nabla \bar{\omega}_\xi(x) d\xi \\ &\quad + \frac{2}{\epsilon \omega_d} \int_{H_1(0)} \bar{\omega}_\xi(x) J(|\xi|) \sqrt{s_\xi} F_1'''(\sqrt{s_\xi} S_\xi(u)) S_\xi(v) e_\xi \otimes \nabla S_\xi(u) d\xi \\ &=: G_1(v; u)(x) + G_2(v; u)(x) + G_3(v; u)(x). \end{aligned} \quad (161)$$

It is straight forward to show that

$$\begin{aligned} \|G_1(v; u) - G_1(w; u)\| &\leq \frac{L_1}{\epsilon^2} \|\nabla v - \nabla w\| \leq \frac{L_1}{\epsilon^2} \|v - w\|_2 \\ \|G_2(v; u) - G_2(w; u)\| &\leq \frac{L_1 C_{\omega_1}}{\epsilon^2} \|v - w\| \leq \frac{L_1 C_{\omega_1}}{\epsilon^2} \|v - w\|_2. \end{aligned}$$

Applying the inequalities $|S_\xi(v) - S_\xi(w)| \leq 2\|v - w\|_\infty / (\epsilon|\xi|) \leq 2C_{e_1}\|v - w\|_2 / (\epsilon|\xi|)$ and $|F_1'''(r)| \leq C_3$, we have

$$|G_3(v; u)(x) - G_3(w; u)(x)| \leq \frac{4C_3 C_{e_1} \|v - w\|_2}{\epsilon^{5/2}} \frac{1}{\omega_d} \int_{H_1(0)} \frac{J(|\xi|)}{|\xi|^{3/2}} \epsilon |\xi| |\nabla S_\xi(u)| d\xi.$$

Using the estimates above one has

$$\|G_3(v; u) - G_3(w; u)\| \leq \frac{8C_3 C_{e_1} \bar{J}_{3/2}}{\epsilon^{5/2}} \|u\|_2 \|v - w\|_2 = \frac{L_2}{\epsilon^{5/2}} \|u\|_2 \|v - w\|_2,$$

where $L_2 = 8C_3 C_{e_1} \bar{J}_{3/2}$. On collecting results, we have shown

$$\|\nabla Q(v; u) - \nabla Q(w; u)\| \leq \frac{\sqrt{\epsilon} L_1 (1 + C_{\omega_1}) + L_2 \|u\|_2}{\epsilon^{5/2}} \|v - w\|_2. \quad (162)$$

Next we take the gradient of [161](#), and write

$$\nabla^2 Q(v; u)(x) = \nabla G_1(v; u)(x) + \nabla G_2(v; u)(x) + \nabla G_3(v; u)(x). \quad (163)$$

Following the steps used in previous subsection, we estimate each term in [163](#) to obtain the following estimate given by

$$\|\nabla^2 Q(v; u) - \nabla^2 Q(w; u)\| \leq \frac{L_5 (1 + \|u\|_2 + \|u\|_2^2)}{\epsilon^3} \|v - w\|_2. \quad (164)$$

The proof of Theorem [A.2](#) is completed on summing up [160](#), [162](#), [164](#) under the hypothesis $\epsilon < 1$.

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