Plane elastodynamic solutions for running cracks as the limit of double well nonlocal dynamics *

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Abstract

A nonlocal field theory is applied to model brittle fracture. The fracture evolution is shown to converge in the limit of vanishing nonlocality to classic plane elastodynamics with a running crack. We carry out our analysis for a single crack in a plate subject to mode one loading.

1 Introduction

Fracture can be viewed as a collective interaction across large and small length scales. With the application of enough stress or strain to a brittle material, atomistic scale bonds will break, leading to fracture of the macroscopic specimen. From a modeling perspective fracture should appear as an emergent phenomena generated by an underlying field theory eliminating the need for a supplemental kinetic relation describing crack growth. The deformation field inside the body for points $\boldsymbol{x}$ at time t is written $\boldsymbol{u}(\boldsymbol{x},t)$. The perydynamic model [24], [25], is described by the nonlocal balance of linear momentum of the form

$$\rho \boldsymbol{u}_{tt}(\boldsymbol{x},t) = \int_{\mathcal{H}_\epsilon(\boldsymbol{x})} \boldsymbol{f}(\boldsymbol{y},\boldsymbol{x})\, d\boldsymbol{y} + \boldsymbol{b}(\boldsymbol{x},t) \tag{1}$$

where $\mathcal{H}_{\epsilon(\boldsymbol{x})}$ is a neighborhood of $\boldsymbol{x}$, ρ is the density, $\boldsymbol{b}$ is the body force density field, and $\boldsymbol{f}$ is a material-dependent constitutive law that represents the force density that a point $\boldsymbol{y}$ inside the neighborhood exerts on $\boldsymbol{x}$ as a result of the deformation field. The radius ϵ of the neighborhood is referred to as the *horizon*. Here all points satisfy the same field equation (1). The displacement fields and fracture evolution predicted by the nonlocal model should agree with the dynamic fracture of specimens when the length scale of non-locality is sufficiently small. In this respect numerical simulations are compelling, see for example [4], [26], and [28].

In this paper we theoretically examine the predictions of the nonlocal theory in the limit of vanishing non-locality. We examine a class of peridynamic models with nonlocal

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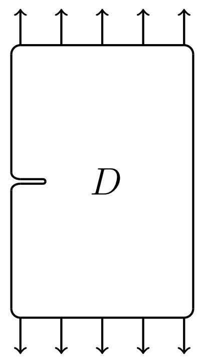

Figure 1: **Single-edge-notch**

forces derived from double well potentials see, [16]. For small strains the nonlocal force is linearly elastic but for larger strains the force begins to soften and then approaches zero after reaching a critical strain. This type of nonlocal model is called the cohesive model. We theoretically investigate the limit of the displacements for the cohesive model as the length scale ϵ of nonlocal interaction goes to zero. All information on this limit is obtained from what is known from the nonlocal peridynamic model for $\epsilon > 0$. Here we consider the single edge notch specimen as given in figure 1 and identify the target theory governing the evolution of displacement fields when $\epsilon = 0$. We are able to describe the interaction between the crack and the surrounding displacement field of intact material in this limit.

Previous work has addressed the convergence of the nonlocal cohesive fracture model to local brittle fracture for dynamic free crack propagation with multiple interacting cracks for arbitrarily shaped specimens in two and three dimensions [15], [16]. There it is shown that the nonlocal cohesive evolution converges to an evolution of sharp cracks with bounded Griffith fracture energy satisfying the linear elastic wave equation off the cracks. In this paper we provide the global description of the limit dynamics. It is shown that as $\epsilon \to 0$ the displacement solution of the nonlocal model converges in mean square uniformly in time to the displacment $\boldsymbol{u}^0(\boldsymbol{x}, t)$ that satisfies:

- Prescribed initial conditions.
- Prescribed inhomogeneous traction boundary conditions.
- Balance of linear momentum as described by the linear elastic wave equation off the crack.
- Zero traction on the sides of the evolving crack.

To summarize for prescribed initial conditions we recover the field equations and traction boundary conditions of modern dynamic Linear Elastic Fracture Mechanics (LEFM) described in [9], [22], [3], [27]. These are the principal results of the paper and are given in theorems 5 and 6. Here Γ convergence methods are no longer applicable as it is a dynamic problem. Instead the paper develops compactness methods appropriately suited to the balance of momentum for nonlocal - nonlinear operators, see lemma 5 of section 7. This method gives the zero traction condition on the crack lips for the fracture model in the local limit.

It needs to be stressed that the LEFM dynamic fracture problem is difficult to simulate numerically. This is due to the coupling of the crack tip velocity to the elastic wave equation. For remote boundaries or in the absence of elastic waves scattering off the crack tip this coupling is given by the well known kinetic relation for LEFM [9], [22], [3], [27]. The kinetic relation is deduced as a consequence of Mott's hypothesis [21] relating the energy release rate of the growing crack to the elastic power delivered to the crack tip. The power into the crack tip is described by the semi-explicit dynamic stress intensity factors developed in [1], [8], [14], [29]. On the other hand this coupling is handled autonomously in the nonlocal model (1). Recently the kinetic relation of LEFM describing crack tip motion is obtained directly from the nonlocal cohesive dynamic model in the in the $\epsilon = 0$ limit using applied mathematics arguments, see [12].

Last we point out that for stationary boundary value problems in the absence of fracture a Γ convergence approach to nonlocal problems can be employed. Boundary value problems associated with convex nonlocal potentials for material specimens in equilibrium are shown to converge to equilibrium boundary value problems for hyperelastic and elastic materials as $\epsilon \to 0$, see [5], [20].

The paper is organized as follows: In section 2 we describe the nonlocal constitutive law derived from a double well potential and present the nonlocal boundary value problem describing crack evolution. Section 3 outlines how the fracture toughness and elastic properties of a material are contained in the description of the double well potential. Section 4 provides the principle results of the paper and describes the convergence of the displacement fields given in the nonlocal model to the displacement fields seen in LFEM [9], [22], [3], [27]. The hypotheses on the emergence and nature of the softening zone where the force between points decreases with increasing strain follows from the symmetry of the loading and domain and are corroborated by the numerical simulations in [12]. The existence and uniqueness of the nonlocal evolution is established in section 5. The relation between the softening zone and jump set for the limit evolution is given by lemma 3 is proved in section 6. This relation is applied to prove theorems 2 and 3. Theorems 4, 5, and 6 are proved in section 7. We summarize results in the conclusion section 8.

2 Nonlocal Dynamics

In this section we formulate the nonlocal dynamics as an initial boundary value problem driven by a layer of force adjacent to the boundary. Here all quantities are non-dimensional. Define the region by D given by a notched rectangle with rounded corners, see figure 1. The domain lies within the rectangle $\{0 < x_1 < a; -b/2 < x_2 < b/2\}$ and the notch originates on the left side of the specimen and is of thickness $2d$ and total length $\ell(0)$ with a circular tip and rounded corners see figure 1. The domain is subject to plane strain loading and we will assume small (infinitesimal) deformations so the deformed configuration is the same as the reference configuration. We have $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ as a function of space and time but will suppress the $\boldsymbol{x}$ dependence when convenient and write $\boldsymbol{u}(t)$. The tensile strain S between two points $\boldsymbol{x}, \boldsymbol{y}$ in D along the direction $\boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}}$ is defined as

$$S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}(t)) = \frac{\boldsymbol{u}(\boldsymbol{y}, t) - \boldsymbol{u}(\boldsymbol{x}, t)}{|\boldsymbol{y} - \boldsymbol{x}|} \cdot \boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}}, \tag{2}$$

where $\boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}} = \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}$ is a unit vector and "$\cdot$" is the dot product. The influence function $J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|)$ is a measure of the influence that the point $\boldsymbol{y}$ has on $\boldsymbol{x}$. Only points inside the horizon can influence $\boldsymbol{x}$ so $J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|)$ nonzero for $|\boldsymbol{y}-\boldsymbol{x}| < \epsilon$ and zero otherwise. We take J^ϵ to be of the form: $J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|) = J(\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon})$ with $J(r) = 0$ for $r \geq 1$ and $0 \leq J(r) \leq M < \infty$ for $r < 1$.

2.1 The class of nonlocal potentials

The nonlocal force is defined in terms of a double well potential. The force potential is a function of the strain and is defined for all $\boldsymbol{x}, \boldsymbol{y}$ in D by

$$\mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t))) = J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|)\frac{1}{\epsilon^3\omega_2|\boldsymbol{y}-\boldsymbol{x}|}g(\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t))) \tag{3}$$

where $\mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t)))$ is the pairwise force potential per unit length between two points $\boldsymbol{x}$ and $\boldsymbol{y}$. It is described in terms of its potential function g, given by $g(r) = h(r^2)$ where h is concave, see figure 2(a). Here ω_2 is the area of the unit disk and $\epsilon^2\omega_2$ is the area of the horizon $\mathcal{H}_\epsilon(\boldsymbol{x})$.

The potential function g represents a convex-concave potential such that the associated force acting between material points $\boldsymbol{x}$ and $\boldsymbol{y}$ are initially elastic and then soften and decay to zero as the strain between points increases, see figure 2(b). The first well for $\mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t)))$ is at zero tensile strain and the potential function satisfies

$$g(0) = g'(0) = 0. \tag{4}$$

The well for $\mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t)))$ in the neighborhood of infinity is characterized by the horizontal asymptote $\lim_{S\to\infty} g(S) = C^+$, see figure 2(a). The critical tensile strain $S_c > 0$ for which the force begins to soften is given by the inflection point $r^c > 0$ of g and is

$$S_c = \frac{r^c}{\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}}, \tag{5}$$

and S_+ is the strain at which the force goes to zero

$$S_+ = \frac{r^+}{\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}}. \tag{6}$$

We assume here that the potential functions are bounded and are smooth.

2.2 Peridynamic equation of motion

The potential energy of the motion is given by

$$PD^\epsilon(\boldsymbol{u}) = \int_D \int_{\mathcal{H}_\epsilon(\boldsymbol{x})\cap D} |\boldsymbol{y}-\boldsymbol{x}|\mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t)))\, d\boldsymbol{y}d\boldsymbol{x}. \tag{7}$$

The set notation $H_\epsilon(\boldsymbol{x}) \cap D$ means if $\boldsymbol{x}$ belongs to D and if the line connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ crosses the boundary ∂D then the force $\partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}(t)))$ on $\boldsymbol{x}$ due to $\boldsymbol{y}$ is zero and vice versa. We consider single edge notched specimen D pulled apart by an ϵ thickness layer of body force on the top and bottom of the domain consistent with plain strain loading.

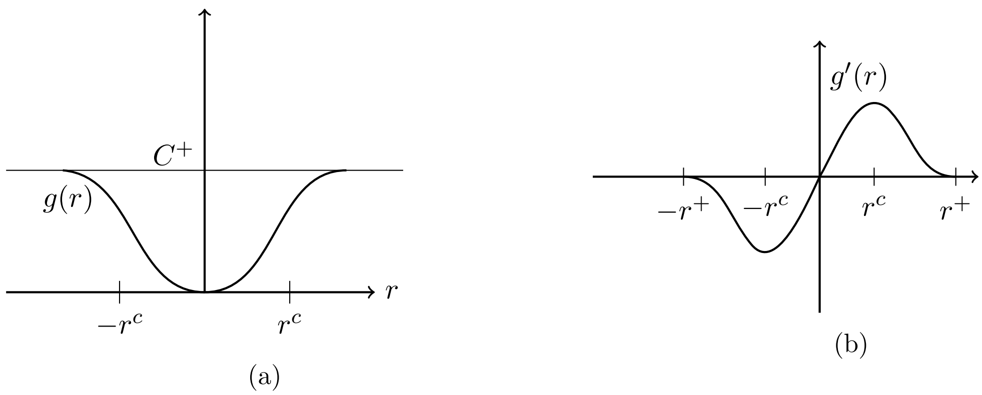

Figure 2: **(a) The potential function $g(r)$ for tensile force. Here C^+ is the asymptotic value of g. (b) Cohesive force. The derivative of the force potential goes smoothly to zero at $\pm r^+$.**

In the nonlocal setting the "traction" is given by the layer of body force on the top and bottom of the domain. For this case the body force is written as

$$\begin{aligned} \boldsymbol{b}^\epsilon(\boldsymbol{x},t) &= \boldsymbol{e}^2\epsilon^{-1}g_+(x_1,t)\chi^\epsilon_+(x_1,x_2) \text{ on the top layer and} \\ \boldsymbol{b}^\epsilon(\boldsymbol{x},t) &= \boldsymbol{e}^2\epsilon^{-1}g_-(x_1,t)\chi^\epsilon_-(x_1,x_2) \text{ on the bottom layer,} \end{aligned} \tag{8}$$

where $\boldsymbol{e}^2$ is the unit vector in the vertical direction, χ^ϵ_+ and χ^ϵ_- are the characteristic functions of the boundary layers given by

$$\begin{aligned} \chi^\epsilon_+(x_1,x_2) &= 1 \text{ on } \{\theta < x_1 < a-\theta,\, b/2-\epsilon < x_2 < b/2\} \text{ and 0 otherwise,} \\ \chi^\epsilon_-(x_1,x_2) &= 1 \text{ on } \{\theta < x_1 < a-\theta,\, -b/2 < x_2 < -b/2+\epsilon\} \text{ and 0 otherwise,} \end{aligned} \tag{9}$$

where θ is the radius of curvature of the rounded corners of D. The top and bottom traction forces are equal and in opposite directions, ie., $g_-(x_1,t) = -g_+(x_1,t)$ and $g_+(x_1,t) > 0$. We take the functions g_- and g_+ to be smooth and bounded in the variables x_1 and t and define $\boldsymbol{g}$ on ∂D such that

$$\boldsymbol{g} = \boldsymbol{e}^2 g_\pm \text{ on } \{\theta \le x_1 \le a-\theta,\ x_2 = \pm b/2\} \text{ and } \boldsymbol{g} = 0 \text{ elsewhere on } \partial D. \tag{10}$$

For any in-plane rigid body motion $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{\Omega}\times\boldsymbol{x}+\boldsymbol{c}$ where $\boldsymbol{\Omega}$ and $\boldsymbol{c}$ are constant vectors we see that

$$\int_D \boldsymbol{b}^\epsilon\cdot\boldsymbol{w}\,d\boldsymbol{x} = 0 \text{ and } S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{w}) = 0, \tag{11}$$

and we show in lemma 4 that $\boldsymbol{b}^\epsilon$, is a bounded linear functional on an appropriate Sobolev space and converges as $\epsilon\to 0$ to a boundary traction.

For future reference we denote the space of all square integrable fields orthogonal to rigid body motions in the L^2 inner product by

$$\dot{L}^2(D;\mathbb{R}^2). \tag{12}$$

In this treatment the density ρ is assumed constant and we define the Lagrangian

$$\mathrm{L}(\boldsymbol{u},\partial_t\boldsymbol{u},t) = \frac{\rho}{2}||\dot{\boldsymbol{u}}||^2_{L^2(D;\mathbb{R}^2)} - PD^\epsilon(\boldsymbol{u}) + \int_D \boldsymbol{b}^\epsilon\cdot\boldsymbol{u}\,d\boldsymbol{x},$$

where $\dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t}$ is the velocity. The action integral for a time evolution over the interval $0 < t < T$, is given by

$$I = \int_0^T \mathrm{L}(\boldsymbol{u}, \partial_t \boldsymbol{u}, t)\, dt. \tag{13}$$

We suppose $\boldsymbol{u}^\epsilon(t)$ is a stationary point $\boldsymbol{w}(t)$ is a perturbation and applying the principal of least action gives the nonlocal dynamics

$$\begin{aligned}
&\rho \int_0^T \int_D \dot{\boldsymbol{u}}^\epsilon(\boldsymbol{x}, t) \cdot \dot{\boldsymbol{w}}(\boldsymbol{x}, t) d\boldsymbol{x}\, dt \\
&= \int_0^T \int_D \int_{H_\epsilon(\boldsymbol{x}) \cap D} |\boldsymbol{y} - \boldsymbol{x}| \partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^\epsilon(t))) S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{w}(t))\, d\boldsymbol{y} d\boldsymbol{x}\, dt \\
&- \int_0^T \int_D \boldsymbol{b}^\epsilon(\boldsymbol{x}, t) \cdot \boldsymbol{w}(\boldsymbol{x}, t) d\boldsymbol{x}\, dt.
\end{aligned} \tag{14}$$

and an integration by parts gives the strong form

$$\rho \ddot{\boldsymbol{u}}^\epsilon(\boldsymbol{x}, t) = \mathcal{L}^\epsilon(\boldsymbol{u}^\epsilon)(\boldsymbol{x}, t) + \boldsymbol{b}^\epsilon(\boldsymbol{x}, t), \text{ for } \boldsymbol{x} \in D. \tag{15}$$

Here $\mathcal{L}^\epsilon(\boldsymbol{u}^\epsilon)$ is the peridynamic force

$$\mathcal{L}^\epsilon(\boldsymbol{u}^\epsilon) = \int_{\mathcal{H}_\epsilon(x) \cap D} \boldsymbol{f}^\epsilon(\boldsymbol{y}, \boldsymbol{x})\, d\boldsymbol{y} \tag{16}$$

and $\boldsymbol{f}^\epsilon(\boldsymbol{x}, \boldsymbol{y})$ is given by

$$\boldsymbol{f}^\epsilon(\boldsymbol{x}, \boldsymbol{y}) = 2\partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^\epsilon(t))) \boldsymbol{e}_{\boldsymbol{y} - \boldsymbol{x}}, \tag{17}$$

where

$$\partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^\epsilon(t))) = \frac{1}{\epsilon^3 \omega_2} \frac{J^\epsilon(|\boldsymbol{y} - \boldsymbol{x}|)}{|\boldsymbol{y} - \boldsymbol{x}|} \partial_S g(\sqrt{|\boldsymbol{y} - \boldsymbol{x}|} S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^\epsilon(t))). \tag{18}$$

The dynamics is complemented with the initial data

$$\boldsymbol{u}^\epsilon(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \partial_t \boldsymbol{u}^\epsilon(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}). \tag{19}$$

Where $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ lie in $\dot{L}^2(D; \mathbb{R}^2)$.

The initial value problem for the nonlocal evolution given by (15) and (19) or equivalently by (14) and (19) has a unique solution in $C^2([0, T]; \dot{L}^2(D; \mathbb{R}^2))$, see section 5. Application of Gronwall's inequality shows that the nonlocal evolution $\boldsymbol{u}^\epsilon(\boldsymbol{x}, t)$ is uniformly bounded in the mean square norm over bounded time intervals $0 < t < T$, i.e.,

$$\max_{0 < t < T} \left\{ \|\boldsymbol{u}^\epsilon(\boldsymbol{x}, t)\|^2_{L^2(D; \mathbb{R}^2)} \right\} < K, \tag{20}$$

where the upper bound K is independent of ϵ and depends only on the initial conditions and body force applied up to time T, see [16].

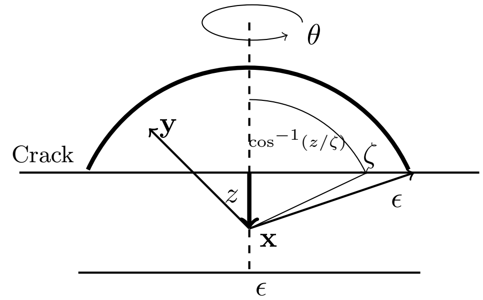

Figure 3: **Evaluation of fracture toughness $\mathcal{G}_c$. For each point $\boldsymbol{x}$ along the dashed line, $0 \leq z \leq \epsilon$, the work required to break the interaction between $\boldsymbol{x}$ and $\boldsymbol{y}$ in the spherical cap is summed up in (21) using spherical coordinates centered at $\boldsymbol{x}$.**

3 Fracture toughness and elastic properties for the cohesive model specified through the force potential

For finite horizon $\epsilon > 0$ the fracture toughness and elastic moduli are recovered directly from the cohesive strain potential $\mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}(t)))$. Here the fracture toughness $\mathcal{G}_c$ is defined to be the energy per unit length required eliminate interaction between each point $\boldsymbol{x}$ and $\boldsymbol{y}$ on either side of a line in $\mathbb{R}^2$. Because of the finite length scale of interaction only the force between pairs of points within an ϵ distance from the line are considered. The fracture toughness $\mathcal{G}_c$ is calculated in [16]. It is given by the formula

$$\mathcal{G}_c = 2 \int_0^\epsilon \int_z^\epsilon \int_0^{\arccos(z/\zeta)} \mathcal{W}^\epsilon(\mathcal{S}_+)\zeta^2 \, d\psi \, d\zeta \, dz \tag{21}$$

where $\zeta = |\boldsymbol{y} - \boldsymbol{x}|$, see figure 3. Substitution of $\mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}(t)))$ given by (3) into (21) and calculation delivers the formula

$$\mathcal{G}_c = \frac{4}{\pi} \int_0^1 h(S_+) r^2 J(r) dr. \tag{22}$$

It is evident from this calculation that the fracture toughness is the same for all choice of horizons. This provides the rational behind the ϵ scaling of the potential (3) for the cohesive model. Moreover the layer width on either side of the crack centerline over which the force is applied to create new surface tends to zero with ϵ. In this way ϵ can be interpreted as a parameter associated with the extent of the process zone of the material.

To find the elastic moduli associated with the cohesive potential $\mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}(t)))$ we suppose the displacement inside $\mathcal{H}_\epsilon(\boldsymbol{x})$ is affine, that is, $u(\boldsymbol{x}) = F\boldsymbol{x}$ where F is a constant matrix. For small strains, i.e., $\mathcal{S} = Fe \cdot e \ll \mathcal{S}_c$, a Taylor series expansion at zero strain [17] shows that the strain potential is linear elastic to leading order and characterized by elastic moduli μ and λ associated with a linear elastic isotropic material

$$\begin{aligned} W(\boldsymbol{x}) &= \int_{\mathcal{H}_\epsilon(\boldsymbol{x})} |\boldsymbol{y} - \boldsymbol{x}| \mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u})) \, d\boldsymbol{y} \\ &= 2\mu |F|^2 + \lambda |Tr\{F\}|^2 + O(\epsilon |F|^4). \end{aligned} \tag{23}$$

The elastic moduli λ and μ are calculated directly from the strain energy density and are given by

$$\mu = \lambda = M\frac{1}{4}\Phi'(0)\,, \tag{24}$$

where the constant $M = \int_0^1 r^2 J(r)dr$. The elasticity tensor is given by

$$\mathbb{C}_{ijkl} = 2\mu\left(\frac{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}}{2}\right) + \lambda\delta_{ij}\delta_{kl}. \tag{25}$$

4 Convergence of the Peridynamic evolution to the field equations of Linear Elastic Fracture Mechanics

In this section we the show solutions of the nonlocal cohesive model converge to the elastic fields of LEFM described in [9]. It is known that the in limit of small horizon $\epsilon \to 0$ the solutions $\boldsymbol{u}^\epsilon$ of the nonlocal model converge in the L^2 norm to a displacement field $\boldsymbol{u}^0$ that is linear elastodynamic off the crack set in a distributional sense and a special function of bounded deformation in the spatial variables for almost all times. The distributional form of the elastodynamic balance laws are characterized by the elastic moduli μ, λ defined in the previous section. Moreover the evolving crack set possesses bounded Griffith surface free energy associated with the fracture toughness $\mathcal{G}_c$ as defined in the previous section. These properties for the $\epsilon = 0$ limit are established in [15] and [16]. What is lacking is the global description of the limit dynamics that includes boundary conditions, crack geometry, and traction conditions on the crack faces.

To get the global description for $\boldsymbol{u}^0$ we begin by applying the results of [15] and [16] to the initial value problem for the single edge notch (8), (14), and (19). Consider solutions $\boldsymbol{u}^\epsilon$ of the initial value problem for progressively smaller peridynamic horizons ϵ. It is assumed as in [16] that the magnitude of the displacement $\boldsymbol{u}^\epsilon$ is bounded uniformly in $(\boldsymbol{x}, t)$ for all horizons $\epsilon > 0$. On passing to a subsequence $\{\epsilon_n\}_{n=1}^\infty$ if necessary the peridynamic evolutions $\boldsymbol{u}^{\epsilon_n}$ converge in mean square uniformly in time to a limit evolution $\boldsymbol{u}^0(\boldsymbol{x}, t)$ in $C([0,T]; \dot{L}^2(D;\mathbb{R}^2))$ i.e.,

$$\lim_{\epsilon_n\to 0}\max_{0\le t\le T}\int_D |\boldsymbol{u}^{\epsilon_n}(\boldsymbol{x},t) - \boldsymbol{u}^0(\boldsymbol{x},t)|^2\, d\boldsymbol{x} = 0, \tag{26}$$

moreover $\boldsymbol{u}^0_t(\boldsymbol{x}, t)$ belongs to $L^2(0,T; \dot{L}^2(D;\mathbb{R}^2))$. The limit evolution $\boldsymbol{u}^0(\boldsymbol{x}, t)$ is found to be a special function of bounded deformation $SBV(D)$ for almost all times. Functions $\boldsymbol{u} \in SBD(D)$ belong to $L^1(D;\mathbb{R}^d)$ (where $d = 2$ in this work) and are approximately continuous, i.e., have Lebesgue limits for almost every $\boldsymbol{x} \in D$ given by

$$\lim_{\epsilon\searrow 0}\frac{1}{\omega_2\epsilon^2}\int_{\mathcal{H}_\epsilon(\boldsymbol{x})} |\boldsymbol{u}(\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})|\, d\boldsymbol{y} = 0, \tag{27}$$

where $\mathcal{H}_\epsilon(\boldsymbol{x})$ is the ball of radius ϵ centered at $\boldsymbol{x}$ and $\omega_2\epsilon^2$ is its area given in terms of the area of the unit disk ω_2 times ϵ^2. The jump set $\mathcal{J}_{\boldsymbol{u}}$ for elements of $SBD(D)$ is defined to be the set of points of discontinuity which have two different one sided Lebesgue limits. One sided Lebesgue limits of $\boldsymbol{u}$ with respect to a direction $\nu_{\boldsymbol{u}(\boldsymbol{x})}$ are denoted by $\boldsymbol{u}^-(\boldsymbol{x})$, $\boldsymbol{u}^+(\boldsymbol{x})$

and are given by

$$
\begin{aligned}
&\lim_{\epsilon\searrow 0}\frac{1}{\epsilon^2\omega_2}\int_{\mathcal{H}_\epsilon^-(\boldsymbol{x})}|\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}^-(\boldsymbol{x})|\,d\boldsymbol{y}=0,\\
&\lim_{\epsilon\searrow 0}\frac{1}{\epsilon^2\omega_2}\int_{\mathcal{H}_\epsilon^+(\boldsymbol{x})}|\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}^+(\boldsymbol{x})|\,d\boldsymbol{y}=0,
\end{aligned}
\tag{28}
$$

where $\mathcal{H}_\epsilon^-(\boldsymbol{x})$ and $\mathcal{H}_\epsilon^+(\boldsymbol{x})$ are given by the intersection of $\mathcal{H}_\epsilon(\boldsymbol{x})$ with the half spaces $(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{u}(\boldsymbol{x})}<0$ and $(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{u}(\boldsymbol{x})}>0$ respectively. $SBD(D)$ functions have jump sets $\mathcal{J}_{\boldsymbol{u}}$, that are countably rectifiable. Hence they are described by a countable number of components $K_1, K_2, \ldots$, contained within smooth manifolds, with the exception of a set K_0 that has zero 1 dimensional Hausdorff measure [2]. For the case treated here we take the jump set to be a one dimensional Lebesgue measurable set. The one dimensional Hausdorff measure of $\mathcal{J}_{\boldsymbol{u}}$ agrees with the Lesbegue measure and $\mathcal{H}^1(\mathcal{J}_{\boldsymbol{u}})=\sum_i\mathcal{H}^1(K_i)$. The strain of a displacement $\boldsymbol{u}$ belonging to $SBD(D)$, written as $\mathcal{E}_{ij}\boldsymbol{u}^0(t)=(\partial_{x_i}\boldsymbol{u}_j^0+\partial_{x_j}\boldsymbol{u}_i^0)/2$, is a generalization of the classic local strain tensor and is related to the nonlocal strain $S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^0)$ by

$$
\lim_{\epsilon\to 0}\frac{1}{\epsilon^2\omega_2}\int_{\mathcal{H}_\epsilon(x)}|S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^0)-\mathcal{E}\boldsymbol{u}^0(\boldsymbol{x})\boldsymbol{e}\cdot\boldsymbol{e}|\,d\boldsymbol{y}=0,
\tag{29}
$$

for almost every $\boldsymbol{x}$ in D with respect to 2-dimensional Lebesgue measure $\mathcal{L}^2$. The symmetric part of the distributional derivative of $\boldsymbol{u}$, $E\boldsymbol{u}=1/2(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ for $SBD(D)$ functions is a 2×2 matrix valued Radon measure with absolutely continuous part described by the density $\mathcal{E}u$ and singular part described by the jump set [2] and

$$
\langle Eu,\Phi\rangle=\int_D\sum_{i,j=1}^d\mathcal{E}u_{ij}\Phi_{ij}\,d\boldsymbol{x}+\int_{J_u}\sum_{i,j=1}^d(\boldsymbol{u}_i^+-\boldsymbol{u}_i^-)\boldsymbol{n}_j\Phi_{ij}\,d\mathcal{H}^1,
\tag{30}
$$

for every continuous, symmetric matrix valued test function Φ. In the sequel we will write $[\boldsymbol{u}]=\boldsymbol{u}^+-\boldsymbol{u}^-$. The limit evolution has a bounded Griffith surface energy and elastic energy given by

$$
\int_D 2\mu|\mathcal{E}\boldsymbol{u}^0(t)|^2+\lambda|\mathrm{div}\,\boldsymbol{u}^0(t)|^2\,d\boldsymbol{x}+\mathcal{G}|\mathcal{J}_{\boldsymbol{u}^0(t)}|\leq C,
\tag{31}
$$

for $0\leq t\leq T$, where $\mathcal{J}_{\boldsymbol{u}^0(t)}$ denotes the evolving jump set inside the domain D, across which the displacement $\boldsymbol{u}^0$ has a jump discontinuity and $|\mathcal{J}_{\boldsymbol{u}^0(t)}|$ is its length. Here $|A|$ is the one dimensional Lesbegue measure on measurable sets A. Because $\boldsymbol{u}^0$ has bounded Griffith energy (31) we see that $\boldsymbol{u}^0$ also belongs to $SBD^2(D)$, that is the set of $SBD(D)$ functions with square integrable strains $\mathcal{E}\boldsymbol{u}$ and jump set with bounded Hausdorff $\mathcal{H}^1$ measure.

To proceed we identify a link between the zone of instability inside the domain where the force between two points is decreasing with increasing strain for the $\epsilon>0$ and the jumpset $\mathcal{J}_{\boldsymbol{u}^0(t)}$. For the $\epsilon>0$ dynamics we describe the zone of instability inside the domain where the force between two points is decreasing with increasing strain. First we fix the time t and decompose the strain $S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))$ as

$$
S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))=S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))^-+S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))^+
\tag{32}
$$

where

$$
S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))^-=\begin{cases}S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t)), & \text{if } |S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))|<S_c\\ 0, & \text{otherwise}\end{cases}
\tag{33}
$$

and

$$S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))^+ = \begin{cases} S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t)), & \text{if } |S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))| \geq S_c \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

The subset of points $\boldsymbol{x} \in D$ for which there is at least one $\boldsymbol{y} \in \mathcal{H}_\epsilon(\boldsymbol{x}) \cap D$ such that $|S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))| \geq S_c$ is denoted by $SZ^\epsilon(t)$. We will often suppress the dependence on time and write SZ^ϵ. This is the set of points in D for which there are always points $\boldsymbol{y}$ inside $\mathcal{H}_\epsilon(\boldsymbol{x})$ for which the force between $\boldsymbol{x}$ and $\boldsymbol{y}$ is decreasing with increasing strain. The two dimensional Lesbegue measure of the zone of instability or softening zone decreases with the horizon [16]. Although the following theorem [16] is not used here we state it to illustrate that the rate of decrease is consistent with the softening zone concentrating on a curve.

Theorem 1 *Let $SZ^\epsilon_\eta(t)$ be the set of points in D for which there are a fixed area fraction $\eta > 0$ points $\boldsymbol{y}$ inside $\mathcal{H}_\epsilon(\boldsymbol{x})$ for which the force between $\boldsymbol{x}$ and $\boldsymbol{y}$ is decreasing with increasing strain then*

$$\mathcal{L}^2(SZ^\epsilon_\eta(t)) \leq C\epsilon, \tag{35}$$

where $\mathcal{L}^2$ is two dimensional Lebesgue measure and C is independent of ϵ and depends only on the time interval of evolution $[0,T]$, geometry of the domain, and loading.

We now adopt hypothesis on the softening zone that are consistent with results of numerical simulations and carried out expressly for the single edge notch specimen subjected to the boundary loads described here in [12]. Motivated by the symmetry of the domain and the loading on the top and bottom boundaries we make the following geometric hypothesis on $SZ^\epsilon(t)$:

Hypothesis 1 *The softening zone SZ^ϵ is centered on the $x_2 = 0$ axis. It originates at the notch and is symmetric with respect to the $x_2 = 0$ axis and its intersection with the $x_2 = 0$ axis is the interval adjacent to the notch given by $\ell(0) \leq x_1 \leq \ell^\epsilon(t) < a$. For points $\boldsymbol{x}$ inside SZ^ϵ if $\boldsymbol{y} \in \mathcal{H}_\epsilon(\boldsymbol{x}) \cap D$ and $\boldsymbol{x}$ lie on different sides of the $x_2 = 0$ axis then $S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t)) \geq S_c$ on the other hand if the points $\boldsymbol{y} \in \mathcal{H}_\epsilon(\boldsymbol{x}) \cap D$ and $\boldsymbol{x}$ lie on the same side then $|S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))| < S_c$.*

This hypothesis is supported by the numerical simulations given in [12] where SZ^ϵ emerges from the simulation.

Definition 1 *The softening zone center line for the nonlocal model is given by the interval*

$$C^\epsilon(t) = \{\ell(0) \leq x_1 \leq \ell^\epsilon(t),\, x_2 = 0\} \tag{36}$$

Since $C^\epsilon(t)$ is contained in $SZ^\epsilon(t)$ the force between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ is decreasing for increasing strain and unstable if the line segment connecting them intersects this interval. It is reiterated that in the nonlocal formulation the softening zone center line is part of the solution and its location, shape, and evolution emerges from the numerical simulations.

The set $F^\epsilon(t) \subset SZ^\epsilon(t)$ is defined to be the collection of $\boldsymbol{x}$ such that the force between at least one of its neighbors $\boldsymbol{y}$ is zero, i.e., there is a $\boldsymbol{y} \in \mathcal{H}_\epsilon(\boldsymbol{x}) \cap D$ such that $S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t)) > S_+$. We remark that after the critical strain is exceeded the force then decreases with increasing strain and the interaction between the two points is unstable and the force between them goes to zero. We shall call the force interaction between two points $\boldsymbol{y}$ and $\boldsymbol{x}$ a bond. This is formalized in the following hypothesis.

Hypothesis 2 *We suppose that $SZ^\epsilon(t) = F^\epsilon(t)$ for all $0 < t < T$, i.e., once bonds soften they fail.*

From this hypothesis all pairs $\boldsymbol{x}$ and $\boldsymbol{y}$ in F^ϵ that lie on on either side of $C^\epsilon(t)$ have no force between them. Thus the displacements on one side of $C^\epsilon(t)$ are not influenced by forces on the other side of $C^\epsilon(t)$. Motivated by this consideration we introduce the set $H^\epsilon \subset F^\epsilon$ given by

$$H^\epsilon(t) = \{\ell(0) \leq x_1 < \ell^\epsilon(t), \, -\epsilon/2 < x_2 < \epsilon/2\}, \tag{37}$$

and state the final hypothesis on the displacement and strain across $C^\epsilon(t)$ given by

Hypothesis 3 *For $\boldsymbol{x} \in H^\epsilon(t)$ the strain grows as ϵ^{-1}, i.e.,*

$$\frac{1}{\epsilon^2\omega_2}\int_{\mathcal{H}_\epsilon(\boldsymbol{x})\cap D} \frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon} J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|) S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon)^+ \, d\boldsymbol{y} > \frac{\alpha}{\epsilon} > 0. \tag{38}$$

It follows from the hypotheses that the magnitude of the strain remains below $r^c/\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}$ for $\boldsymbol{x}$ and $\boldsymbol{y}$ not separated from each other by $C^\epsilon(t)$.

With the hypotheses for the $\epsilon > 0$ dynamics in hand we now provide the global description of the $\epsilon = 0$ dynamics. As ϵ approaches zero and taking subsequences indexed by ϵ_n if necessary we arrive at a limit $C^0(t)$. The limit of the softening zone center line $C^0(t)$ is related to the jump set $\mathcal{J}_{\boldsymbol{u}^0(t)}$.

Theorem 2 *Suppose hypothesis 1 and 3 hold, then the limit of the sets $C^{\epsilon_n} = \{\ell(0) \leq x_1 \leq \ell^{\epsilon_n}(t)\,,\, x_2 = 0\}$ exists and is given by $C^0(t) = \{\ell(0) \leq x_1 < \ell^0(t)\,,\, x_2 = 0\}$ where*

$$|C^0(t)\Delta\mathcal{J}_{\boldsymbol{u}^0(t)}| = 0 \textit{ and } \quad \ell^0(t) = |\mathcal{J}_{\boldsymbol{u}^0(t)}| \tag{39}$$

and

$$[\boldsymbol{u}^0]\cdot\boldsymbol{n} > 0, \tag{40}$$

almost everywhere on $\mathcal{J}_{\boldsymbol{u}^0(t)}$ with respect to one dimensional Lebesgue measure and $\boldsymbol{n} = \boldsymbol{e}^2$.

This theorem is proved in section 6, see figure 4.

The limit evolution $\boldsymbol{u}^0$ has symmetries with respect to the $x_2 = 0$ axis.

Theorem 3 *The first component of the displacement $\boldsymbol{u}^0$ denoted by $u_1^0(x_1,x_2)$ is even with respect to the $x_2 = 0$ axis and the second component of the displacement denoted by $u_2^0(x_1,x_2)$ is odd with respect to the $x_2 = 0$ axis so $u_2^0(x_1,x_2) = 0$ off the jump set $\mathcal{J}_{\boldsymbol{u}^0(t)}$.*

This theorem is proved in section 6. We mention here that for ease of exposition the proofs of the following theorems are deferred to section 7.

We now set $D_t = D \setminus C^0(t)$, see figure 4. The global description of $\ddot{\boldsymbol{u}}^0$ is in terms of suitable Sobolev spaces defined on D_t with boundary denoted by ∂D_t. Choose a positive number β, such that $\ell(0) < \ell^0(t) - \beta$ and consider the set $\Gamma_t^\beta = \{\ell(0) \leq x_1 \leq \ell^0(t) - \beta;\, x_2 = 0\}$. Define $D_{\beta,t} = D \setminus \Gamma_t^\beta$ and its boundary by $\partial D_{\beta,t} \subset \partial D_t$. The subsets of $\partial D_{\beta,t}$ with $\pm x_2 \geq 0$ are denoted by $\partial D_{\beta,t}^\pm$ and define the layers $L_\beta^\pm(t) \subset D_t$ given by the open sets adjacent to the boundary $\partial D_{\beta,t}^\pm$ with $\pm x_2 > 0$. To fix ideas we portray the layer $L_\beta^+(t)$ in figure 5. The component of boundary of $L_\beta^+(t)$ interior to D_t is denoted by ∂L^+. Similarly the layer $L_\beta^-(t) \subset D_t$ is the open set adjacent to the boundary of ∂D_t with $x_2 < 0$ given by

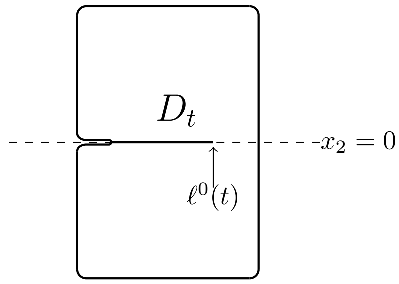

Figure 4: **Single-edge-notch and crack corresponding to $\epsilon = 0$ limit.**

reflection of $L^+_\beta(t)$ across the $x_2 = 0$ axis. The component of boundary of $L^-_\beta(t)$ interior to D_t is denoted by ∂L^-. Set

$$H^{1,0}(L^\pm_\beta(t), \mathbb{R}^2) = \left\{ \boldsymbol{w} \in H^1(L^\pm_\beta(t), \mathbb{R}^2) \text{ and } \gamma\boldsymbol{w} = 0 \text{ on } \partial L^\pm \right\}, \tag{41}$$

here γ is the trace operator mapping functions in $H^1(L^\pm_\beta(t), \mathbb{R}^2)$ to functions defined on the boundary. The Hilbert space dual to $H^{1,0}(L^\pm_\beta(t), \mathbb{R}^2)$ is denoted by $H^{1,0}(L^\pm_\beta(t), \mathbb{R}^2)'$.

Recall $H^1_0(D, \mathbb{R}^2)$ and its dual $H^{-1}(D, \mathbb{R}^2)$ as well as $H^1(D, \mathbb{R}^2)$ and its dual $H^1(D, \mathbb{R}^2)'$. The limit acceleration $\ddot{\boldsymbol{u}}^0$ has the regularity given by the following theorem.

Theorem 4 *Suppose hypotheses 1 through 3 hold then the limit acceleration $\ddot{\boldsymbol{u}}^0(\boldsymbol{x}, t)$ belongs to $H^1(D, \mathbb{R}^2)'$, $H^{-1}(D, \mathbb{R}^2)$ and $H^{1,0}(L^\pm_\beta(t), \mathbb{R}^2)'$ for almost all $t \in [0, T]$.*

Since $\ddot{\boldsymbol{u}}^0$ is not an element of $L^2(D, \mathbb{R}^2)$ we introduce the the normal traction $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}$ defined on the crack lips and ∂D in the generalized sense, [19]. In order to describe the generalized traction we introduce trace spaces compatible with the symmetry and jump set of $\boldsymbol{u}^0$. We introduce the weight defined on $\partial D^\pm_{\beta,t}$ given by

$$\alpha_\pm(x_1, x_2, \beta) = \begin{cases} \min\{1, \sqrt{(\ell^0(t) - \beta - x_1)}\}, & \text{on } x_2 = 0 \\ \min\{1, \sqrt{\pm x_2}\}, & \text{on } x_1 = a,\ \pm x_2 > 0 \\ 1, & \text{otherwise.} \end{cases} \tag{42}$$

and the trace spaces $H^{1/2}_{00}(\partial D^\pm_{\beta,t})^2$ given in [18] are defined by all functions $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x})$ in $H^{1/2}(\partial D^\pm_{\beta,t})^2$ with

$$\int_{\partial D^\pm_{\beta,t}} |\boldsymbol{w}(\boldsymbol{x})|^2 \alpha^{-1}_\pm(\boldsymbol{x}, \beta)\, ds < \infty. \tag{43}$$

Its dual is denoted by $H^{-1/2}_{00}(\partial D_{\beta,t})^2$. This type of trace space is employed for problems of mechanical contact in [13], see also [23]. The trace operator γ is a continuous linear map from $H^{1,0}(L^\pm_\beta(t), \mathbb{R}^2)$ onto $H^{1/2}_{00}(\partial D^\pm_{\beta,t})^2$, see [18]. Additionally the trace operator γ is a continuous linear map from $H^1(D, \mathbb{R}^2)$ onto $H^{1/2}(\partial D)^2$.

In what follows we define the duality bracket for Hilbert spaces H and their dual H' by $\langle \cdot, \cdot \rangle$, where the first argument is an element of H' and the second an element of H. We first introduce the generalized traction $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}$ on ∂D as an element of $H^{-1/2}(\partial D)^2$.

Lemma 1 *If $\ddot{\boldsymbol{u}}^0$ belongs to $H^1(D,\mathbb{R}^2)'$ and $\boldsymbol{u}^0$ is in $SBD^2(D)$ then the generalized traction on the boundary ∂D given by*

$$\langle \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}, \gamma\boldsymbol{w}\rangle = \int_D \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\,d\boldsymbol{x} + \rho\langle\ddot{\boldsymbol{u}}^0, \boldsymbol{w}\rangle, \tag{44}$$

for all test functions $\boldsymbol{w}$ in $H^1(D,\mathbb{R}^2)$ is uniquely defined and agrees with the usual definition when $\ddot{\boldsymbol{u}}^0$ is in $L^2(D,\mathbb{R}^2)$.

On the crack faces the generalized traction is given by the lemma.

Lemma 2 *If $\ddot{\boldsymbol{u}}^0$ belongs to $H^{1,0}(L^{\pm}_{\beta}(t),\mathbb{R}^2)'$ and $\boldsymbol{u}^0$ is in $SBD^2(D)$ then the generalized traction $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}^{\pm}$ on the upper and lower sides of the crack $C^0(t)$ are uniquely defined as as elements of $H^{-1/2}_{00}(\partial D^{\pm}_{t,\beta})^2$ by*

$$\langle \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}^{\pm}, \gamma\boldsymbol{w}\rangle = \int_{L^{\pm}_{\beta}(t))} \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\,d\boldsymbol{x} + \rho\langle\ddot{\boldsymbol{u}}^0, \boldsymbol{w}\rangle, \tag{45}$$

for all test functions $\boldsymbol{w}$ in $H^{1,0}(L^{\pm}_{\beta}(t),\mathbb{R}^2)$. The tractions agree with the usual definition when $\ddot{\boldsymbol{u}}^0$ is in $L^2(L^{\pm}_{\beta}(t),\mathbb{R}^2)$.

Lemmas 1 and 2 are proved in section 7.

The global dynamics for $\boldsymbol{u}^0(\boldsymbol{x},t)$ is given by the following theorem.

Theorem 5 *Suppose hypotheses 1 through 3 hold then:*

$$\rho\ddot{\boldsymbol{u}}^0 = div\left(\mathbb{C}\mathcal{E}\boldsymbol{u}^0\right) \tag{46}$$

as elements of $H^{-1}(D,\mathbb{R}^2)$, and

$$\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n} = \boldsymbol{g} \text{ on } \partial D, \tag{47}$$

where the traction $\boldsymbol{g}$ is given by (10) and equality holds as elements of $H^{-1/2}(\partial D)^2$. Moreover there is zero traction on the upper and lower sides of the crack, this is given by

$$\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}^{\pm} = 0 \tag{48}$$

for all $0 < \beta < \ell^0(t) - \ell(0)$ as an element of $H^{-1/2}_{00}(\partial D^{\pm}_{t,\beta})^2$.

Here the normal tractions (47) and (48) are defined in the generalized sense (44), (45) respectively.

Additionally we have that initial data given in $\dot{L}^2(D;\mathbb{R}^2)$ is well defined for the limit evolution $\boldsymbol{u}^0$ and $\dot{\boldsymbol{u}}^0$. This is stated in the following theorem.

Theorem 6 *The displacement $\boldsymbol{u}^0$ and velocity $\dot{\boldsymbol{u}}^0$ belong to $C([0,T];\dot{L}^2(D;\mathbb{R}^2))$ and $C([0,T];\dot{H}^1(D;\mathbb{R}^2)')$ respectively so initial values $\boldsymbol{u}^0(\boldsymbol{x},0)$ and $\dot{\boldsymbol{u}}^0(\boldsymbol{x},0)$ prescribed in $\dot{L}^2(D;\mathbb{R}^2)$ are well defined.*

To summarize theorems 5 and 6 constitute the global description of the displacement fields inside the cracking body. Together they deliver the elastodynamic equations, initial values, and homogeneous traction boundary conditions on the crack faces given in LEFM [9], [22], [3], and [27].

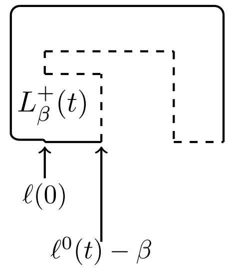

Figure 5: **Layer $L^+_\beta(t)$ adjacent to $\partial D^+_{\beta,t}$. The boundary of $L^+_\beta(t)$ interior to D_t is denoted by the dashed line.**

5 Existence and uniqueness of the nonlocal evolution

We assert the existence and uniqueness for a solution $\boldsymbol{u}^\epsilon(\boldsymbol{x},t)$ of the nonlocal evolution with the balance of momentum given in strong form (15).

Theorem 7 Existence and uniqueness of the nonlocal evolution. *The initial value problem given by* (15) *and* (19) *has a unique solution* $\boldsymbol{u}(\boldsymbol{x},t)$ *such that for every* $t\in[0,T]$, $\boldsymbol{u}$ *takes values in* $\dot{L}^2(D;\mathbb{R}^2)$ *and belongs to the space* $C^2([0,T];\dot{L}^2(D;\mathbb{R}^2))$.

The proof of this proposition follows from the Lipschitz continuity of $\mathcal{L}^\epsilon(\boldsymbol{u}^\epsilon)(\boldsymbol{x},t)+\boldsymbol{b}(\boldsymbol{x},t)$ as a function of $\boldsymbol{u}^\epsilon$ with respect to the $L^2(D;\mathbb{R}^2)$ norm and the Banach fixed point theorem, see e.g. [17]. It is pointed out that SZ^ϵ describes an unstable phase of the material however because the peridynamic force is a Lipschitz function on $\dot{L}^2(D;\mathbb{R}^2)$ the model can be viewed as an ODE for vectors in $\dot{L}^2(D;\mathbb{R}^2)$ and is well posed.

6 Relation between the softening zone and the jump set

In this section we establish the relation between the fracture set and the softening zone given by theorem 2. We then prove theorem 3. From now on we explicitly consider a denumerable sequence ϵ_n converging to zero for index $n=1,2,3\dots$. To prove theorem 2 we need the following lemma.

Lemma 3 *Suppose hypothesis 1 holds then*

$$\begin{aligned}
&\lim_{\epsilon_n\to 0}\frac{1}{{\epsilon_n}^2\omega_2}\int_D\int_{\mathcal{H}_\epsilon(\boldsymbol{x})\cap D}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^- d\boldsymbol{y}\,\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&=\int_D div\,\boldsymbol{u}^0(\boldsymbol{x},t)\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&\lim_{\epsilon_n\to 0}\frac{1}{{\epsilon_n}^2\omega_2}\int_{SZ^\epsilon}\int_{\mathcal{H}_\epsilon(\boldsymbol{x})\cap D}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^+ d\boldsymbol{y}\,\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&=C\int_{\mathcal{J}_{\boldsymbol{u}^0(t)}}[\boldsymbol{u}^0(\boldsymbol{x},t)]\cdot\boldsymbol{n}\,\varphi(\boldsymbol{x})d\mathcal{H}^1(\boldsymbol{x})
\end{aligned}\tag{49}$$

for all scalar test functions φ that are differentiable with support in D. Moreover the jump set lies on the $x_2=0$ axis. Here $[\boldsymbol{u}^0(\boldsymbol{x},t)]$ denotes the jump in displacement across $\mathcal{J}_{\boldsymbol{u}^0(t)}$ and $\boldsymbol{n}$ is the unit normal to $\mathcal{J}_{\boldsymbol{u}^0(t)}$ and points in the vertical direction $\boldsymbol{e}^2$, and $C=\omega_2\int_0^1 r^2dr$.

We prove lemma 3. It is convenient to make the change of variables $\boldsymbol{y}=\boldsymbol{x}+\epsilon\xi$ where ξ belongs to the unit disk at the origin $\mathcal{H}_1(0)=\{|\xi|<1\}$ and $\boldsymbol{e}=\xi/|\xi|$. The strain is written

$$
\begin{aligned}
&\frac{\boldsymbol{u}^\epsilon(\boldsymbol{x}+\epsilon\xi)-\boldsymbol{u}^\epsilon(\boldsymbol{x})}{\epsilon|\xi|}:=D_{\boldsymbol{e}}^{\epsilon|\xi|}\boldsymbol{u}^\epsilon, \text{ and}\\
&S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))=D_{\boldsymbol{e}}^{\epsilon|\xi|}\boldsymbol{u}^\epsilon\cdot\boldsymbol{e},
\end{aligned}
\tag{50}
$$

and for infinitely differentiable scalar valued functions φ and vector valued functions $\boldsymbol{w}$ with compact support in D we have

$$
\lim_{\epsilon\rightarrow 0} D_{-\boldsymbol{e}}^{\epsilon|\xi|}\varphi=-\nabla\varphi\cdot\boldsymbol{e},
\tag{51}
$$

and

$$
\lim_{\epsilon\rightarrow 0} D_{\boldsymbol{e}}^{\epsilon|\xi|}\boldsymbol{w}\cdot\boldsymbol{e}=\mathcal{E}\boldsymbol{w}\,\boldsymbol{e}\cdot\boldsymbol{e}
\tag{52}
$$

where the convergence is uniform in D. We now recall $S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^\epsilon(t))^-=D_{\boldsymbol{e}}^{\epsilon|\xi|}\boldsymbol{u}^\epsilon\cdot\boldsymbol{e}^-$ defined by (33). We extend $D_{\boldsymbol{e}}^{\epsilon|\xi|}\boldsymbol{u}^\epsilon\cdot\boldsymbol{e}^-$ by zero when $\boldsymbol{x}\in D$ and $\boldsymbol{x}+\epsilon\xi\notin D$ and

$$
\begin{aligned}
&\frac{1}{{\epsilon_n}^2\omega_2}\int_D\int_{\mathcal{H}_\epsilon(\boldsymbol{x})\cap D}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)|S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^-|^2 d\boldsymbol{y}\,d\boldsymbol{x}\\
&=\int_{D\times\mathcal{H}_1(0)}|\xi|J(|\xi|)|(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n}\cdot\boldsymbol{e})^-|^2\,d\xi\,d\boldsymbol{x}.
\end{aligned}
\tag{53}
$$

Then as in inequality (6.73) of [16] we have that

$$
\int_{D\times\mathcal{H}_1(0)}|\xi|J(|\xi|)|(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n}\cdot\boldsymbol{e})^-|^2\,d\xi\,d\boldsymbol{x}<C,
\tag{54}
$$

for all $\epsilon>0$. From this we can conclude there exists a function $g(\boldsymbol{x},\xi)$ such that a subsequence $D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n}\cdot\boldsymbol{e}^-\rightharpoonup g(\boldsymbol{x},\xi)$ converges weakly in $L^2(D\times\mathcal{H}_1(\boldsymbol{x}),\mathbb{R}^2)$ where the L^2 norm and inner product are with respect to the weighted measure $|\xi|J(|\xi|)d\xi d\boldsymbol{x}$. Now for any positive number η and and any subset D' compactly contained in D we can argue as in ([16] proof of lemma 6.6) that $g(\boldsymbol{x},\xi)=\mathcal{E}\boldsymbol{u}^0\boldsymbol{e}\cdot\boldsymbol{e}$ for all points in D' with $|x_2|>\eta$. Since D' and η is arbitrary we get that

$$
g(\boldsymbol{x},\xi)=\mathcal{E}\boldsymbol{u}^0\boldsymbol{e}\cdot\boldsymbol{e}
\tag{55}
$$

almost everywhere in D. Additionally for any smooth scalar test function $\varphi(\boldsymbol{x})$ with compact support in D straight forward computation gives

$$
\begin{aligned}
&\lim_{\epsilon_n\rightarrow 0}\int_{D\times\mathcal{H}_1(0)}|\xi|J(|\xi|)D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n}\cdot\boldsymbol{e}^-\,d\xi\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&=\int_{D\times\mathcal{H}_1(0)}|\xi|J(|\xi|)g(\boldsymbol{x},\xi)\,d\xi\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&=\int_{D\times\mathcal{H}_1(0)}|\xi|J(|\xi|)\mathcal{E}\boldsymbol{u}^0(\boldsymbol{x})\boldsymbol{e}\cdot\boldsymbol{e}\,d\xi\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&=C\int_D div\boldsymbol{u}^0(\boldsymbol{x})\varphi(\boldsymbol{x})d\boldsymbol{x},
\end{aligned}
\tag{56}
$$

Here $C = \omega_2 \int_0^1 r^2 dr$ and we have used

$$\frac{1}{\omega_2} \int_{\mathcal{H}_1(0)} |\xi| J(|\xi|) \boldsymbol{e}_i e_j \, d\xi = \delta_{ij} \int_0^1 r^2 J(r) \, dr. \tag{57}$$

On the other hand for any smooth test function φ with compact support in D we can integrate by parts and use (51) to write

$$\begin{aligned}
&\lim_{\epsilon_n \to 0} \int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) D_{\boldsymbol{e}}^{\epsilon_n |\xi|} \boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e} \varphi(\boldsymbol{x}) \, d\xi \, d\boldsymbol{x} \\
&= \lim_{\epsilon_n \to 0} \int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) D_{-\boldsymbol{e}}^{\epsilon_n |\xi|} \varphi(\boldsymbol{x}) \boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e}, d\xi \, d\boldsymbol{x} \\
&= -\int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) \boldsymbol{u}^0 \cdot \boldsymbol{e} \, \nabla \varphi(\boldsymbol{x}) \cdot \boldsymbol{e} \, d\xi \, d\boldsymbol{x} \\
&= -C \int_D \boldsymbol{u}^0 \cdot \nabla \varphi(\boldsymbol{x}) \, d\boldsymbol{x} \\
&= C \int_D tr E \boldsymbol{u}^0 \varphi(\boldsymbol{x}) \, d\boldsymbol{x},
\end{aligned} \tag{58}$$

where $E\boldsymbol{u}^0$ is the strain of the SBD^2 limit displacement $\boldsymbol{u}^0$. Now since $\boldsymbol{u}^0$ is in SBD its weak derivitave satisfies (30) and it follows on choosing $\Phi_{ij} = \delta_{ij} \varphi$ that

$$\int_D tr E \boldsymbol{u}^0 \varphi \, d\boldsymbol{x} = \int_D div \boldsymbol{u}^0 \varphi \, d\boldsymbol{x} + \int_{\mathcal{J}_{\boldsymbol{u}^0(t)}} [\boldsymbol{u}^0] \cdot \boldsymbol{n} \, \varphi \mathcal{H}^1(\boldsymbol{x}), \tag{59}$$

and note further that

$$\begin{aligned}
&\int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) D_{\boldsymbol{e}}^{\epsilon_n |\xi|} \boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e} \, d\xi \varphi(\boldsymbol{x}) \, d\boldsymbol{x} \\
&= \int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) (D_{\boldsymbol{e}}^{\epsilon_n |\xi|} \boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e})^- d\xi \varphi(\boldsymbol{x}) \, d\boldsymbol{x} \\
&+ \int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) (D_{\boldsymbol{e}}^{\epsilon_n |\xi|} \boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e})^+ d\xi \varphi(\boldsymbol{x}) \, d\boldsymbol{x}
\end{aligned} \tag{60}$$

to conclude

$$\begin{aligned}
&\lim_{\epsilon_n \to 0} \int_{D \times \mathcal{H}_1(0)} |\xi| J(|\xi|) (D_{\boldsymbol{e}}^{\epsilon_n |\xi|} \boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e})^+ d\xi \varphi(\boldsymbol{x}) \, d\boldsymbol{x} \\
&= C \int_{\mathcal{J}_{\boldsymbol{u}^0(t)}} [\boldsymbol{u}^0] \cdot \boldsymbol{n} \, \varphi \mathcal{H}^1(\boldsymbol{x}).
\end{aligned} \tag{61}$$

On changing variables we obtain the identities:

$$\begin{aligned}
&\lim_{\epsilon_n \to 0} \frac{1}{\epsilon_n^2} \int_D \int_{\mathcal{H}_{\epsilon_n}(\boldsymbol{x})} \frac{|\boldsymbol{y} - \boldsymbol{x}|}{\epsilon_n} J^{\epsilon_n}(|\boldsymbol{y} - \boldsymbol{x}|) S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^{\epsilon_n}(t))^+ \, d\boldsymbol{y} \, \varphi(\boldsymbol{x}) \, d\boldsymbol{x} \\
&= C \int_{\mathcal{J}_{\boldsymbol{u}^0(t)}} [\boldsymbol{u}^0] \cdot \boldsymbol{n} \, \varphi \mathcal{H}^1(\boldsymbol{x}).
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
&\lim_{\epsilon_n\to 0}\frac{1}{\epsilon_n^2}\int_D\int_{\mathcal{H}_{\epsilon_n}(\boldsymbol{x})}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^-\,d\boldsymbol{y}\,\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&=C\int_D div\boldsymbol{u}^0(\boldsymbol{x})\varphi(\boldsymbol{x})d\boldsymbol{x}.
\end{aligned}\tag{63}$$

Since $S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^+\neq 0$ for $\boldsymbol{x}$ in SZ^{ϵ_n} and zero otherwise the jump set lies on the $x_2=0$ axis and lemma 3 is proved.

We now prove theorem 2. The claim on the positivity of the jump across the crack (40) follows easily from the one sided Lesbegue limits (28) and the odd symmetry of $u_2^0(x_1,x_2)$ given by theorem 3. Next we use lemma 3 together with hypothesis 1 to get (39). Fix t and recall that the crack centerline is $C^{\epsilon_n}(t)=\{\ell(0)\leq x_1\leq \ell^{\epsilon_n}(t),\,x_2=0\}$. The sequence of numbers $\{\ell^{\epsilon_n}(t)\}_{n=1}^\infty$ is bounded so there exists at least one limit point and call it $\ell^0(t)$. Then there are at most two possibilities: a non-decreasing subsequence converging to $\ell^0(t)$ or a non-increasing subsequence converging to $\ell^0(t)$. Then for either possibility if a positive smooth test function φ has support set that only intersects $\{\ell^0(t)<x_1<a;x_2=0\}$ on a set of nonzero one dimensional Lesbegue measure, then from hypothesis 1 we have

$$\lim_{\epsilon_n\to 0}\frac{1}{\epsilon^2\omega_2}\int_{SZ^{\epsilon_n}}\int_{\mathcal{H}_{\epsilon_n}(\boldsymbol{x})}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^+\,d\boldsymbol{y}\,\varphi(\boldsymbol{x})\,d\boldsymbol{x}=0,\tag{64}$$

so from lemma 3 one concludes

$$\int_{\mathcal{J}_{\boldsymbol{u}^0(t)}}[\boldsymbol{u}^0]\cdot\boldsymbol{n}\,\varphi\mathcal{H}^1(\boldsymbol{x})=0.\tag{65}$$

Since this is true for all tests with support intersecting $\ell^0(t)<x_1<a$ we conclude $|\mathcal{J}_{\boldsymbol{u}^0(t)}\setminus C^0(t)|=0$.

Then for $\varphi\geq 0$ with support on $C^0(t)$ and applying hypothesis 1 and 3 we see that

$$\begin{aligned}
&\lim_{\epsilon_n\to 0}\frac{1}{\epsilon_n^2\omega_2}\int_{SZ^{\epsilon_n}}\int_{\mathcal{H}_{\epsilon_n}(\boldsymbol{x})}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^+\,d\boldsymbol{y}\,\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&\geq\lim_{\epsilon_n\to 0}\frac{1}{\epsilon_n^2\omega_2}\int_{H^{\epsilon_n}}\int_{\mathcal{H}_{\epsilon_n}(\boldsymbol{x})}\frac{|\boldsymbol{y}-\boldsymbol{x}|}{\epsilon_n}J^{\epsilon_n}(|\boldsymbol{y}-\boldsymbol{x}|)S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t))^+\,d\boldsymbol{y}\,\varphi(\boldsymbol{x})\,d\boldsymbol{x}\\
&\geq\lim_{\epsilon_n\to 0}\frac{\alpha}{\epsilon_n}\int_{H^{\epsilon_n}}\varphi(x_1,x_2)\,dx_1dx_2=\lim_{\epsilon_n\to 0}\alpha\int_0^{\ell^{\epsilon_n}(t)-\epsilon_n}\int_{-1/2}^{1/2}\varphi(x_1,\epsilon_n z)\,dx_1dz\\
&=\alpha\int_0^{\ell^0(t)}\varphi(x_1,0)\,dx_1.
\end{aligned}\tag{66}$$

Since (66) holds for all nonnegative test functions we see from (49) that (40) holds and $|C^0(t)\setminus\mathcal{J}_{\boldsymbol{u}^0(t)}|=0$ follows. It also follows that the whole sequence$\{\ell^{\epsilon_n}(t)\}$ converges to $\ell^0(t)$ and theorem 2 is proved.

We conclude by proving theorem 3. The sequence $\{\boldsymbol{u}^\epsilon\}_{\epsilon>0}$ converges in $L^2(D,\mathbb{R}^2)$ to $\boldsymbol{u}^0$ so we may pass to a subsequence that converges almost everywhere to $\boldsymbol{u}^0$. Since the subsequence u_1^ϵ is even with respect to $x_2=0$ we discover that u_1^0 is also even, a.e. Similarly since the subsequence u_2^ϵ is odd we find that u_2^0 is odd a.e. and the theorem is established.

7 Convergence of nonlocal evolution to classic brittle fracture models

The convergence of the elastic displacement field, velocity field and acceleration field are described in terms of suitable Hilbert space topologies. The space of strongly measurable functions $\boldsymbol{w} : [0,T] \to \dot{L}^2(D;\mathbb{R}^2)$ that are square integrable in time is denoted by $L^2(0,T;\dot{L}^2(D;\mathbb{R}^2))$. Additionally we recall the Sobolev space $H^1(D;\mathbb{R}^2)$ with norm

$$\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)} := \left(\int_D |\boldsymbol{w}|^2\, d\boldsymbol{x} + \int_D |\nabla \boldsymbol{w}|^2\, d\boldsymbol{x}\right)^2. \tag{67}$$

The subspace of $H^1(D;\mathbb{R}^2)$ containing all vector fields orthogonal to the rigid motions with respect to the $L^2(D;\mathbb{R}^2)$ inner product is written

$$\dot{H}^1(D;\mathbb{R}^2). \tag{68}$$

The Hilbert space dual to $\dot{H}^1(D;\mathbb{R}^2)$ is denoted by $\dot{H}^1(D;\mathbb{R}^2)'$. The set of functions strongly square integrable in time taking values in $\dot{H}^1(D;\mathbb{R}^2)'$ for $0 \leq t \leq T$ is denoted by $L^2(0,T;\dot{H}^1(D;\mathbb{R}^2)')$. These Hilbert spaces are well known and related to the wave equation, see [6].

We write $\boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)$ as $\boldsymbol{b}^{\epsilon_n}(t)$ and the sequence $\{\boldsymbol{b}^{\epsilon_n}(t)\}$ is uniformly bounded as linear functionals on $\dot{H}^1(D,\mathbb{R}^2)$ and converge to a traction defined on the boundary of D. This is stated in the following lemma.

Lemma 4 *There is a positive constant C independent of ϵ_n and $t \in [0,T]$ such that*

$$|\langle \boldsymbol{b}^{\epsilon_n}(t), \boldsymbol{w}\rangle| \leq C\|\boldsymbol{w}\|_{H^1(D,\mathbb{R})}, \text{ for all } \epsilon_n > 0 \text{ and } \boldsymbol{w} \in \dot{H}^1(D,\mathbb{R}^2), \tag{69}$$

where $\langle \cdot,\cdot\rangle$ is the duality paring between $\dot{H}^1(D,\mathbb{R}^2)$ and its Hilbert space dual $\dot{H}^1(D,\mathbb{R}^2)'$. Then there exists $\boldsymbol{b}^0(t)$ such that $\boldsymbol{b}^{\epsilon_n} \rightharpoonup \boldsymbol{b}^0$ in $L^2(0,T;\dot{H}^1(D;\mathbb{R}^2)')$ and

$$\langle \boldsymbol{b}^0(t), \boldsymbol{w}\rangle = \int_{\partial D} \boldsymbol{g}(t)\cdot \boldsymbol{w}\, ds, \tag{70}$$

for all $\boldsymbol{w} \in \dot{H}^1(D,\mathbb{R}^2)$, where $\boldsymbol{g}(t)$ is defined by (10) *and $\boldsymbol{g} \in H^{-1/2}(\partial D)^2$.*

For ease of exposition we defer the proof of lemma 4 as well as proofs of all other lemmas introduced here to the end of this section.

Passing to subsequences as necessary we obtain the convergence of the elastic displacement field, velocity field, and acceleration field given by

Lemma 5

$$\begin{aligned}
&\boldsymbol{u}^{\epsilon_n} \to \boldsymbol{u}^0 \text{ strong in } C([0,T];\dot{L}^2(D;\mathbb{R}^2))\\
&\dot{\boldsymbol{u}}^{\epsilon_n} \rightharpoonup \dot{\boldsymbol{u}}^0 \text{ weakly in } L^2(0,T;\dot{L}^2(D;\mathbb{R}^2))\\
&\ddot{\boldsymbol{u}}^{\epsilon_n} \rightharpoonup \ddot{\boldsymbol{u}}^0 \text{ weakly in } L^2(0,T;\dot{H}^1(D;\mathbb{R}^2)').
\end{aligned} \tag{71}$$

In order to derive theorem 5 the following variational identities over properly chosen test spaces are introduced. The first variational identity over the domain D is given in the following theorem.

Lemma 6 *For a.e. $t \in [0,T]$ we have*

$$\rho\langle \ddot{\boldsymbol{u}}^0, \boldsymbol{w}\rangle = -\int_D \mathbb{C}\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\, dx + \int_{\partial D} \boldsymbol{g}\cdot\boldsymbol{w}\, ds, \text{ for all } \boldsymbol{w} \in \dot{H}^1(D,\mathbb{R}^2), \tag{72}$$

where $\langle\cdot,\cdot\rangle$ is the duality paring between $\dot{H}^1(D,\mathbb{R}^2)$ and its Hilbert space dual $\dot{H}^1(D,\mathbb{R}^2)'$.

The next variational identity applies to the domains $L^{\pm}_{\beta}(t)$ adjacent to the moving crack.

Lemma 7 *The field $\ddot{\boldsymbol{u}}^0(t)$ is in fact a bounded linear functional on the spaces $H^{1,0}(L^{\pm}_{\beta}(t),\mathbb{R}^2)$ for a.e. $t \in [0,T]$ and we have*

$$\begin{aligned}\rho\langle \ddot{\boldsymbol{u}}^0, \boldsymbol{w}\rangle = -\int_{L^{\pm}_{\beta}(t)} \mathbb{C}\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\ d\boldsymbol{x} + \int_{\partial D^{\pm}_{\beta,t}} \boldsymbol{g}\cdot\boldsymbol{w}\, ds,\\ \text{for all } \boldsymbol{w} \in H^{1,0}(L^{\pm}_{\beta}(t),\mathbb{R}^2).\end{aligned} \tag{73}$$

We now prove theorem 5 using lemmas 1 and 2 and the variational identities given above. We may choose test functions $\boldsymbol{w}$ in $H^1_0(D,\mathbb{R}^2) \subset \dot{H}^1(D,\mathbb{R}^2)$ in (72) to see that

$$\rho\ddot{\boldsymbol{u}}^0 = div\left(\mathbb{C}\mathcal{E}\boldsymbol{u}^0\right) \tag{74}$$

as elements of $H^{-1}(D,\mathbb{R}^2)$ and (46) of theorem 5 is established. The traction on ∂D given by (47) now follows immediately from lemma 1 and lemma 6. Similarly the zero traction force on the component of $\partial D^{\pm}_{\beta,t}$ corresponding to the crack faces given by (48) now follows immediately from lemma 2 and lemma 7. This concludes the proof of theorem 5.

Lemmas 1 and 2 will be shown to follow from a generalized trace formula on the boundary of a Lipschitz domain Ω. We call the domain Ω a polygon when it is a Lipschitz domain with smooth curvilinear arcs for edges E_i, $i = 1,\dots M$ connected by vertices. We introduce the Sobolev space defined on Ω given by

$$H^{1,0}(\Omega,\mathbb{R}^2) = \left\{\boldsymbol{w} \in H^1(\Omega,\mathbb{R}^2) \text{ and } \gamma\boldsymbol{w} = 0 \text{ on one of the edges } E_j\right\}, \tag{75}$$

here $H^{1,0}(\Omega,\mathbb{R}^2) \subset \dot{H}^1(\Omega,\mathbb{R}^2)$.

Lemma 8 *Given a domain Ω with Lipschitz boundary and let $\boldsymbol{u}^0$ be an element of $SBV^2(\Omega)$, let $\boldsymbol{f}$ be an element of $\dot{H}^1(\Omega,\mathbb{R}^2)'$, and*

$$div\left(\mathbb{C}\mathcal{E}\boldsymbol{u}^0\right) = \boldsymbol{f} \tag{76}$$

as elements of $H^{-1}(\Omega,\mathbb{R}^2)$. Suppose first that test functions $\boldsymbol{w}$ belong to $\dot{H}^1(\Omega,\mathbb{R}^2)$ and define $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}$ on $\partial\Omega$ by

$$\langle \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}, \gamma\boldsymbol{w}\rangle = \int_\Omega \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\, d\boldsymbol{x} + \langle \boldsymbol{f}, \boldsymbol{w}\rangle \tag{77}$$

for all $\boldsymbol{w}$ in $\dot{H}^1(\Omega,\mathbb{R}^2)$. Then the functional $\langle \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}, \gamma\boldsymbol{w}\rangle$ is uniquely defined for all test functions $\boldsymbol{w}$ in $\dot{H}^1(\Omega,\mathbb{R}^2)$, hence $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}$ belongs to $H^{-1/2}(\partial\Omega)$.

Next suppose Ω is a polygon. Let $\boldsymbol{w}$ belong to $H^{1,0}(\Omega,\mathbb{R}^2)$ and let $\boldsymbol{f}$ be an element of $H^{1,0}(\Omega,\mathbb{R}^2)'$ and let $div\left(\mathbb{C}\mathcal{E}\boldsymbol{u}^0\right)$ and $\boldsymbol{f}$ satisfy (76) as elements of $H^{-1}(\Omega,\mathbb{R}^2)$. Define $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}$ on $\partial\Omega$ by

$$\langle \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}, \gamma\boldsymbol{w}\rangle = \int_\Omega \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\, d\boldsymbol{x} + \langle \boldsymbol{f}, \boldsymbol{w}\rangle \tag{78}$$

for all $\boldsymbol{w}$ in $H^{1,0}(\Omega,\mathbb{R}^2)$. The functional $\langle \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}, \gamma\boldsymbol{w}\rangle$ is uniquely defined for all test functions $\boldsymbol{w}$ in $H^{1,0}(\Omega,\mathbb{R}^2)$, hence $\mathbb{C}\,\mathcal{E}\boldsymbol{u}^0\boldsymbol{n}$ belongs to the dual space $H^{-1/2}_{00}(\partial\Omega)$.

We now prove lemmas 1 and 2. With the hypothesis of lemma 1 we apply lemma 6 with test functions $\boldsymbol{w}$ in $H^1_0(D,\mathbb{R}^2) \subset \dot{H}^1(D,\mathbb{R}^2)$ in (72) to see as before

$$\rho \ddot{\boldsymbol{u}}^0 = div\left(\mathbb{C}\mathcal{E}\boldsymbol{u}^0\right), \tag{79}$$

as elements of $H^{-1}(\Omega,\mathbb{R}^2)$. Then we set $\boldsymbol{f} = \rho\ddot{\boldsymbol{u}}^0$ and lemma 1 follows immediately from the first part of lemma 8. Now we see that the domains $L^{\pm}_{\beta}(t)$ of lemma 2 are polygons. With the hypothesis of lemma 2 we apply lemma 7 with test functions $\boldsymbol{w}$ in $H^1_0(L^{\pm}_{\beta}(t),\mathbb{R}^2) \subset H^{1,0}(L^{\pm}_{\beta}(t),\mathbb{R}^2)$ in (73) to see

$$\rho \ddot{\boldsymbol{u}}^0 = div\left(\mathbb{C}\mathcal{E}\boldsymbol{u}^0\right), \tag{80}$$

as elements of $H^{-1}(L^{\pm}_{\beta}(t),\mathbb{R}^2)$. Then we set we set $\boldsymbol{f} = \rho\ddot{\boldsymbol{u}}^0$ and lemma 2 follows immediately from the second part of lemma 8.

We now prove the lemmas introduced in this section. We begin with the proof of lemma 8 following the approach of [19]. To fix ideas we prove the second part of lemma 8 noting the first part follows identical lines. First note if $\boldsymbol{u}^0$ belongs to $SBD^2(\Omega)$ then $\int_\Omega \mathbb{C}\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\, d\boldsymbol{x}$ as a map from $\boldsymbol{w} \in H^{1,0}(\Omega,\mathbb{R}^2)$ to $\mathbb{R}$ belongs to $H^{1,0}(\Omega,\mathbb{R}^2)'$. Second note that the trace operator mapping $H^{1,0}(\Omega,\mathbb{R}^2)$ to $H^{-1/2}_{00}(\Omega)$ has a continuous right inverse denoted by τ. We define $\tilde{\boldsymbol{g}}$ by

$$\langle \tilde{\boldsymbol{g}}, \boldsymbol{v}\rangle = \int_\Omega \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\tau\boldsymbol{v}\, d\boldsymbol{x} + \langle \boldsymbol{f}, \tau\boldsymbol{v}\rangle \tag{81}$$

for all $\boldsymbol{v}$ in $H^{-1/2}_{00}(\partial\Omega)$ to show

$$\langle \tilde{\boldsymbol{g}}, \gamma\boldsymbol{w}\rangle = \int_\Omega \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\, d\boldsymbol{x} + \langle \boldsymbol{f}, \boldsymbol{w}\rangle \tag{82}$$

for all $\boldsymbol{w}$ in $H^{1,0}(\Omega,\mathbb{R}^2)$. To see this pick $\boldsymbol{w}$ in $H^{1,0}(\Omega,\mathbb{R}^2)$ and set $\boldsymbol{w}_0 = \boldsymbol{w} - \tau\gamma\boldsymbol{w}$ so $\boldsymbol{w}_0$ is in $H^1_0(\Omega,\mathbb{R}^2)$ and from (76) we have

$$-\int_\Omega \mathbb{C}\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}_0\, d\boldsymbol{x} = \langle \boldsymbol{w}_0, \boldsymbol{f}\rangle, \tag{83}$$

so

$$-\int_\Omega \mathbb{C}\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\boldsymbol{w}\, d\boldsymbol{x} + \int_\Omega \mathbb{C}\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\tau\gamma\boldsymbol{w}\, d\boldsymbol{x} = \langle \boldsymbol{w}, \boldsymbol{f}\rangle - \langle \tau\gamma\boldsymbol{w}, \boldsymbol{f}\rangle. \tag{84}$$

Equation (82) follows directly from (84), (81), and manipulation. Now we show that the definition of $\tilde{\boldsymbol{g}}$ given by (81) is unique and independent of the choice of right inverse (lift) τ. Suppose we have $\boldsymbol{g}^*$ defined by the lift τ^* given by

$$\langle \boldsymbol{g}^*, \boldsymbol{v}\rangle = \int_\Omega \mathbb{C}\,\mathcal{E}\boldsymbol{u}^0 : \mathcal{E}\tau^*\boldsymbol{v}\, d\boldsymbol{x} + \langle \boldsymbol{f}, \tau^*\boldsymbol{v}\rangle \tag{85}$$

for all $\boldsymbol{v}$ in $H^{-1/2}_{00}(\partial\Omega)$. From (82) and linearity we get

$$\langle \tilde{\boldsymbol{g}} - \boldsymbol{g}^*, \gamma\boldsymbol{w}\rangle = 0, \tag{86}$$

for all $\boldsymbol{w}$ in $H^{1,0}(\Omega,\mathbb{R}^2)$ and uniqueness follows. We define $\mathbb{C}\mathcal{E}\boldsymbol{u}^0\boldsymbol{n} = \tilde{\boldsymbol{g}}$ and the second part of lemma 8 is proved.

Next we give the proof of lemma 4. First we show that the sequence $\{\boldsymbol{b}^{\epsilon_n}(t)\}$ is uniformly bounded in $H^1(D,\mathbb{R}^2)'$ for $t \in [0,T]$. Let $\chi^{\epsilon_n} = \chi_+^{\epsilon_n} + \chi_-^{\epsilon_n}$ where $\chi_\pm^{\epsilon_n}$ are the indicator functions of the body force layers defined in (9) so recalling (10) we have for any $\boldsymbol{w} \in H^1(D,\mathbb{R}^2)$,

$$\begin{aligned}
&\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x} = \int_D \frac{1}{\epsilon_n}\chi^{\epsilon_n}(\boldsymbol{x})\boldsymbol{g}(x_1,t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x}\\
&= \int_D \frac{1}{\sqrt{\epsilon_n}}\chi^{\epsilon_n}(\boldsymbol{x})\boldsymbol{g}(x_1,t)\cdot\frac{1}{\sqrt{\epsilon_n}}\chi^{\epsilon_n}(\boldsymbol{x})\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x}\\
&\leq \left(\int_D \frac{1}{\epsilon_n}\chi^{\epsilon_n}|\boldsymbol{g}(t)|^2\,d\boldsymbol{x}\right)^{1/2}\left(\int_D \frac{1}{\epsilon_n}\chi^{\epsilon_n}(\boldsymbol{x})|\boldsymbol{w}|^2\,d\boldsymbol{x}\right)^{1/2}\\
&\leq 2\|g_+(t)\|_{L^2(\theta,a-\theta)}I_{\epsilon_n}.
\end{aligned} \tag{87}$$

Here I_{ϵ_n} is given by

$$\begin{aligned}
I_{\epsilon_n} &= \left(\int_D \frac{1}{\epsilon_n}\chi^{\epsilon_n}(\boldsymbol{x})|\boldsymbol{w}|^2\,d\boldsymbol{x}\right)^{1/2}\\
&= \left(\int_0^1\int_\theta^{a-\theta}|\boldsymbol{w}(x_1,\frac{b}{2}+\epsilon_n(y_2-1))|^2 dx_1dy_2\right.\\
&\quad \left.+\int_0^1\int_\theta^{a-\theta}|\boldsymbol{w}(x_1,-\frac{b}{2}+\epsilon_n(1-y_2)|^2 dx_1dy_2\right)^{1/2}
\end{aligned} \tag{88}$$

where the change of variables $x_2 = \pm\frac{b}{2} \mp \epsilon_n \pm \epsilon_n y_2$ has been made. From the change of variable it is evident that the factor I_{ϵ_n} is bounded above by

$$I_{\epsilon_n} \leq \left(\int_0^1\int_{\partial D_{\delta(y)}}|\boldsymbol{w}|^2\,ds\,dy\right)^{1/2} \tag{89}$$

where $D_{\delta(y)} = \{\boldsymbol{x} \in D:\ \mathrm{dist}(\boldsymbol{x},\partial D) > \delta(y)\}$ and $\delta(y) = \epsilon_n(1-y)$. Since the trace operator is a bounded linear transformation between $H^1(D_{\delta(y)},\mathbb{R}^2)$ and $L^2(\partial D_{\delta(y)})^2$ we have

$$\int_{\partial D_{\delta(y)}}|\boldsymbol{w}|^2\,ds \leq C_{\delta(y)}\|\boldsymbol{w}\|^2_{H^1(D_{\delta(y)},\mathbb{R}^2)} \leq C_{\delta(y)}\|\boldsymbol{w}\|^2_{H^1(D,\mathbb{R}^2)}. \tag{90}$$

Additionally $C_{\delta(y)}$ depends only on the Lipschitz constant of the boundary [7] so for the case at hand we see that

$$\sup_{y\in[0,1]}\{C_{\delta(y)}\} < \infty, \tag{91}$$

and from (87), (89), and (91) we conclude that there is a constant C independent of t and ϵ_n such that

$$|\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x}| \leq C\|\boldsymbol{w}\|^2_{H^1(D,\mathbb{R}^2)}, \tag{92}$$

so

$$\sup_{\epsilon_n>0}\int_0^T \|\boldsymbol{b}^{\epsilon_n}(t)\|^2_{H^1(D;\mathbb{R}^2)'}dt < \infty. \tag{93}$$

Thus we can pass to a subsequence also denoted by $\{\boldsymbol{b}^{\epsilon_n}\}_{n=1}^{\infty}$ that converges weakly to $\boldsymbol{b}^0$ in $L^2(0,T;H^1(D;\mathbb{R}^2)')$. Next we identify the weak limit $\boldsymbol{b}^0(t)$ for a dense set of trial fields. Let $\boldsymbol{w} \in C^1(\overline{D},\mathbb{R}^2)$ then a change of variables $x_2 = \pm\frac{b}{2} \mp \epsilon_n \pm \epsilon_n y_2$ gives

$$\begin{aligned}\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x} &= \int_D \frac{1}{\epsilon_n}\chi^{\epsilon_n}(\boldsymbol{x})\boldsymbol{g}(x_1,t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x}\\ &= \int_0^1\int_\theta^{a-\theta} g_+(x_1,t)\boldsymbol{e}^2\cdot\boldsymbol{w}(x_1,\frac{b}{2}+\epsilon_n(y_2-1)\,dx_1\,dy_2 \\ &+ \int_0^1\int_\theta^{a-\theta} g_-(x_1,t)\boldsymbol{e}^2\cdot\boldsymbol{w}(x_1,-\frac{b}{2}+\epsilon_n(1-y_2)\,dx_1\,dy_2.\end{aligned} \tag{94}$$

One passes to the $\epsilon_n \to 0$ limit in (94) applying the uniform continuity of $\boldsymbol{w}$ to obtain

$$\lim_{\epsilon_n\to 0}\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\partial D}\boldsymbol{g}\cdot\boldsymbol{w}\,ds. \tag{95}$$

Lemma 4 now follows noting that $C^1(\overline{D},\mathbb{R}^2)$ is dense in $H^1(D,\mathbb{R}^2)$.

We now establish lemma 5. The strong convergence

$$\boldsymbol{u}^{\epsilon_n} \to \boldsymbol{u}^0 \text{ strong in } C([0,T];\dot{L}^2(D;\mathbb{R}^2)) \tag{96}$$

follows immediately from the same arguments used to establish theorem 5.1 of [16]. The weak convergence

$$\dot{\boldsymbol{u}}^{\epsilon_n} \rightharpoonup \dot{\boldsymbol{u}}^0 \text{ weakly in } L^2(0,T;\dot{L}^2(D;\mathbb{R}^2)) \tag{97}$$

follows noting that theorem 2.2 of [16] shows that

$$\sup_{\epsilon_n>0}\int_0^T \|\dot{\boldsymbol{u}}^{\epsilon_n}(t)\|^2_{L^2(D;\mathbb{R}^2)}dt < \infty. \tag{98}$$

Thus we can pass to a subsequence also denoted by $\{\dot{\boldsymbol{u}}^{\epsilon_n}\}_{n=1}^{\infty}$ that converges weakly to $\dot{\boldsymbol{u}}^0$ in $L^2(0,T;\dot{L}^2(D;\mathbb{R}^2))$.

To prove

$$\ddot{\boldsymbol{u}}^{\epsilon_n} \rightharpoonup \ddot{\boldsymbol{u}}^0 \text{ weakly in } L^2(0,T;\dot{H}^1(D;\mathbb{R}^2)') \tag{99}$$

we must show that

$$\sup_{\epsilon_n>0}\int_0^T \|\ddot{\boldsymbol{u}}^{\epsilon_n}(t)\|^2_{\dot{H}^1(D;\mathbb{R}^2)'}\,dt < \infty, \tag{100}$$

and existence of a weakly converging sequence follows. To do this we consider the strong form of the evolution (15) which is an identity in $\dot{L}^2(D;\mathbb{R}^2)$ for all times t in $[0,T]$. We multiply (15) with a test function $\boldsymbol{w}$ from $\dot{H}^1(D;\mathbb{R}^2)$ and integrate over D.

A straightforward integration by parts gives

$$\begin{aligned}&\int_D \ddot{\boldsymbol{u}}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}\\ &= -\frac{1}{\rho}\int_D\int_{H_{\epsilon_n}(\boldsymbol{x})\cap D}|\boldsymbol{y}-\boldsymbol{x}|\partial_S\mathcal{W}^{\epsilon_n}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t)))S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{w})\,d\boldsymbol{y}d\boldsymbol{x}\\ &+ \frac{1}{\rho}\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x},\end{aligned} \tag{101}$$

and we now estimate the right hand side of (101). For the first term on the righthand side we change variables $\boldsymbol{y} = \boldsymbol{x} + \epsilon\xi$, $|\xi| < 1$, with $d\boldsymbol{y} = \epsilon_n^2 d\xi$ and write out $\partial_S \mathcal{W}^\epsilon(S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u}^\epsilon(t)))$ to get

$$
\begin{aligned}
I = &-\frac{1}{\rho\omega_2} \int_{D\times\mathcal{H}_1(0)} \omega(\boldsymbol{x}, \xi)|\xi| J(|\xi|) h'\left(\epsilon_n|\xi||D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e}|^2\right) \\
&\times 2(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e})(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w} \cdot \boldsymbol{e})\, d\xi\, d\boldsymbol{x},
\end{aligned}
\tag{102}
$$

where $\omega(\boldsymbol{x}, \xi)$ is unity if $\boldsymbol{x} + \epsilon\xi$ is in D and zero otherwise. We define the sets

$$
\begin{aligned}
A_{\epsilon_n}^- &= \left\{(\boldsymbol{x}, \xi) \text{ in } D \times \mathcal{H}_1(0);\ |D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e}| < \frac{r^c}{\sqrt{\epsilon_n|\xi|}}\right\} \\
A_{\epsilon_n}^+ &= \left\{(\boldsymbol{x}, \xi) \text{ in } D \times \mathcal{H}_1(0);\ |D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e}| \geq \frac{r^c}{\sqrt{\epsilon_n|\xi|}}\right\},
\end{aligned}
\tag{103}
$$

with $D \times \mathcal{H}_1(0) = A_{\epsilon_n}^- \cup A_{\epsilon_n}^+$ and we write

$$
I = I_1 + I_2, \tag{104}
$$

where

$$
\begin{aligned}
I_1 = &-\frac{1}{\rho\omega_2} \int_{D\times\mathcal{H}_1(0)\cap A_{\epsilon_n}^-} \omega(\boldsymbol{x}, \xi)|\xi| J(|\xi|) h'\left(\epsilon_n|\xi||D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e}|^2\right) \\
&\qquad \times 2(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e})(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w} \cdot \boldsymbol{e})\, d\xi\, d\boldsymbol{x}, \\
I_2 = &-\frac{1}{\rho\omega_2} \int_{D\times\mathcal{H}_1(0)\cap A_{\epsilon_n}^+} \omega(\boldsymbol{x}, \xi)|\xi| J(|\xi|) h'\left(\epsilon_n|\xi||D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e}|^2\right) \\
&\qquad \times 2(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{u}^{\epsilon_n} \cdot \boldsymbol{e})(D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w} \cdot \boldsymbol{e})\, d\xi\, d\boldsymbol{x},
\end{aligned}
\tag{105}
$$

In what follows we introduce the the generic constant $C > 0$ that is independent of $\boldsymbol{u}^{\epsilon_n}$ and $\boldsymbol{w} \in \dot{H}^1(D; \mathbb{R}^2)$. First note that h is concave so $h'(r)$ is monotone decreasing for $r \geq 0$ and from Cauchy's inequality, and (54) one has

$$
\begin{aligned}
|I_1| &\leq \frac{2h'(0)C}{\rho\omega_2} \left(\int_{D\times\mathcal{H}_1(0)\cap A_{\epsilon_n}^-} |D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w} \cdot \boldsymbol{e})|^2\, d\xi\, d\boldsymbol{x}\right)^{1/2}, \\
&\leq \frac{2h'(0)C}{\rho\omega_2} \left(\int_{\mathcal{H}_1(0)} \int_D |D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w} \cdot \boldsymbol{e})|^2\, d\boldsymbol{x}\, d\xi\right)^{1/2},
\end{aligned}
\tag{106}
$$

The function $\boldsymbol{w}$ is extended as an H^1 function to a larger domain $\tilde{D}$ containing D such that there is a positive η such that $0 < \eta < dist(D, \tilde{D})$ and $\|\boldsymbol{w}\|_{H^1(\tilde{D};\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)}$. For $\epsilon_n < \eta$ we have

$$
\|D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w} \cdot \boldsymbol{e})\|_{L^2(D;\mathbb{R}^2)} \leq \|\boldsymbol{w}\|_{H^1(\tilde{D};\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)}, \tag{107}
$$

for all $\xi \in \mathcal{H}_1(0)$. to conclude

$$
|I_1| \leq C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)}. \tag{108}
$$

Elementary calculation gives the estimate
(see equation (6.53) of [16])

$$\sup_{0\le x<\infty}|h'(\epsilon_n|\xi|x^2)2x| \le \frac{2h'(\overline{r}^2)\overline{r}}{\sqrt{\epsilon_n|\xi|}}, \tag{109}$$

and we also have (see equation (6.78) of [16])

$$\int_{D\times\mathcal{H}_1(0)\cap A^+_{\epsilon_n}} \omega(\boldsymbol{x},\xi)J(|\xi|)\,d\xi\,d\boldsymbol{x} < C\epsilon_n, \tag{110}$$

so Cauchy's inequality and the inequalities (107), (109), (110) give

$$\begin{aligned}
|I_2| &\le \frac{1}{\rho\omega_2}\int_{D\times\mathcal{H}_1(0)\cap A^+_{\epsilon_n}} \omega(\boldsymbol{x},\xi)|\xi|J(|\xi|)\frac{2h'(\overline{r}^2)\overline{r}}{\sqrt{\epsilon_n|\xi|}}|D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w}\cdot\boldsymbol{e}|\,d\xi\,d\boldsymbol{x},\\
&\le \frac{1}{\rho\omega_2}\left(\int_{D\times\mathcal{H}_1(0)\cap A^+_{\epsilon_n}} \omega(\boldsymbol{x},\xi)|\xi|J(|\xi|)\frac{(2h'(\overline{r}^2)\overline{r})^2}{\epsilon_n|\xi|}\,d\xi\,d\boldsymbol{x}\right)^{1/2}\times\\
&\left(\int_{D\times\mathcal{H}_1(0)\cap A^+_{\epsilon_n}} \omega(\boldsymbol{x},\xi)|\xi|J(|\xi|)|D_{\boldsymbol{e}}^{\epsilon_n|\xi|}\boldsymbol{w}\cdot\boldsymbol{e}|^2\,d\xi\,d\boldsymbol{x}\,dt\right)^{1/2}\\
&\le C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)},
\end{aligned} \tag{111}$$

and we conclude that the first term on the right hand side of (101) admits the estimate

$$|I| \le |I_1| + |I_2| \le C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)}, \tag{112}$$

for all $\boldsymbol{w}\in H^1(D;\mathbb{R}^2)$.

It follows immediately from lemma 4 that the second term on the right hand side of (101) satisfies the estimate

$$\frac{1}{\rho}\left|\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x}\right| \le C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)},\text{ for all }\boldsymbol{w}\in H^1(D;\mathbb{R}^2) \tag{113}$$

From (112) and (113) we conclude that there exists a $C>0$ so that

$$\left|\int_D \ddot{\boldsymbol{u}}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})\,d\boldsymbol{x}\right| \le C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)},\text{ for all }\boldsymbol{w}\in \dot{H}^1(D;\mathbb{R}^2) \tag{114}$$

so

$$\sup_{\epsilon_n>0}\sup_{t\in[0,T]}\frac{\int_D \ddot{\boldsymbol{u}}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}}{\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)}} < C,\text{ for all }\boldsymbol{w}\in\dot{H}^1(D;\mathbb{R}^2), \tag{115}$$

or

$$\sup_{t\in[0,T]}\|\ddot{\boldsymbol{u}}^{\epsilon_n}(t)\|_{H^1(D;\mathbb{R}^2)'} < C,\text{ for all }\epsilon_n \tag{116}$$

and (100) follows. The estimate (100) implies weak compactness and passing to subsequences if necessary we deduce that $\ddot{\boldsymbol{u}}^{\epsilon_n}\rightharpoonup\ddot{\boldsymbol{u}}^0$ weakly in
$L^2(0,T;\dot{H}^1(D;\mathbb{R}^2)')$ and lemma 5 is proved.

To establish lemma 6 we take a test function $\boldsymbol{w}(\boldsymbol{x})$ that is infinitely differentiable on D and orthogonal to rigid body motions in the L^2 inner product. Extending this function as before and multiplying (15) by this test function and integration by parts gives

$$\begin{aligned}&\rho\int_D \ddot{\boldsymbol{u}}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}\\&=-\int_D\int_{\mathcal{H}_{\epsilon_n}(\boldsymbol{x})\cap D}|\boldsymbol{y}-\boldsymbol{x}|\partial_S\mathcal{W}^{\epsilon_n}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t)))S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{w})\,d\boldsymbol{y}d\boldsymbol{x}\\&+\int_D \boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x},\end{aligned}\tag{117}$$

The goal is to pass to the $\epsilon_n=0$ in this equation to recover (72). Using arguments identical to those above we find that for fixed t that on passage to a possible subsequence also denoted by $\{\epsilon_n\}$ one recovers the term on the left hand side of (72), i.e.,

$$\lim_{\epsilon_n\to 0}\rho\int_D \ddot{\boldsymbol{u}}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}=\rho\langle\ddot{\boldsymbol{u}}^0,\boldsymbol{w}\rangle.\tag{118}$$

To recover the $\epsilon_n=0$ limit of the first term on the right hand side of (117) we see that (52) and (55) hold. theorem 6.7 of [15] shows that on passage to a further subsequence if necessary one obtains

$$\begin{aligned}&-\lim_{\epsilon_n\to 0}\int_D\int_{H_{\epsilon_n}(\boldsymbol{x})\cap D}|\boldsymbol{y}-\boldsymbol{x}|\partial_S\mathcal{W}^{\epsilon_n}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t)))S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{w})\,d\boldsymbol{y}d\boldsymbol{x}\\&=-\int_D \mathbb{C}\mathcal{E}\boldsymbol{u}^0:\mathcal{E}\boldsymbol{w}\;d\boldsymbol{x}.\end{aligned}\tag{119}$$

We pass to the limit in the second term on the right hand side of (117) using lemma 4 and the last term on the righthand side of (72) follows.

This shows that (72) holds for all infinitely differentiable test functions and lemma 6 now follows from the density of the test functions in $\dot{H}^1(D,\mathbb{R}^2)$.

To establish lemma 7 we first show that $\ddot{\boldsymbol{u}}^0(t)$ is a bounded linear functional on the spaces $H^{1,0}(L^{\pm}_\beta(t),\mathbb{R}^2)$ for a.e. $t\in[0,T]$. We recall $\ell^{\epsilon_n}(t)\to\ell^0(t)$ and for β such that $\ell(0)\leq\ell^0(t)-\beta$ we only consider ϵ_n so that $\ell^0(t)-\beta/2<\ell^{\epsilon_n}(t)$ and $\epsilon_n<\beta/2$. We make this choice so that the interval $\{\ell(0)\leq x_1<\ell^0(t)-\beta;\,x_2=0\}$ is now included in the softening zone center line $C^{\epsilon_n}(t)$ see Definition 1. We multiply (15) with a test function $\boldsymbol{w}$ from $H^{1,0}(L^{\pm}_\beta(t),\mathbb{R}^2)$ extended to the interior of D_t by zero, integrate over $L^{\pm}_\beta(t)$, apply hypothesis 2, and perform a straight forward integration by parts to get

$$\begin{aligned}&\int_{L^{\pm}_\beta(t)}\ddot{\boldsymbol{u}}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}\\&=-\frac{1}{\rho}\int_{L^{\pm}_\beta(t)}\int_{H_{\epsilon_n}(\boldsymbol{x})\cap L^{\pm}_\beta(t)}|\boldsymbol{y}-\boldsymbol{x}|\partial_S\mathcal{W}^{\epsilon_n}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}^{\epsilon_n}(t)))S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{w})\,d\boldsymbol{y}d\boldsymbol{x}\\&+\int_{L^{\pm}_\beta(t)}\boldsymbol{b}^{\epsilon_n}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x},\end{aligned}\tag{120}$$

We can bound the terms on the righthand side of (120) using the same arguments used to bound the right hand side of (101). The only difference is in the extension of the test function from the polygons $L^+_\beta(t)$ or $L^-_\beta(t)$ to larger polygons. Here, given a fixed $\eta > 0$ for $t \in [0,T]$ the function $\boldsymbol{w}$ is extended as an H^1 function on $L^+_\beta(t)$ to the larger polygon $\tilde{L}^+_\beta(t)$ containing $L^+_\beta(t)$ given by $\tilde{L}^+_\beta(t) = \{\boldsymbol{x} \in \mathbb{R}^2; \operatorname{dist}(\boldsymbol{x}, L^+_\beta(t)) < \eta\}$ and $\|\boldsymbol{w}\|_{H^1(\tilde{L}^+_\beta(t),\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{H^1(L^+_\beta(t),\mathbb{R}^2)}$. Similarly for $t \in [0,T]$ the function $\boldsymbol{w}$ is extended as an H^1 function on $L^-_\beta(t)$ to the larger rectangle $\tilde{L}^-_\beta(t)$ containing $L^-_\beta(t)$ given by $\tilde{L}^-_\beta(t) = \{\boldsymbol{x} \in \mathbb{R}^2; \operatorname{dist}(\boldsymbol{x}, L^-_\beta(t)) < \eta\}$ and $\|\boldsymbol{w}\|_{H^1(\tilde{L}^-_\beta(t),\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{H^1(L^-_\beta(t),\mathbb{R}^2)}$. Here we fix η sufficiently small so that $\ell^{\epsilon_n} > \ell^0(t) - \beta + \eta$ for $\epsilon_n < \eta$. The strain now satisfies

$$\|S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{w}(t))\|_{L^2(D;\mathbb{R}^2)} \leq \|\boldsymbol{w}\|_{H^1(\tilde{L}^\pm_\beta,;\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{H^1(D;\mathbb{R}^2)}, \tag{121}$$

for all $\boldsymbol{y} \in \mathcal{H}_\epsilon(\boldsymbol{x})$. On changing variables from $\boldsymbol{y}$ to ξ given by $\boldsymbol{y} = \boldsymbol{x} + \epsilon_n \xi$, $\xi \in \mathcal{H}_1(0)$ the difference quotient satisfies

$$\|D^{\epsilon_n|\xi|}_{\boldsymbol{e}} \boldsymbol{w} \cdot \boldsymbol{e})\|_{L^2(L^\pm_\beta,\mathbb{R}^2)} \leq \|\boldsymbol{w}\|_{H^1(\tilde{L}^\pm_\beta,\mathbb{R}^2)} \leq C\|\boldsymbol{w}\|_{H^1(L^\pm_\beta,\mathbb{R}^2)}, \tag{122}$$

for all $\xi \in \mathcal{H}_1(0)$. We can then proceed as before to find for a.e. $t \in [0,T]$ that

$$\sup_{\epsilon_n > 0} \sup_{t \in [0,T]} \|\ddot{\boldsymbol{u}}^{\epsilon_n}(t)\|^2_{H^1(L^\pm_\beta(t),\mathbb{R}^2)'} < \infty, \tag{123}$$

The estimate (123) implies compactness with respect to weak convergence and passing to subsequences if necessary we deduce that $\ddot{\boldsymbol{u}}^{\epsilon_n}(t) \rightharpoonup \ddot{\boldsymbol{u}}^0(t)$ weakly in $H^1(L^\pm_\beta(t),\mathbb{R}^2)'$ and we see that $\ddot{\boldsymbol{u}}^0(t)$ is a bounded linear functional on the spaces $H^1(L^\pm_\beta(t),\mathbb{R}^2)$ for a.e. $t \in [0,T]$ and theorem 4 follows.

To illustrate the ideas we now recover (73) on $L^+_\beta(t)$. We first consider (120) with infinitely differentiable test functions $\boldsymbol{w}(\boldsymbol{x})$ on $L^+_\beta(t)$ with support sets that do not intersect the boundary component ∂L^+. Passing to subsequences as necessary we recover the limit equation (73) using the same arguments that were used to pass to the limit in (72). lemma 7 now follows using the density of these trial fields in $H^1(L^+_\beta(t),\mathbb{R}^2)$. An identical procedure works for the polygons $L^-_\beta(t)$ and lemma 7 is proved.

Last theorem 6 is established. From lemma 5 the limit displacement $\boldsymbol{u}^0$ belongs to $C([0,T]; \dot{L}^2(D;\mathbb{R}^2))$. We now show $\dot{\boldsymbol{u}}^0(t)$ belongs to $C([0,T]; \dot{H}^1(D;\mathbb{R}^2)')$. From theorem 7 we have that $\boldsymbol{u}^{\epsilon_k}$ belongs to $C^2([0,T]; \dot{L}^2(D;\mathbb{R}^2))$ so for $0 \leq t_1 \leq t_2 \leq T$

$$\dot{\boldsymbol{u}}^{\epsilon_k}(t_2) - \dot{\boldsymbol{u}}^{\epsilon_k}(t_1) = \int_{t_1}^{t_2} \ddot{\boldsymbol{u}}^{\epsilon_k}(t)\,dt, \tag{124}$$

We take the $L^2(D,\mathbb{R}^2)$ inner product of both sides with a test function $\boldsymbol{w}$ in $H^1(D,\mathbb{R}^2)$

$$\int_D (\dot{\boldsymbol{u}}^{\epsilon_k}(t_2) - \dot{\boldsymbol{u}}^{\epsilon_k}(t_1)) \cdot \boldsymbol{w}\,d\boldsymbol{x} = \int_D \int_{t_1}^{t_2} \ddot{\boldsymbol{u}}^{\epsilon_k}(t)\,dt \cdot \boldsymbol{w}\,d\boldsymbol{x}, \tag{125}$$

so

$$|\int_D (\dot{\boldsymbol{u}}^{\epsilon_k}(t_2) - \dot{\boldsymbol{u}}^{\epsilon_k}(t_1)) \cdot \boldsymbol{w}\,d\boldsymbol{x}| \leq \|\int_{t_1}^{t_2} \ddot{\boldsymbol{u}}^{\epsilon_k}(t)\,dt\|_{\dot{H}^1(D;\mathbb{R}^2)'} \|\boldsymbol{w}\|_{H^1(D,\mathbb{R}^2)}. \tag{126}$$

By (116) there is a $K > 0$ independent of ϵ_k such that

$$|\int_D (\dot{\boldsymbol{u}}^{\epsilon_k}(t_2) - \dot{\boldsymbol{u}}^{\epsilon_k}(t_1)) \cdot \boldsymbol{w}\,d\boldsymbol{x}| \leq K|t_2 - t_1| \|\boldsymbol{w}\|_{H^1(D,\mathbb{R}^2)}. \tag{127}$$

Theorem 2.2 of [16] shows that

$$\sup_{0\le t\le T}\sup_n \|\dot{\boldsymbol{u}}^{\epsilon_n}(t)\|_{L^2(D;\mathbb{R}^2)} < \infty, \tag{128}$$

so on passing to subsequences if necessary $\dot{\boldsymbol{u}}^{\epsilon_k}(t) \rightharpoonup \dot{\boldsymbol{u}}^0(t)$ in $L^2(D;\mathbb{R}^2)$ and

$$|\int_D (\dot{\boldsymbol{u}}^0(t_2) - \dot{\boldsymbol{u}}^0(t_1)) \cdot \boldsymbol{w}\, d\boldsymbol{x}| \le K|t_2 - t_1|\|\boldsymbol{w}\|_{H^1(D,\mathbb{R}^2)}. \tag{129}$$

This implies Lipschitz continuity in time for $\dot{\boldsymbol{u}}^0$ in the $H^1(D,\mathbb{R}^2)'$ norm

$$\|\dot{\boldsymbol{u}}^0(t_2) - \dot{\boldsymbol{u}}^0(t_1)\|_{H^1(D,\mathbb{R}^2)'} \le K|t_2 - t_1|, \tag{130}$$

for $0 \le t_1 < t_2 \le T$ and theorem 6 is proved.

8 Conclusions

In this paper we use the double well energy in a nonlocal perydynamic formulation. We provide the global description of the limit dynamics. It is shown that as $\epsilon \to 0$ the displacement solution of the nonlocal model converges in mean square uniformly in time to a displacement $\boldsymbol{u}^0(\boldsymbol{x},t)$ that satisfies the dynamic brittle fracture boundary value problem given by

- Prescribed initial conditions.
- Prescribed inhomogeneous traction boundary conditions.
- Balance of linear momentum as described by the linear elastic wave equation.
- Zero traction on the sides of the evolving crack.

We recover the field equations of the modern dynamic Linear Elastic Fracture Mechanics (LEFM) developed and described in [9], [22], [3], [27].

The kinetic relation governing the crack tip motion should also be obtained from the nonlocal model in the limit of vanishing nonlocality. This is recently demonstrated through formal mathematical arguments relating crack tip velocity to the energy flowing into the crack tip. These calculations and the numerical experiments corroborating them are given in recent work by the authors in [12]. Future work aims to establish this rigorously. For a-priori convergence rates of finite difference and finite element implementations of the nonlocal model treated here see [10], [11].

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