Elasticity of Demand

Consider the demand function $Q_D(P) = 30 - 5P$. For some values of P, the quantity demanded given by $Q = Q_D(P)$ and the total revenue are listed in the following table.

Price (p)	Quantity demanded (Q)	Revenue (R) = PQ
1	25	25
2	20	40
3	15	45
4	10	40

The increase in the price of the commodity results into the decrease in its demand according to the law of demand. However, it does not necessarily mean the increase in the revenue as well. As we see in the table, the revenue is increasing when P increases from 1 to 3, but it decreases when P increases from 3 to 4. The increase or decrease of the revenue depends on what is known as the elasticity of demand. **Elasticityofdemand**(or the price elasticity of demand) is the ratio of the percentage change in the quantity demanded to the percentage change in the price.

Let ΔP be the change in price, and ΔQ be the corresponding change in the quantity demanded.

The percentage change in the price $= \frac{|\Delta P|}{P} \times 100\%$

The percentage change in the quantity $= \frac{|\Delta Q|}{Q} \times 100\%$

So, the elasticity of demand is$E_D = \frac{|\Delta Q|/Q}{|\Delta P|/P} = \frac{|\Delta Q|}{|\Delta P|} \times \frac{P}{Q}$

If P increases, then Q decreases, i.e. if ΔP is positive, ΔQ is negative, and vice versa. So, removal of the absolute value sign yields

$$E_D = -\frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

When $\Delta P \rightarrow 0$, we may write

$$E_D = -\frac{P}{Q}\frac{dQ}{dP}$$

The demand is called **inelastic**, **unitelastic**, or **elastic** according as $\eta_D < 1$, $\eta_D = 1$ or $\eta_D > 1$.

Note 1: For the linear demand function given by $Q = mP + c$ (in the form $y = mx + c$), since $\frac{dQ}{dP} = m$, the elasticity of demand is $-m\frac{P}{Q}$.

Note 2: Arc elasticity of demand:If the increase in the price from P_1 to P_2 results into the decrease in the quantity demanded from Q_1 to Q_2, then the arc elasticity of demand is given by

$$\text{Arc elasticity} = \frac{\dfrac{Q_1 - Q_2}{Q_1 + Q_2}}{\dfrac{P_2 - P_1}{P_1 + P_2}}.$$

Relation between Revenue and Elasticity of Demand

Let P be the price, Q be the quantity demanded, and TR be the total revenue, then $TR = PQ$

Differentiating,

$$\frac{d}{dP}TR = P\frac{dQ}{dP} + Q\frac{dP}{dP}$$

$$= P\frac{dQ}{dP} + Q = \frac{P}{Q}\frac{dQ}{dP}Q + Q = -E_D Q + Q = Q(1 - E_D)$$

Considering $Q > 0$,

$E_D > 1$ implies $\frac{dR}{dP} < 0$,

$E_D < 1$ implies $\frac{dR}{dP} > 0$, and

$E_D = 1$ implies $\frac{dR}{dP} = 0$,

leading to the following conclusions.

- When the demand is elastic, the revenue decreases.
- When the demand is inelastic, the revenue increases.
- When the demand is unit elastic, the revenue neither increases nor decreases.

Income Elasticity of Demand

The ratio of the percentage change in the quantity demanded to the percentage change in the income is known as income elasticity of demand. When income increases, the demand also increases. Let the quantity demanded be increased from Q to Q + ΔQ, as a result of increase in the income from Y to Y + ΔY, then the income elasticity of demand is:

$$E_Y = \frac{\Delta Q}{\Delta Y} \frac{Y}{Q}.$$

When $\Delta Y \to 0$,

$$E_Y = \frac{dQ}{dY} \frac{Y}{Q}.$$

Elasticity of Supply

Let ΔP be the change in price, and ΔQ be the corresponding change in the quantity supplied.

The percentage change in the price $= \frac{|\Delta P|}{P} \times 100\%$.

The percentage change in the quantity $= \frac{|\Delta Q|}{Q} \times 100\%$.

So, the price elasticity of supply is

$$E_S = \frac{|\Delta Q|/Q}{|\Delta P|/P} = \frac{|\Delta Q|}{|\Delta P|} \times \frac{P}{Q}$$

If P increases, then Qalso increases,

$$E_S = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

When $\Delta P \rightarrow 0$, we may write

$$E_S = \frac{P}{Q}\frac{dQ}{dP}$$

WORKED OUT EXAMPLE

EXAMPLE 1

Calculate the elasticity of demand when the demand function is given by $Q = 50 - 4P$. For what values of P is the demand inelastic?

Solution:

We have,

$$Q = 50 - 4P$$

So, $\frac{dQ}{dP} = -4$

Elasticity of demand,

$$E_D = -\frac{P}{Q}\frac{dQ}{dP} = -\frac{P}{Q}(-4) = \frac{4P}{Q} = \frac{4P}{50 - 4P}$$

For the demand to be inelastic,

$$E_D < 1$$

i.e., $\frac{4P}{50 - 4P} < 1$

or, $4P < 50 - 4P$

or, $8P < 50$

or, $P < 50/8 = 25/4$

As $P \geq 0$, the the demand is inelastic for $0 \leq P < 25/4$.

EXAMPLE 2

Given the demand equation $P =- Q^2 - 10Q + 61$, find the price elasticity of demand when P=100. Estimate the percentage change in price needed to increase demand by 10%.

Solution:

We have,

$$P = - Q^2 - 10Q + 156$$

Differentiating w.r.t. Q,

$$\frac{dP}{dQ} = - 2Q - 10$$

When p = 100,

$$5 = - Q^2 - 10Q + 156$$

or, $Q^2 + 10Q - 56 = 0$

or, $(Q + 14)(Q - 4) = 0$

or, $Q = - 14, 4$

As the quantity demanded is not negative, Q = 4

$$\text{Now,} E_D = - \frac{P}{Q}\frac{dQ}{dP} = - \frac{100}{4} \cdot \frac{1}{- 2Q - 10} = - 25 \cdot \frac{1}{- 8 - 10} = \frac{25}{18}.$$

By the definition of elasticity of demand,

$$E_D = \frac{\text{percentage change in the quantity demanded}}{\text{percentage change is the price}}$$

$$\frac{25}{18} = \frac{10\%}{\text{percentage change is the price}}$$

$$\text{Percentage change in price} = \frac{18}{25} \times 10\% = 7.2\%.$$

So, increase the demand by 10%, the price needs to be decreased by 7.2%.

EXAMPLE 3

At the price that maximizes revenue, prove that the elasticity of demand is 1.

Solution:

We know that revenue

$TR = PQ$

where P is the price and Q is the quantity.

$$\frac{d}{dP}TR = P\frac{dQ}{dP} + Q\frac{dP}{dP} = P\frac{dQ}{dP} + Q = \frac{P}{Q}\frac{dQ}{dP}Q + Q = E_D Q + Q = Q(1 - E_D)$$

So, $\frac{d}{dP}TR = 0$ for $E_D = 1$. Since $\frac{d}{dP}TR > 0$ for $E_D < 1$, and $\frac{d}{dP}TR < 0$ for $E_D > 1$, TR increases for $E_D < 1$ and decreases for $E_D > 1$. Hence using the first derivative test, TR is maximum at $E_D = 1$.

Exercise 23.4

1. A certain movie theatre increases its ticket price by 10%, there is a drop in the demand by 2%. Find the elasticity of demand.
1. Find the elasticity of demand in the following cases. Also mention whether the demand is elastic, inelastic, or unit elastic.

   a. $Q = 20 - \frac{P}{3}$ at $P = 12$ b. $P = 50 - 2Q$ at $P = 30$ c. $Q = \frac{5}{P}$ at $P = 1$
2. For the demand function $Q_D(P) = 100 - P$, calculate the price elasticity of demand for the price

   a. 10 b. 50 c. 90

   Also determine whether the demand is elastic, unit elastic or inelastic.
3. For the demand function $Q_D(P) = \sqrt{1200 - 2P}$, find the elasticity of demand when the quantity is 30 and interpret your result.
4. Given the demand function $Q_D(p) = 60 - 4P$, find the values of P for which the demand is (a) elastic, (b) inelastic (c) unit elastic.
5. For the demand function given by $PQ = c$, where c is a positive constant, show that the elasticity of demand is always unit elastic.
6. Show that the price elasticity of demand is constant for the demand function defined by $P = \frac{a}{Q^n}$ where a and n are positive constants.
7. The demand function for a good is given by $Q(1 + P^2) = k$, where k is a constant. Prove that the demand is inelastic for $0 \le P < 1$.
8. If the demand function for a good is given by $Q = a - bP$ $(a, b > 0)$, show that the demand is inelastic for $0 < P < \frac{a}{2b}$.
9. Show that $\frac{MR}{AR} = 1 - \frac{1}{E_D}$ where MR = marginal revenue = $\frac{d}{dQ}TR$ and $AR = \frac{TR}{Q}$.
10. Consider a supply function given by $P = 200 + 5Q$, find the price elasticity of supply at $P = 250$.
11. The demand Q of a person with income Rs. Y is given by the equation $Q = ½ Y+500$. Find the income elasticity of demand, when the income is Rs. 20,000.

Answers:

1. 0.2 2. a 0.25, inelastic, b. 1.5, elastic c. 1, unit elastic
3. a. 1/9 inelastic b. 1 unit elastic c. 9, elastic
4. 1/6; 6% increase in price results into 1% decrease in the quantity demanded& vice-versa 5. P> 7.5, P< 7.5, P = 7.5
11. 5 12. $\frac{20}{21}$

Exercise 23.4

1. A certain movie theatre increases its ticket price by 10%, there is a drop in the demand by 2%. Find the elasticity of demand.

Solution:

Here, % change in price = 10%

% change in demand = 2%

elasticity of demand (e_d) or η = ?

Since, $e_d = \eta = \dfrac{\text{\% change in quantity deamand}}{\text{\% change in price}} = \dfrac{2\%}{10\%} = 0.2$

2. Find the elasticity of demand in the following cases. Also mention whether the demand is elastic, inelastic, or unit elastic.

a. $q = 20 - \frac{p}{3}$ at $p = 12$ b. $p = 50 - 2q$ at $p = 30$ c. $q = \frac{5}{p}$ at $p = 1$

Solution:

a. Here, $q = 20 - \frac{p}{3}$ at $p = 12$

When $p = 12$, $q = 20 - \frac{12}{3} = 20 - 4 = 16$ and $\frac{dq}{dp} = -\frac{1}{3}$

we have, $\eta = -\frac{p}{q}\frac{dq}{dp} = -\frac{12}{16} \times -\frac{1}{3} = \frac{1}{4} = 0.25 < 1$

Since, $\eta = 0.25 < 1$

the elasticity of demand is inelastic.

b. $p = 50 - 2q$ at $p = 30$

when $p = 30$, $30 + 2q = 50$ or, $q = 10$

and $2q = 50 - p$, i.e. $q = 25 - \frac{p}{2} \Rightarrow \frac{dq}{dp} = -\frac{1}{2}$

we have, $\eta = -\frac{p}{q}\frac{dq}{dp} = -\frac{30}{10} \times -\frac{1}{2} = 1.5 > 1$

Since, $\eta = 1.5 > 1$, it is elastic.

c. $q = \frac{5}{p}$ at $p = 1$

when $p = 1$, $q = \frac{5}{1} = 5$

and $\frac{dq}{dp} = -\frac{5}{p^2}$

we have, $\eta = -\frac{p}{q} \times \frac{dq}{dp} = -\frac{p}{q} \times -\frac{5}{p^2} = -\frac{1}{5} \times -\frac{5}{1^2} = 1$

Since, $\eta = 1$, it is unit elastic.

3. For the demand function $Q_D(p) = 100 - p$, calculate the price elasticity of demand for the price

a. 10 b. 50 c. 90

Also determine whether the demand is elastic, unit elastic or inelastic.

Solution:

Here, demand function is $Q_D(p) = 100 - p$

a. For price (p) = 10, price elasticity of demand = ?

when $p = 10$, $Q = 100 - 10 = 90$ and, $\frac{dQ}{dp} = -1$

We have, $\eta = -\frac{dQ}{dp} \times \frac{p}{Q} = -(-1) \times \frac{10}{90} = \frac{1}{9}$

Since, $\eta = \frac{1}{9} < 1$, it is inelastic.

b and c. Similar to 'a'.

4. For the demand function $Q_D(p) = \sqrt{1200 - 2p}$, find the elasticity of demand when the quantity is 30 and interpret your result.

Solution:

Here, demand function is $Q_D(p) = \sqrt{1200 - 2p}$

for quantity (Q) = 30, price elasticity of demand (η) = ?

Now, when Q = 30, then $30^2 = 1200 - 2p$

or, $2p = 300 = 150$

Also, $\frac{dQ}{dp} = \frac{d\sqrt{1200 - 2p}}{dp} = \frac{d(1200 - 2p)^{\frac{1}{2}}}{d(1200 - 2p)} \times \frac{d(1200 - 2p)}{dp}$

$= \frac{1}{2}(1200 - 2p)^{\frac{1}{2} - 1} \times -2 = \frac{-1}{\sqrt{1200 - 2p}}$

we have,

$\eta = -\frac{dQ}{dp} \times \frac{p}{Q} = -\frac{-1}{\sqrt{1200 - 2p}} \times \frac{p}{Q} = \frac{-1}{\sqrt{1200 - 2p \times 150}} \times \frac{150}{30} = \frac{1}{30} \times 5 = \frac{1}{6}$

Since, $\eta = \frac{1}{6} < 1$, it is inelastic.

Interpretation: when price increase (or decreases) by 6%, then quantity demand decreases (or increases) by 1%.

5. Given the demand function $Q_D(p) = 60 - 4p$, find the values of p for which the demand is (a) elastic, (b) inelastic (c) unit elastic.

Solution:

Here, demand function $Q_D(p) = 60 - 4p$

Price (p) = ?

a. If price elasticity of demand (η) is elastic, then n > 1

Now, $\frac{dQ}{dp} = -4$ and $\eta = -\frac{dQ}{dp} \times \frac{p}{Q}$

$$\Rightarrow -\frac{dQ}{dp} \times \frac{p}{Q} > 1 \qquad [\because \eta > 1]$$

$$\text{or, } -(-4) \times \frac{p}{60 - 4p} > 1$$

$$\text{or, } \frac{4p}{4(15 - p)} > 1$$

or, $p > 15 - p$

or, $2p > 15$

i.e. $p > 7.5$

b. and c. Similar ot'a'.

6. For the demand function given by pq = c, where c is a positive constant, show that the elasticity of demand is always unit elastic.

Solution:

Here, demand functionis pq = C, C is positive constant.

We have,

Price elasticity of demand $(\eta) = -\frac{dq}{dp} \times \frac{p}{q}$

$$\therefore \eta = -\frac{-C}{p^2} \times \frac{p}{\frac{C}{p}} = \frac{-C}{p^2} \times \frac{p^2}{C} = 1$$

Hence, the elasticity of demand is unit elastic.

7. Show that the price elasticity of demand is constant for the demand function defined by $p = \frac{a}{q^n}$ where a and n are positive constants.

Solution:

Here, demand function is $p = \frac{a}{q^n}$,

a and n are positive constants.

Now, $\frac{dp}{dq} = \frac{d}{dq} aq^{-n} = -an \,.\, q^{-n-1} = \frac{-an}{q^{1+n}}$

i.e. $\frac{dq}{dp} = \frac{q^{1+n}}{-an}$

Since, $\eta = -\frac{dq}{dp} \times \frac{p}{q}$, So,

$\eta = -\left(-\frac{q^{1+n}}{an}\right) \times \frac{a}{q^n.q} = \frac{q^{1+n}}{n\,q^{1+n}} = \frac{1}{n}$ which is constant.

8. The demand function for a good is given by $q(1 + p^2) = k$, where k is a constant. Prove that the demand is inelastic for $0 \le p < 1$.

Solution:

Here, demand function $q(1 + p^2) = k$,
k is a constant.

Now, $q = \dfrac{k}{1 + p^2} = k\,(1 + p^2)^{-1}$

or, $\dfrac{dq}{dp} = -k\,(1 + p^2)^{-2} \times 2p = \dfrac{-k}{(1 + p^2)^2} \times 2p$

Since, demand elasticity $(\eta) = -\dfrac{dq}{dp} \times \dfrac{p}{q}$

So, $\eta = -\dfrac{-2pk}{(1 + p^2)^2} \times \dfrac{p}{\dfrac{k}{1 + p^2}} = \dfrac{2p^2}{(1 + p^2)^2} \times (1 + p^2) = \dfrac{2p^2}{1 + p^2} < 1$ for all $0 \le p < 1$.

Hence the price elasticity is inelastic for all $0 \le p < 1$.

9. If the demand function for a good is given by $q = a - bp$ $(a, b > 0)$, show that the demand is inelastic for $0 < p < \frac{a}{2b}$

Solution:

Here, demand function is $q = a - bp$, $(a, b > 0)$

Now, $\frac{dq}{dp} = -b$.

we have, demand elasticity $(\eta) = -\frac{dq}{dp} \times \frac{p}{q} = -(-b) \times \frac{p}{a - bp} = \frac{bp}{a - bp}$

To be η inelastic, $bp < a - bp$

or, $2bp < a$

or, $p < \frac{a}{2b}$ & $p \nless 0$ because p is in price unit.

So, η is inelastic for $0 < p < \frac{a}{2b}$

10. Show that $\frac{MR}{AR} = 1 - \frac{1}{\eta}$ where MR = marginal revenue = $\frac{dR}{dq}$ and AR = $\frac{R}{q}$.

Solution:

$$\text{We have, } MR = \frac{dR}{dq} = \frac{d(pa)}{dq} = p\frac{dq}{dq} + q\frac{dp}{dq} = p + \frac{q}{p}\frac{dp}{dq} \cdot p = p - \frac{1p}{E_d} = p\left(1 - \frac{1}{E_d}\right).$$

$$\text{and } AR = \frac{R}{q} = \frac{pq}{q} = p$$

$$\therefore \frac{MR}{AR} = \frac{p\left(1 - \frac{1}{E_d}\right)}{p} = 1 - \frac{1}{E_d} \text{proved.}$$

11. Consider a supply function given by P = 200 + 5Q, find the price elasticity of supply at P = 250.

Solution:

Here, Supply function P = 200 + 5Q

Price elasticity of supply (E_s) = ? when P = 250

Now, If P = 250, then 5Q = 250 – 200 ∵ Q = 10

Since, $E_S = \frac{1}{d}\frac{P}{Q}$ (Where d = 5 as compared P = 200 + 5Q with P = C + dQ

$$\text{or, } E_s = \frac{1}{5} \times \frac{200 + 5Q}{Q} = \frac{40 + Q}{Q} = \frac{40 + 10}{10} = 5$$

12. The demand Q of a person with income Rs. Y is given by the equation Q = ½ Y+500. Find the income elasticity of demand, when the income is Rs. 20,000.

Solution:

Here, demand $Q = \frac{1}{2}y + 500$

For Y = Rs. 20,000, Income elasticity of demand E_y= ?

When Y = Rs. 20,000, then $Q = \frac{1}{2} \times 20000 + 500 = 10500$

Now, $E_y = \frac{\Delta Q}{\Delta Y} \times \frac{Y}{Q} = \frac{1}{2} \times \frac{20000}{10500}$ [$\because$ Slop $\frac{\Delta Q}{\Delta Y} = \frac{1}{2}$) $= \frac{100}{105} = \frac{20}{21}$