Revenue, Cost and Profit Functions

Selling 5 units of something at the rate of Rs 2 per unit yields a revenue of Rs 5 × 2 = Rs 10. We call the revenue earned from q units of goods or services at the price P as the **total revenue** TR, which is given by

$$TR = PQ$$

For a linear demand function represented by $P = a - bQ$ $(a, b > 0)$ the total revenue function is

$$TR = PQ$$
$$= (a - bQ)Q = aQ - bQ^2$$

The curve representing this function is a parabolapassing through the origin, and opens downwards as the coefficient of Q^2 is negative.

The **total cost** function relates the production costs to the level of production. The total cost rises as the level of production rises. However, in a short run, there are some costs (cost of land, equipment, rent, etc.) which remain fixed. The variable costs are directly proportional to the level output. If VC is the variable cost per unit of output, then the **total variable cost (TVC)** corresponding to the level of output Q, is given by

$$TVC = VC \times Q$$

The **total cost(TC)** is the sum of **fixed costs(FC)** and **total variable cost(TVC)**, i.e.

$$TC = FC + (VC)Q$$

The **average cost function (AC)** is obtained by dividing TC by Q. So,

$$AC = \frac{TC}{Q} = \frac{FC + (VC)Q}{Q}$$

$$\text{Thus, } AC = \frac{FC}{Q} + VC$$

The graph of a total cost function is a straight line with slope V and P-intercept FC. The graph of the average cost is the portion of a rectangular hyperbola in the first quadrant.

The difference between total revenue and total cost is the **profit**. The profit function is given by

$$\pi = TR - TC$$

Break-even

Break-even is a situation, when there is neither gain nor loss. So, for a break even to occur,

$$\pi = 0$$

or, $TR - TC = 0$

or, $TR = TC$

i.e. the total revenue equals the total cost.

Note: Some other functions to related to cost and revenue are:

- Average revenue (AR) = $\frac{TR}{Q}$
- Marginal cost function (MC) = $\frac{dC}{dQ}$
- Marginal average cost function (MAC) = $\frac{d}{dQ}AC$
- Marginal revenue function (MR) = $\frac{d}{dQ}TR$

WORKED OUT EXAMPLES

EXAMPLE 1

Suppose that the demand function of a particular good is given by $P = 20 - Q$. If the fixed cost is 25, and the variable cost per unit is 2, find

a. the levels of output which gives a profit of 31.
b. the maximum profit and the value of the output at which it is achieved.

Solution:

We have,

$$P = 20 - Q,\ FC = 25,\ VC = 2$$

The total revenue

$$TR = PQ = (20 - Q)Q = 20Q - Q^2$$

The total cost

$$TC = FC + (VC)Q = 25 + 2Q$$

The profit function is

$$\pi = TR - TC = 20Q - Q^2 - 25 - 2Q = -Q^2 + 18Q - 25$$

For profit (π) to be 31,

$$31 = -Q^2 + 18Q - 25$$

or, $Q^2 - 18Q + 56 = 0$

or, $(Q - 4)(Q - 14) = 0$

$\therefore$ $Q = 4, 14$

Now,

$$\frac{d\pi}{dQ} = -2Q + 18$$

$$\frac{d^2\pi}{dQ^2} = -2 < 0$$

So, the profit is maximum, when

$$\frac{d\pi}{dQ} = 0$$

or, $-2Q + 18 = 0$

or, $2Q = 18$

$\therefore$ $Q = 9$

For $Q = 9$, $\pi = -(9^2) + 18(9) - 25 = -81 + 162 - 25 = 56$.

So,

a. the levels of output for which the profit is 31 are 4 units and 14 units.
b. the maximum profit is 56 for the level of output $Q = 9$.

EXAMPLE 2

If the demand function is given by the equation $2Q + P = 25$ and the average cost function is $AC = \frac{32}{Q} + 5$, find the value of Q for (a) a break even, (b) a loss of 432 units,and (c) maximum profit

Solution:

From the demand equation,

$$P = 25 - 2Q$$

So, $TR = PQ = (25 - 2Q)Q = 25Q - 2Q^2$

As $AC = TC/Q$,

$$TC = Q(AC) = 32 + 5Q$$

The profit function is given by

$$\pi = TR - TC$$
$$= 25Q - 2Q^2 - 32 - 5Q = -2Q^2 + 20Q - 32$$

a. For a break even,

$$\pi = 0$$

or, $-2Q^2 + 20Q - 32 = 0$

or, $Q^2 - 10 + 16 = 0$

or, $(Q - 2)(Q - 8) = 0$

$\therefore$ $Q = 2, 8$

b. For a loss of 432 units,

$\pi = -432$

or, $-2Q^2 + 20Q - 32 = -432$

or, $-2Q^2 + 20Q + 400 = 0$

or, $Q^2 - 10Q - 200 = 0$

or, $(Q - 20)(Q + 10) = 0$

$\therefore$ $Q = 20, -10$

As $Q \geq 0$, $Q \neq -10$

c. To find the maximum profit,

$$\frac{d\pi}{dq} = -4Q + 20$$

$$\frac{d^2\pi}{dq^2} = -4 < 0$$

So, the profit is maximum when

$$\frac{d\pi}{dq} = 0$$

or, $-4Q + 20 = 0$

or, $4Q = 20$

$\therefore$ $Q = 5$

Hence,(a) for a break-even, Q = either 2 or 8, (b) for a loss of 432 units, Q = 20, (c) for maximum profit Q = 5.

EXAMPLE 3

The demand function of a good is $P = 15 - Q$ and the total cost function is $20 + 3Q$. Determine the range of prices for which the firm's profit is at least 15 units.

Solution:

We have

$TC = 20 + 3Q$

$P = 15 - Q$

We know that

$TR = PQ = (15 - Q)Q = 15Q - Q^2$

Profit function

$\pi = TR - TC = 15Q - Q^2 - 20 - 3Q = -Q^2 + 12Q - 20$

For the profit to be at least 15 units,

$\pi \geq 15$

or, $-Q^2 + 12Q - 20 \geq 15$

or, $-Q^2 + 12Q - 35 \geq 0$

or, $Q^2 - 12Q + 35 \leq 0$

$\therefore$ $(Q - 5)(Q - 7) \leq 0$

Since $P = 15 - Q$,

$Q = 15 - p$

So, for the required profit,

$(15 - P - 5)(15 - P - 7) \leq 0$

$(10 - P)(8 - P) \leq 0$

This is possible wheneither $(10 - P \leq 0$ and $8 - P \geq 0)$ or $(10 - P \geq 0$ and $8 - P \leq 0)$

i.e. either $(10 \leq P$ and $8 \geq P)$ or $(10 \geq P$ and $8 \leq P)$

i.e. either $(P \geq 10$ and $P \leq 8)$ or $(P \leq 10$ and $P \geq 8)$

i.e. $P \in \{\} \cup [8, 10]$

i.e. $P \in [8, 10]$

So, for the profit to be at least 15 units, $8 \leq P \leq 10$.

Exercise 23.3

1. If fixed costs are Rs 100 and variable costs are Rs 2 per unit. Express the total cost and average cost as a function of Q. Also draw their graphs.
2. Find the demand function for each of the following revenue functions.
   a. $TR = 40Q - 5Q^2$ b. $TR = 10$
3. Given the following demand functions, express the total revenue as a function of Q, and hence sketch the graph of total revenue.
   a. $P = 3/Q$ b. $P = 5 - 2Q$ c. $P = 5$
4. Given fixed costs 4, variable costs per unit 1, and the demand function $Q_D(P) = 5 - P/2$, express the profit as a function of Q and hence draw its graph.
5. A demand function is given by $P = 50 - 2Q$. Express the total revenue (TR) as a function of Q and graph it. Find the value of Q for which TR is zero. Also find the maximum total revenue. Also find the corresponding price.
6. The total cost, TC, of producing 10 units of a product is Rs. 60 and the total cost of producing 15 units is Rs 85. Assuming that the total cost function is linear, find an expression for TC in terms of Q.
7. If the profit function is given by $\pi = 2(1 - Q)(Q - 15)$, find the values of Q for (a) break-even (b) maximum profit.
8. If the demand function is given by $2Q + P = 25$, and the average cost function is $AC = 32/Q + 5$, find the value of Q for which the firm (a) breaks even, (b) makes a loss of 432 units, (c) maximizes profit.
9. The demand function is given by the equation $aP + bQ = c$, fixed cost is d and variable cost is e per unit. Express each of the following as a function of Q: (a) total revenue, (b) total cost, (c) average cost, (d) profit.
10. A firm's average cost function is given by $AC = 800/Q + 2Q + 18$.
    Find, to the nearest whole number, the value of Q at the lowest point on the graph of AC plotted against Q, in the interval, $0 \le Q \le 30$.
11. Given the total revenue $TR = 5Q$, and total cost $TC = 20 + 3Q$, calculate the total revenue and the total cost at the break-even point. Show the total revenue function, total cost function and the break-even point on a graph.
12. The initial set-up cost for to manufacture an item is Rs. 20,000. If the additional cost for producing each of the item is Rs. 100, and each item can be sold at Rs. 150, find (a) the total cost function, (b) total revenue function, and (c) break-even point (d) the quantity to be produced for the profit of Rs. 5000.

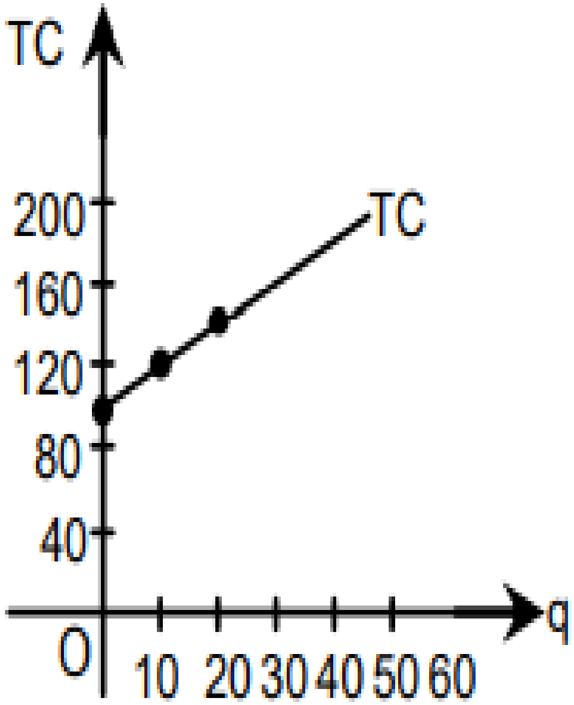
Answers:

1. $100 + 2Q, \frac{100}{q} + 2$
2. $40 - 5Q, \frac{10}{q}$
3. $3, 5Q - 2Q^2, 5Q$
4. $\pi = -2Q^2 + 9Q - 4$
5. $R = 50Q - 2Q^2$, $Q = 0$ or 25, 312.5, 25
6. $TC = 10 + 5Q$
7. a. 1 or 15, b. 8
8. a. 2 or 8, b. 20, c. 5
9. $\frac{CQ}{a} - \frac{bQ^2}{a}$, $d + eQ$, $\frac{d}{Q} + e$, $\frac{c}{a}Q - \frac{b}{a}Q^2 - d - eQ$
10. 20
11. 50
12. a. $TC = 20000 + 100Q$, b. $150Q$, c. $Q = 400$, (d) $Q = 500$

Exercise 23.3

1. If fixed costs are Rs 100 and variable costs are Rs 2 per unit. Express the total cost and average cost as a function of q. Also draw their graphs.

Solution:



Here, fixed cost (FC) = Rs.100, variable cost (VC) = Rs.2 per unit

Now, total cost (TC) = FC + VC = 100 + 2q (q be the number of quantities)

and average cost (AC) = $\frac{TC}{q} = \frac{100 + 29}{q} = \frac{100}{q} + 2, q \neq 0$

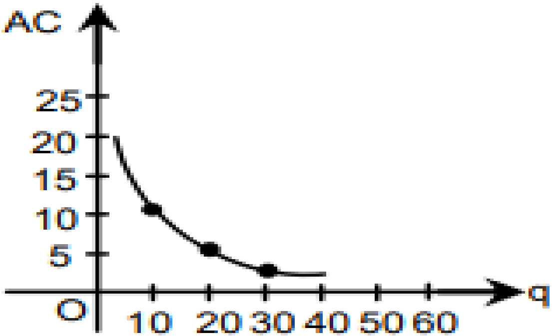
In graph: For TC

When q =	0	10	20
TC =	100	120	140

For AC:

When q =	10	20	40
AC =	12	7	4.5

2. Find the demand function for each of the following revenue functions.
   a. $R = 40q - 5q^2$ b. $R = 10$



Solution:

Here, revenue function is $R = 40q - 5q^2$ and $R = 10$

Now,

a. demand function (p) $= \dfrac{R}{q} = \dfrac{40q - 5q^2}{q} = 40 - 5q.$

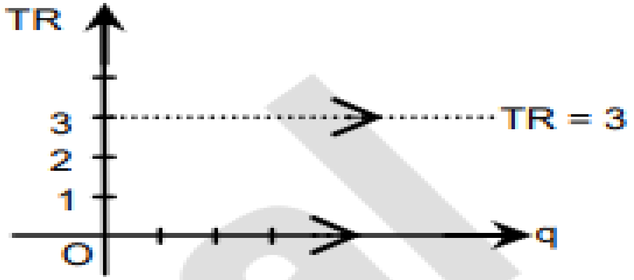
b. demand function (p) $= \dfrac{R}{q} = \dfrac{10}{q}$

3. Given the following demand functions, express the total revenue as a function of q, and hence sketch the graph of total revenue.
   a. $p = 3/q$ b. $p = 5 - 2q$ c. $p = 5$

Solution:

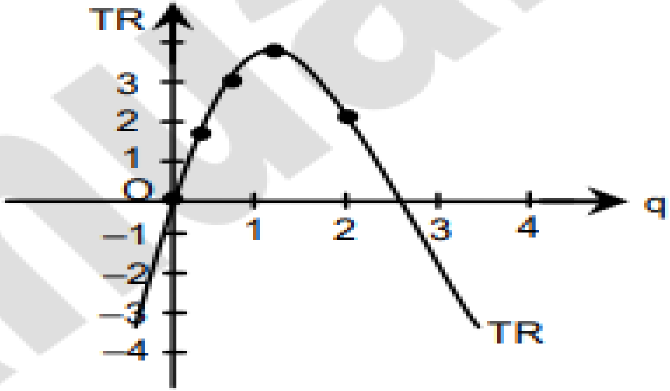
a. Here, demand function (p) $= \dfrac{3}{q}$

Now, total revenue (TR) $= pq = \dfrac{3}{q} \cdot q = 3$



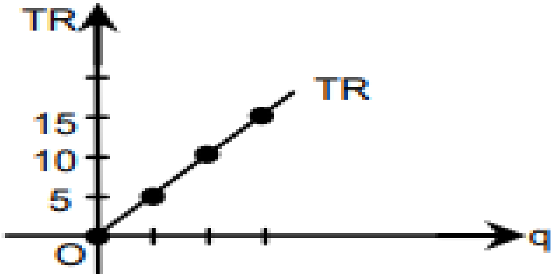
b. Here, demand function (p) $= 5 - 2q$

Now, total revenue (TR) $= pq = (5 - 2q)\, q = 5q - 2q^2$



c. Here, demand function (p) $= 5$

Now, total revenue (TR) $= pq = 5q$



4. Given fixed costs 4, variable costs per unit 1, and the demand function $Q_D(p) = 5 - p/2$, express the profit as a function of q and hence draw its graph.

Solution:

Here, fixed cost (FC) = 4, variable cost (VC) = 1

per unit, demand function $Q_D\ (p) = 5 - \frac{p}{2}$

Now, total cost (TC) = FC + VE = 4 + q [q is the quantity]

and total revenue (TR) = pq where,

$\frac{p}{2} = (10 - 2q)\ q \Rightarrow p = 10q - 2q$

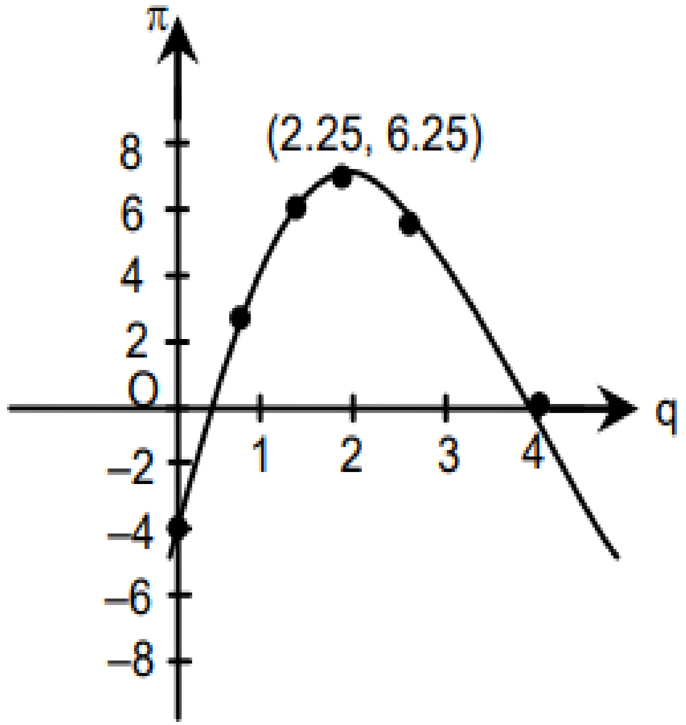
$\therefore$ TR = $(10 - 2q)\ q = 109 - 2q^2$

Since, profit function (π) $= TR - TC = 10q - 2q^2 - 4 - q$

$= -2q^2 + 9q - 4$

In graph:

q:	0	1	2	2.25	3	4
π:	–4	3	6	12.25	5	0



Where, vetrex = $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$

a = –2, b = 9, c = –4

5. A demand function is given by p = 50 – 2q. Express the total revenue (R) as a function of q and graph it. Find the value of q for which R is zero. Also find the maximum total revenue. Also find the corresponding price.

Solution:

Here, demand function is p = 50 – 2q

Now, revenue function (R) = p . q = (50 – 2q) q = $50q - 2q^2$

graph is similar to question no.4

Also, if R = 0, then

$2q^2 - 50q = 0$

or, $q^2 - 25q = 0$

or, q (q – 25) = 0

$\therefore$ q = 0 or 25.

For maximum revenue, $\frac{dR}{dq} = 50 - 4q$ and $\frac{d^2R}{dq^2} = -4 < 0$

Let, $\frac{dR}{dq} = 0$

Then, $50 - 4q = 0 \Rightarrow 4q = 50 \therefore q = 12.5$

So, maximum revenue is $50 \times 12.5 - 2 \times 12.5^2 = 625 - 312.5 = 312.5$

and corresponding price (p) = $\frac{R}{q} = \frac{312.5}{12.5} = 25$

6. The total cost, TC, of producing 10 units of a product is Rs 60 and the total cost of producing 15 units is Rs 85. Assuming that the total cost function is linear, find an expression for TC in terms of q.

Solution:

Let the total cost (TC) which is linear be in the form of TC = a + bQ (being direct relation between cost and quantity, so slope is + ve)

If cost = 60, q = 10

$\therefore a + 10b = 60$ ... (i)

if cost = 85, Q = 15

$\therefore a + 15b = 85$ ... (ii)

Solving equation (i) and (ii), we get

$$
\begin{array}{rcl}
a + 10b &=& 60 \\
a + 15b &=& 85 \\
- \quad - & & - \\
\hline
-5b &=& -25
\end{array}
$$

$\therefore b = 5$

Put b = 5 in (ii), then

$a + 15 \times 5 = 85$ or, $a = 10$

So, TC in terms of Q is 10 + 5Q.

7. If the profit function is given by $\pi = 2(1 - q)(q - 15)$, find the values of q for (a) break-even (b) maximum profit.

Solution:

Here, profit function $\pi = (1 - q)(q - 15) = (16q - 15 - q^2) = -q^2 + 16q - 15$ ... (i)

Which is quadratic in q.

Compare (i) with $\pi = aq^2 + bq + c$, we get

$a = -1$, $b = 16$, $c = -15$

Now,

a. For break even : $\pi = 0$

$\Rightarrow q^2 - 16q + 15 = 0$

or, $q^2 - 15q - q + 15 = 0$

or, $q(q - 15) - 1(q - 15) = 0$

or, $(q - 15)(q - 1) = 0$

$\therefore q = 1$ or 15

b. For maximum profit : $\frac{d\pi}{dq} = -2q + 16$

and $\frac{d^2\pi}{dq^2} = -2 < 0$ (maximum profit)

Let $\frac{d\pi}{dq} = 0$

Then, $-2q + 16 = 0$

or, $2q = 16$

$\therefore q = 8$

Hence, at q = 1 or 15, there is neither profit nor loss and at q = 8 , there is maximum profit.

8. If the demand function is given by $2q + p = 25$, and the average cost function is $AC = 32/q+5$, find the value of q for which the firm (a) breaks even, (b) makes a loss of 432 units, (c) maximizes profit.

Solution:

Here, the demand function is $2q + p = 25$, average cost function is $AC = \dfrac{32}{q} + 5$

a. $p = 25 - 2q$, $TC = AC \times q = \left(\dfrac{32}{q} + 5\right) q = 32 + 5q$

So, $TR = p \,.\, q = (25 - 2q)q = 25q - 2q^2$

For break even, $TR - TC = 0$

or, $25q - 2q^2 - 32 - 5q = 0$

or, $-2q^2 + 20q - 32 = 0$

or, $q^2 - 10q + 16 = 0$

or, $q^2 - 8q - 2q + 16 = 0$

or, $q\,(q - 8) - 2(q - 8) = 0$

or, $(q - 8)\,(q - 2) = 0$

$\therefore$ $Q = 2$ or, 8.

b. When profit = –432, then

$\pi = TR - TC$

or, $-432 = -2q^2 + 20q - 32$

or, $q^2 - 10q - 200 = 0$

or, $q^2 - 20q + 10q - 200 = 0$

or, $q(q - 20) + 10\,(q - 20) = 0$

or, $(q - 20)\,(q + 10) = 0$

$\therefore$ $q = 20$ or, -10

but q must be +ve, so $q = 20$ only.

c. Here, $\pi = TR - TC = 25q - 2q^2 - 32 - 5q = -2q^2 + 20q - 32$

$\therefore \frac{d\pi}{dq} = -4q + 20$ and $\frac{d^2\pi}{dq^2} = -4 < 0$ (profit is maximu)

let $\frac{d\pi}{dq} = 0 \Rightarrow -4q + 20 = 0$

or, $q = 5$

9. The demand function is given by the equation ap + bq = c, fixed costs are d and variable costs are e per unit. Express each of the following as a function of q: (a) total revenue, (b) total cost, (c) average cost, (d) profit.

Solution

a. Total renveu (TR) $= p\,.\,q = \left(\frac{c - bq}{a}\right).\,q = \frac{cq}{a} - \frac{bq^2}{a}$

b. Total cost (TC) $= FC + VC = d + eq$

c. Average cost (AC) $= \frac{TC}{q} = \frac{d + eq}{q} = \frac{d}{q} + e.$

d. Profit $(\pi) = TR - TC = \frac{c}{a}q - \frac{b}{a}q^2 - \frac{d}{q} - e$

10. A firm's average cost function is given by AC = 800/q + 2q + 18
Find, to the nearest whole number, the value of q at the lowest point on the graph of AC plotted against q, in the interval, $0 \le q \le 30$.

Solution:

Here, $\frac{d\,AC}{dq} = \frac{-800}{92} + 2$ and $\frac{d^2AC}{dq^2} = \frac{1600}{q^3}$

Let $\frac{d\,AC}{dq} = 0 \Rightarrow \frac{-800}{q^2} + 2 = 0$

or, $\frac{800}{q^2} = 2$

or, $q^2 = 400$

$\therefore$ q = 20.

Put q = 20 in $\frac{d^2AC}{dq^2}$, we get

$\frac{d^2AC}{dq^2} = \frac{1600}{20^3} = \frac{1}{5} > 0$

So, the average cost (AC) is minimum at q = 20.
i.e. the nearest whole number of q at which the AC is minimum is 20.

11. Similar as above

12. Similar as above