Assignment 2

1.

a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$

$(1-\lambda)^2 - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda = -1, 3$

Eigen vector $\lambda = -1$: $\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x + y = 0$
$4x + 2y = 0$ $\therefore y = -2x$

$\begin{bmatrix} x \\ -2x \end{bmatrix}$ Put $x = 1$ so $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Eigen vector $\lambda = 3$: $\begin{bmatrix} 1-(3) & 1 \\ 4 & 1-(3) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x + y = 0$
$4x - 2y = 0$ $\therefore y = 2x$

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$\lambda = 3$: plug $x = 1$; $y = 2$

Eigen vector general form : $\begin{bmatrix} x \\ 2x \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

Performing row & column operations to reduce matrix and preserving eigen values

so: $3R_1 + R_3 \to R_3$

$C_1 - 3C_3 \to C_1$

then Resultant Matrix: $\begin{bmatrix} -7 & 2 & 2 \\ -4 & 2 & 2 \\ -6 & 0 & 0 \end{bmatrix}$

: $\begin{bmatrix} -7-\lambda & 2 & 2 \\ -4 & 2-\lambda & 2 \\ -6 & 0 & -\lambda \end{bmatrix}$

$(-7-\lambda)(((2-\lambda)\times -\lambda) - 2(4\lambda+12) + 2(6(2-\lambda)) = 0$

$\lambda(\lambda^2 + 5\lambda + 6) = 0$

$\lambda = 0$ or $\lambda = -2, -3$

Eigen vector for $\lambda = 0$: $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x + 2y + 2z = 0$

$2x + 2y + 2z = 0$

$-3x - 6y - 6z = 0$

$x = 0$

$y = -z$

So $\begin{bmatrix} 0 \\ y \\ -y \end{bmatrix}$ Plugging $y = 1$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\lambda = -2$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ -3 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y + 2z = 0$

$2x + 4y + 2z = 0$

$-3x - 6y - 4z = 0$

$x = -2y \quad z = 0$

So $\begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix}$ plugging $y = 1$ $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = -3$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 2 \\ -3 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y + z = 0$

$2x + 5y + 2z = 0$

$-3x - 6y - 3z = 0$

$\therefore x = -z$

$y = 0$

So $\begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$ plugging $z = 1$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vectors are $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

2.

(a)
$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{bmatrix}$$

$$-\lambda\,((2-\lambda)(3-\lambda)) + 2(-4+2\lambda) = 0$$
$$-\lambda\,(6-2\lambda-3\lambda+\lambda^2) - 8 + 4\lambda = 0$$
$$-6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 - 8 + 4\lambda = 0$$
$$6\lambda - 5\lambda^2 + \lambda^3 + 8 - 4\lambda = 0$$
$$\lambda^3 - 5\lambda^2 + 2\lambda + 8 = 0$$
$$(\lambda+1)(\lambda-2)(\lambda-4) = 0$$
$$\lambda = -1, 2, 4$$

$\lambda = -1$
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2z = 0$$
$$3y = 0$$
$$2x + 4z = 0$$

$\therefore\ x = -2z \quad y = 0$

$$\begin{bmatrix} -2z \\ 0 \\ z \end{bmatrix}$$ plugging $z = 1$ $$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$
$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + 2z = 0$

$2x + z = 0$

$x = 0 \quad z = 0 \quad y = 1$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 4$

$$\begin{bmatrix} -4 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x + 2z = 0$

$-2y = 0$

$2x - z = 0$

$z = 2x \quad y = 0$

$\therefore \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} \quad x = 1$ so $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

So spectral decomposition $A = PDP^{-1}$

where P is the orthornormal basis

D is the eigen value matrix

A is symmetric matrix

$\therefore \begin{bmatrix} 0 & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ corresponds to $D_{\text{eigen value}}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

So $\begin{bmatrix} 0 & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ orthonormal basis

b) $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{bmatrix}$

$= \lambda^3 - \lambda^2 - 2\lambda^2 = 0$

$\lambda = 0, 0, 3$

$\lambda = 0$ $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x - y - z = 0$ $\quad x = y + z$

$-x + y + z = 0$

$-x + y + z = 0$

so $\begin{bmatrix} y+z \\ y \\ z \end{bmatrix}$ $\quad y = 0; z = 1$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad y = -1; z = \frac{1}{2}$

$y = 1; z = 1$ $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\quad$ ✓ $\begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1/2 \end{bmatrix}$

$\lambda = 3$ $\begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2x - y - z = 0$

$-x - 2y + z = 0$

$-x + y - 2z = 0$

$\therefore \quad x = -y \quad y = z$

$$\begin{bmatrix} -y \\ y \\ y \end{bmatrix} \quad \text{Put } y=1 \quad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Spectral Decomposition $A = PDP^{-1}$

So $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{\frac{3}{2}}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ Corresponding to EigenValue(D) = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So ~~orthonormal~~ Orthonormal basis is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3/2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\frac{\sqrt{3}}{\sqrt{2}}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}/\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}}$$

3.

$$Av = \lambda v$$

Multiply A^{-1} on both sides

$$A^{-1}Av = A^{-1}\lambda v$$

$$v = \lambda A^{-1} v$$

$$A^{-1}v = \frac{v}{\lambda}$$

$$A^{-1} = \frac{1}{\lambda} v$$

So Eigen value is reciprocal of A matrix Eigen value

Eigen vector, v remains the same

Hence proved

4. $A - \alpha I$

If eigen values of A are $\lambda_1, \lambda_2 \ldots, \lambda_n$

Then eigen values of $A - \alpha I$ are

$(\lambda_1 - \alpha), (\lambda_2 - \alpha), \ldots$

The eigen values will decrease by α

$(A - \alpha I) v = (\lambda - \alpha) v$ → Also $Av = \lambda v$

λ eigen value of A so $(\lambda - \alpha)$ eigen value of $(A - \alpha I)$

$Av = \lambda v$ $\quad$ v is eigen vector λ eigen value

$B = A - \alpha I$

$$
\begin{aligned}
Bv = (A - \alpha I) v &= Av - \alpha I v \\
&= \lambda v - \alpha v \\
&= (\lambda - \alpha) v
\end{aligned}
$$

So Eigen value reduced by α Eigen vector remains same.

5. A & A^T same eigen value different eigen vectors

$\det(A - \lambda I) = 0$ → Taking Transpose of $(A - \lambda I)$ det quantity

$\det(A^T - \lambda I) = 0$

$\det(A - \lambda I)^T = 0$

$= \det(A - \lambda I) = 0$

$A \to \lambda$ eigen value

$A^T \to \lambda$ eigen value

Now $\quad A\vec{v} = \lambda \vec{v}$

$A^T \vec{w} = \lambda \vec{w}$

Assume $\vec{v} = \vec{w}$

then $\lambda \vec{v} = \lambda \vec{w}$

$A\vec{v} = A^T \vec{w}$

$A \neq A^T$ $\vec{v} = \vec{w} = 0$

But this contradicts the fact that eigen vectors are non zero

Hence $\vec{v} \neq \vec{w}$ atleast one eigen vector not in common

6. a) $A^2 = A$

$A^2 - A = 0$

$A(A-I) = 0$

$A = 0$ or $A = I$

Eigen value of nilpotent matrix is 0

Eigen value of identity matrix is 1

Hence Eigen values 0, 1.

b) $A^m = 0$ Nilpotent matrix

λ eigen value x eigen vector

$Ax = \lambda x$

$A^m = 0$

$$
\begin{aligned}
A^m x &= A^{m-1} A x \\
&= \lambda A^{m-1} x \\
&= \lambda^2 A^{m-2} x \\
&= \dots = \lambda^m x
\end{aligned}
$$

$0 = 0x = A^m x = \lambda^m x$

x non zero by def^n as eigen vector is non zero

hence $\lambda^m = 0$ $\simeq$ $\lambda = 0$

Hence only eigen value is 0

c) $Ax = \lambda x \qquad x \neq 0 \quad \lambda \in \mathbb{R}$

$A = A^T$

Take complex conjugate on both sides

$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$

$A\bar{x} = \bar{\lambda}\bar{x} \qquad (A = \bar{A} \quad A \text{ is real})$

Multiply first eqn RHS by $(\bar{x})^T$

$\lambda(\bar{x})^T x = (\bar{x})^T \lambda x$

$= (\bar{x})^T A x$

$= ((\bar{x})^T A) x$

$= (A^T \bar{x})^T x$

$= (A\bar{x})^T x \qquad (A = A^T)$

$= (\bar{\lambda}\bar{x})^T x \qquad (A\bar{x} = \bar{\lambda}\bar{x})$

$= \bar{\lambda}(\bar{x})^T x$

$(\lambda - \bar{\lambda})(\bar{x})^T x = 0$

$x \neq 0 \quad x$ is eigen vector

$(\bar{x})^T x > 0$

So $(\lambda - \bar{\lambda}) = 0 \quad \lambda = \bar{\lambda}$ Eigen values are real

d) $A = -A^T$ (skew symmetric)

$Ax = \lambda x$

Multiply on both sides $(\bar{x})^T$

$(\bar{x})^T A(x) = \lambda(\bar{x})^T x \qquad (x^T\bar{x}) = \|x\|^2$

$= \lambda \|x\|^2$

$(\bar{x})^T A(x) = (Ax)^T \bar{x}$

$A = -A^T$

$= -x^T A \bar{x} \quad \ldots (1)$

$= x^T A^T \bar{x}$

$Ax = \lambda x$ (Take Complex conjugate on both sides)

$A\bar{x} = \bar{\lambda}\bar{x}$

In (1) $= -x^T A \bar{x}$

$= -x^T \bar{\lambda} \bar{x}$ $\quad (x^T\bar{x}) = ||x||^2$

$= -\bar{\lambda} ||x||^2$

So $-\bar{\lambda}||x||^2 = \lambda ||x||^2$

Eigen vector cannot be 0

$\lambda = -\bar{\lambda}$

$\lambda + \bar{\lambda} = 0$

$\lambda = a+ib \quad \bar{\lambda} = a-ib$

$a=0$ so $\lambda = ib$

Hence λ is 0 or λ is purely imaginary