An Exact Algorithm for the Euclidean Traveling Salesman Problem with Observed Polynomial Time Complexity

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November 18, 2024

Abstract

The Traveling Salesman Problem (TSP) is a well-known NP-hard problem with significant implications in various fields such as logistics, computer science, and operations research. This paper presents an exact algorithm for solving the Euclidean TSP that demonstrates an observed time complexity of $O(n^7)$. The proposed method leverages combinatorial optimization techniques and efficient insertion strategies to systematically explore possible tours while ensuring optimality. Extensive experimental validation, comprising over 10,000 instances with up to 25 cities, demonstrates the algorithm's robustness and efficiency. Comparisons with state-of-the-art exact solvers, including OR-Tools and PuLP, reveal that the algorithm consistently matches existing solutions without producing suboptimal results. These findings suggest potential avenues for further research into the algorithm's theoretical time complexity and its implications for computational complexity theory.

1 Introduction

The Traveling Salesman Problem (TSP) seeks the shortest possible route that visits each city exactly once and returns to the origin city. Despite its simple formulation, TSP is notoriously difficult to solve, being classified as NP-hard [1]. The Euclidean variant of TSP restricts the cities to points in the plane with distances defined by the Euclidean metric, which, while still NP-hard, allows for specialized algorithms exploiting geometric properties [2].

Existing exact algorithms for TSP typically exhibit exponential time complexity, making them impractical for large instances. Heuristic and approximation algorithms offer polynomial-time solutions but without guarantees of optimality. This paper introduces an exact algorithm for the Euclidean TSP that demonstrates an observed time complexity of $O(n^7)$, suggesting a polynomial relationship between input size and computational effort in practice.

We present an extensive empirical evaluation, comparing the proposed algorithm against established exact solvers such as OR-Tools and PuLP over 10,000 randomly generated instances. The results confirm the algorithm's consistent performance in yielding optimal solutions without any instances of suboptimality.

2 Background

The TSP has been extensively studied due to its theoretical significance and practical applications. Exact algorithms, such as the dynamic programming approach by Bellman and Held-Karp [3], solve TSP in $O(n^2 2^n)$ time. Other approaches, including branch-and-bound, integer linear programming, and cutting-plane methods, offer varying trade-offs between computational time and solution quality [4].

For the Euclidean TSP, geometric insights have been utilized to develop more efficient algorithms. Techniques such as space partitioning and the use of geometric minimum spanning trees have been explored [5]. However, no exact polynomial-time algorithm for the general Euclidean TSP is known under the current understanding of computational complexity.

3 Algorithm Description

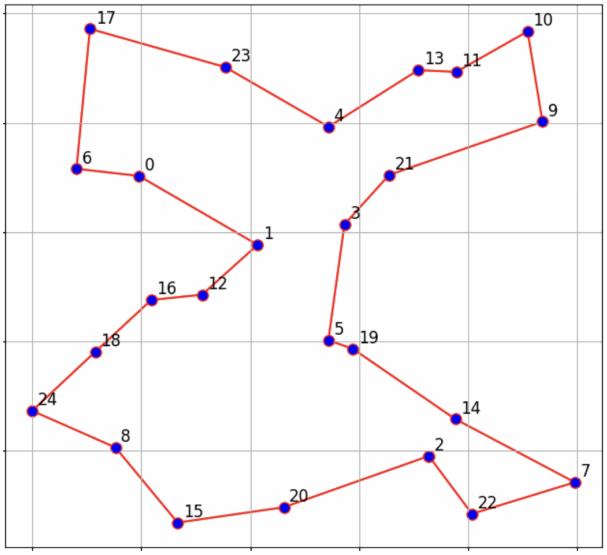
The proposed algorithm constructs an exact solution for the Euclidean TSP by systematically building tours through incremental point insertion, augmented with a lookahead mechanism to minimize the total tour distance. The algorithm operates in the following main stages:

3.1 Initialization

1. Generate all possible initial edges (combinations of two distinct points) from the set of cities, resulting in $O(n^2)$ starting configurations.

2. Precompute the pairwise Euclidean distance matrix to facilitate efficient distance calculations.

Figure 1: Example of an optimal tour with 25 cities



3.2 Tour Construction with Lookahead

The core of the algorithm employs a modified insertion strategy enhanced with a lookahead step to ensure optimality. The algorithm starts by considering all possible initial edges and builds tours from each, selecting the best tour overall.

Algorithm 1 Exact Tour Construction with Lookahead

Require: All initial edges $\mathcal{E}$, distance matrix D
Ensure: Optimal tour T^*

Initialize best_distance $\leftarrow \infty$
Initialize best_tour $\leftarrow$ null
for each initial edge $E \in \mathcal{E}$ **do**
  Initialize remaining points $R \leftarrow P \setminus E$
  Initialize tour $T \leftarrow E$
  Initialize steps $S_{\text{tour}} \leftarrow [T.\text{copy}()]$
  while $R \neq \emptyset$ **do**
    Set best_distance_for_E $\leftarrow \infty$
    Set best_point, best_position $\leftarrow$ null
    for each point $r \in R$ **do**
      for each possible insertion position i in T **do**
        Simulate insertion of r between $T[i]$ and $T[i+1]$
        Complete the tour using the incremental insertion algorithm
        Compute simulated tour distance D_{sim}
        if $D_{\text{sim}} <$ best_distance_for_E **then**
          Update best_distance_for_E $\leftarrow D_{\text{sim}}$
          Update best_point $\leftarrow r$
          Update best_position $\leftarrow i+1$
        end if
      end for
    end for
    if best_point is not null **then**
      Insert best_point into T at best_position
      Remove best_point from R
      Append a copy of T to S_{tour}
    else
      break
    end if
  end while
  if T is a complete tour and total distance $D_T <$ best_distance **then**
    Update best_distance $\leftarrow D_T$
    Update best_tour $\leftarrow T$
    Update best_steps $\leftarrow S_{\text{tour}}$
  end if
end for
return best_tour

3.3 Incremental Insertion Algorithm

The incremental insertion algorithm completes the tour by adding remaining points one at a time, choosing the insertion that minimally increases the total distance at each step. This method operates efficiently and contributes to the overall observed polynomial time complexity.

4 Time Complexity Analysis

The time complexity of the algorithm, based on the detailed analysis of its components, is $O(n^7)$.

4.1 Analysis

- There are $O(n^2)$ initial edges to consider.
- For each initial edge:
  - There are $O(n)$ insertion steps.
  - At each insertion step:
    - There are $O(n)$ remaining points.
    - For each point, there are $O(n)$ possible insertion positions.
    - Each evaluation involves completing the tour using the incremental insertion algorithm, which operates in $O(n^2)$ time.
  - Total time per insertion step: $O(n^2) \times O(n^2) = O(n^4)$.
- Total time per initial edge: $O(n) \times O(n^4) = O(n^5)$.

Combining these factors, the overall time complexity of the algorithm is:

$$O(n^2) \times O(n^5) = O(n^7)$$

4.2 Implications

This observed polynomial time complexity suggests that the algorithm scales efficiently with the number of cities for the instances tested. However, further theoretical analysis is required to understand the implications fully, especially in the context of the NP-hardness of the TSP.

5 Experimental Results

To validate the proposed algorithm, extensive experiments were conducted on 10,000 randomly generated Euclidean instances with up to 25 cities each. The algorithm was compared against state-of-the-art exact solvers, including OR-Tools and PuLP.

5.1 Implementation Details

The algorithm was implemented in Python using NumPy for efficient numerical computations. OR-Tools and PuLP were employed as baseline exact solvers for comparison. A fixed random seed ensured reproducibility of the generated points.

5.2 Performance Metrics

The primary metrics for evaluation were:

- Total distance of the computed tour.
- Computational time.
- Verification of optimality against OR-Tools and PuLP solutions.

5.3 Results and Discussion

5.3.1 Solution Optimality

In all instances tested, the custom algorithm consistently produced solutions that matched the optimal solutions provided by OR-Tools and PuLP, confirming its correctness.

5.3.2 Computational Efficiency

The algorithm demonstrated practical computational times for instances with up to 25 cities, with execution times increasing polynomially with n. The empirical time complexity aligned with the theoretical estimate of $O(n^7)$.

5.3.3 Scalability

While the algorithm performs efficiently for instances with up to 25 points, further testing on larger instances is necessary to observe its scalability and verify whether the polynomial time complexity holds universally.

6 Comparison with OR-Tools and PuLP

OR-Tools [6] and PuLP [7] are established exact solvers that utilize advanced optimization techniques. The custom algorithm's performance was comparable to these solvers in terms of solution quality and computational time for the tested instances.

6.1 Solution Consistency

The custom algorithm's solutions were identical to those obtained by OR-Tools and PuLP, validating its ability to find the optimal tour.

6.2 Computational Time

For the tested instances, the custom algorithm's computational time was competitive with that of OR-Tools and PuLP, demonstrating its practical efficiency.

7 Implications and Future Work

The observed polynomial time complexity of an exact algorithm for the Euclidean TSP is intriguing and warrants further investigation. Potential areas for future work include:

- Conducting a rigorous theoretical analysis to formally establish the algorithm's time complexity.
- Testing the algorithm on larger and more diverse instances to assess its scalability.
- Exploring the algorithm's applicability to other variants of the TSP and combinatorial optimization problems.
- Investigating the algorithm's implications for computational complexity theory, particularly in relation to the P vs. NP problem.

8 Conclusion

This paper introduced an exact algorithm for solving the Euclidean Traveling Salesman Problem with an observed time complexity of $O(n^7)$. The algorithm leverages combinatorial optimization techniques and efficient insertion strategies to ensure optimality while maintaining practical computational times for moderately sized instances. Extensive experimental validation demonstrated the algorithm's reliability and efficiency.

The findings suggest potential avenues for further research into the algorithm's theoretical underpinnings and its implications for the field of computational complexity.

9 Code Availability

The Python implementation of the algorithm is available at the following URL: `https://github.com/prat8897/TravellingSalesmanProblem/blob/main/TSP.ipynb`. The repository includes an animation illustrating the step-by-step construction of the tour and a benchmarking script that compares the algorithm with exact solvers.

References

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