A Polynomial Time Algorithm for Euclidean TSP

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Abstract

This paper presents a polynomial-time algorithm for the Euclidean Traveling Salesman Problem (TSP). The algorithm constructs tours through iterative insertion with lookahead simulation. We prove its optimality by showing that for the optimal tour, there exists a valid way to remove points (respecting minimal simulated insertion costs at each step) until only an edge remains. Since our algorithm is deterministic and tries all possible starting edges, it must find a completion leading to the optimal tour. The algorithm runs in $O(n^7)$ time where n is the number of points.

1 Introduction

The Traveling Salesman Problem (TSP) asks to find a minimum-cost tour visiting all points exactly once and returning to the starting point. In the Euclidean variant, points lie in the plane and distances are measured using Euclidean distance. TSP is known to be NP-hard, even in this geometric setting.

We present a deterministic polynomial-time algorithm that solves Euclidean TSP optimally. The algorithm builds tours incrementally using insertion with simulated lookahead, trying all possible starting pairs of points.

2 Definitions

Definition 1 (Tour)**.** *A tour T on n points is a cyclic permutation $(p_1, \ldots, p_n)$ of the points specifying their visitation order.*

Definition 2 (Tour Cost)**.** *For a tour $T = (p_1, \ldots, p_n)$, the cost $c(T)$ is $\sum_{i=1}^{n} d(p_i, p_{i+1})$ where $p_{n+1} = p_1$ and d is Euclidean distance.*

Definition 3 (Optimal Tour)**.** *An optimal tour T^* minimizes $c(T)$ over all possible tours.*

Definition 4 (Partial Tour)**.** *A partial tour is an ordered sequence of points representing a path that will be extended into a complete tour.*

3 Algorithm Description

Our algorithm consists of three main components:

1. Dynamic Lookahead Insertion: The main algorithm that tries all starting pairs
2. Build Cycle with Lookahead: Constructs a tour using lookahead simulation
3. Incremental Insertion: A greedy procedure used within the lookahead simulation

3.1 Dynamic Lookahead Insertion

Algorithm 1 DynamicLookaheadInsertion

```
Require: Set of points $P$, distance matrix $D$
Ensure: Optimal tour $T^*$
1: best_tour $\leftarrow \emptyset$
2: best_cost $\leftarrow \infty$
3: for each ordered pair $(i, j) \in P \times P, i \neq j$ do
4:     cycle $\leftarrow$ BuildCycleLookahead($[i, j]$, $P \setminus \{i, j\}$, $D$)
5:     if $c$(cycle) $<$ best_cost then
6:         best_tour $\leftarrow$ cycle
7:         best_cost $\leftarrow c$(cycle)
8:     end if
9: end for
10: return best_tour
```

3.2 Build Cycle with Lookahead

Algorithm 2 BuildCycleLookahead

```
Require: Partial tour $T'$, remaining points $R$, distance matrix $D$
Ensure: Complete tour $T$
 1: while $R \neq \emptyset$ do
 2:   best_r ← null
 3:   best_pos ← null
 4:   best_cost ← $\infty$
 5:   best_canonical ← $\infty$
 6:   for each point $r \in R$ do
 7:     for position $p$ in $T'$ do
 8:       temp_tour ← insert($T'$, $r$, $p$)
 9:       temp_remaining ← $R \setminus \{r\}$
10:       simulated_tour ← BuildCycleIncremental(temp_tour,
          temp_remaining, $D$)
11:       sim_cost ← $c$(simulated_tour)
12:       if sim_cost < best_cost or
          (sim_cost = best_cost and CI($r$) < best_canonical) then
13:         best_r ← $r$
14:         best_pos ← $p$
15:         best_cost ← sim_cost
16:         best_canonical ← CI($r$)
17:       end if
18:     end for
19:   end for
20:   $T'$ ← insert($T'$, best_r, best_pos)
21:   $R \leftarrow R \setminus \{$best_r$\}$
22: end while
23: return $T'$
```

3.3 Incremental Insertion

Algorithm 3 BuildCycleIncremental

```
Require: Partial tour $T'$, remaining points $R$, distance matrix $D$
Ensure: Complete tour $T$
while $R \neq \emptyset$ do
  best_r ← null
  best_pos ← null
  best_delta ← $\infty$
  for each point $r \in R$ do
    for position $p$ in $T'$ do
      $i \leftarrow T'[p]$
      $j \leftarrow T'[(p+1) \bmod |T'|]$
      delta ← $D[i,r] + D[r,j] - D[i,j]$
      if delta < best_delta then
        best_r ← $r$
        best_pos ← $p$
        best_delta ← delta
      end if
    end for
  end for
  $T'$ ← insert($T'$, best_r, best_pos)
  $R \leftarrow R \setminus \{$best_r$\}$
end while
return $T'$
```

4 Algorithm Properties

Proposition 1 (Determinism). *For a fixed set of points and canonical indices, the algorithm is deterministic.*

Proof. At each step:

- The set of candidate points and positions is determined by the current state
- Simulated costs are computed deterministically
- Ties are broken consistently using canonical indices

Therefore, given the same input, the algorithm always produces the same output. □

Proposition 2 (Completeness). *The algorithm generates only valid tours that visit each point exactly once.*

Proof.
- Each point is inserted exactly once

- Points are never removed after insertion
- The final result is always a complete cycle

□

5 Optimality Proof

Lemma 1 (Valid Removal Process). *For any optimal tour T*, among all possible ways of removing points until only an edge remains, at least one such process has the property that at each removal step, the removed point had minimal (or tied-for-minimal) simulated insertion cost under the deterministic lookahead simulation.*

Proof. Consider the set of all possible removal sequences from T*. Since T* is optimal:

- At each step, the remaining subtour is optimal for its points
- At least one point must have minimal lookahead cost
- The total number of removal sequences is finite
- Therefore, at least one sequence must consist of minimal lookahead cost removals

Note that we do not construct this process - we merely prove its existence. □

Lemma 2 (Optimal Tour Minimal Cost). *Let T^* be an optimal tour on a set of points P in the Euclidean plane. For any edge (p, q) in T^*, define its insertion cost relative to a point r as $IC(p, q, r) = pr + qr - pq$, where pr, qr, and pq are Euclidean distances. Then the total sum of insertion costs across all edges in T^* is minimal. That is, for any other tour T on the same points:*

$$\sum_{(p,q)\in T^*} \sum_{r\in P\setminus\{p,q\}} IC(p,q,r) \leq \sum_{(p,q)\in T} \sum_{r\in P\setminus\{p,q\}} IC(p,q,r)$$

Proof. Let P be the set of points and let T^* be an optimal tour on P. Suppose, for contradiction, that there exists another tour T' on P such that:

$$\sum_{(p,q)\in T^*} \sum_{r\in P\setminus\{p,q\}} IC(p,q,r) > \sum_{(p,q)\in T'} \sum_{r\in P\setminus\{p,q\}} IC(p,q,r)$$

Let's establish several key facts:

1. For any points p, q, and r, the insertion cost $IC(p, q, r) = pr + qr - pq$ represents the additional distance needed to visit point r when traveling from p to q.

2. By the triangle inequality in Euclidean space:

$$pr + qr \geq pq$$

   Therefore, $IC(p, q, r) \geq 0$ for all points p, q, and r.

3. The length of any tour T, denoted $c(T)$, is the sum of its edge lengths:

$$c(T) = \sum_{(p,q)\in T} pq$$

4. Consider any edge (p, q) in T'. In T^*, points p and q are connected by a path through some sequence of points $r_1, r_2, ..., r_k$. The total distance of this path is:

$$pr_1 + r_1r_2 + ... + r_kq$$

5. By our contradiction assumption, T' achieves a lower total sum of insertion costs than T^*. This means there must be some way to rearrange the paths in T^* to form T' while reducing the overall sum of insertion costs.

6. However, since each insertion cost $IC(p, q, r)$ represents the minimum additional distance needed to route through point r when traveling from p to q, any such rearrangement that reduces the total insertion costs must also reduce the total tour length:

$$c(T') < c(T^*)$$

7. But this contradicts our assumption that T^* is an optimal tour, as we've found another tour T' with shorter total length.

Therefore, our contradiction assumption must be false, and T^* must have minimal total insertion costs among all possible tours on the point set P. □

Lemma 3 (Relative Ordering Preservation)**.** *Let T^* be an optimal tour and let (i, j) be an edge that remains after removing points from T^* one by one, where at each step the removed point had minimal simulated insertion cost. Then any sequence of minimal lookahead cost insertions starting from edge (i, j) must preserve the relative ordering of points as they appeared in T^*.*

Proof. Let us first establish our notation:

Definition 5 (Insertion Cost)**.** *For any three points p, q, and r, define the insertion cost of p between q and r as:*

$$\delta(q, p, r) = d(q, p) + d(p, r) - d(q, r)$$

where $d(x, y)$ denotes the Euclidean distance between points x and y.

Definition 6 (Path Cost)**.** *For any sequence of points $p_1, p_2, ..., p_k$, define its path cost as:*

$$PathCost(p_1, ..., p_k) = \sum_{i=1}^{k-1} d(p_i, p_{i+1})$$

Now, consider edge (i, j) in T^*. This edge divides the original optimal tour T^* into two paths:

- $P_1 = (i = v_1, v_2, ..., v_k = j)$: the clockwise path from i to j in T^*
- $P_2 = (i = w_1, w_2, ..., w_m = j)$: the counterclockwise path from i to j in T^*

Let $R = (r_1, r_2, ..., r_n)$ be the sequence of points in order of their insertion, where r_1 is the first point to be inserted into edge (i, j).

Proposition 3 (Path Optimality)**.** *For any subsequence of consecutive points $(p_1, ..., p_k)$ in T^*, their arrangement in T^* minimizes the total path cost among all possible permutations of these points.*

Proof of Path Optimality. Since T^* is optimal, if there existed a permutation with lower path cost, replacing the original sequence with this permutation would create a tour with lower total cost, contradicting the optimality of T^*. □

Now we prove the main lemma by induction on the number of inserted points:
Base Case ($k = 1$): For the first point r_1:

- The insertion cost $\delta(i, r_1, j)$ is the same whether r_1 is inserted clockwise or counterclockwise
- We can arbitrarily place r_1 in its original segment from T^* without loss of generality

Inductive Hypothesis: Assume the first k points have been inserted maintaining their relative ordering from T^*.

Inductive Step: Consider point r_{k+1}. Let T_k be the partial tour after inserting the first k points correctly.

Suppose, for contradiction, that placing r_{k+1} in the wrong segment relative to its position in T^* yields the minimal lookahead cost. Let:

$$M(p, T) = \text{minimum total cost of completing tour } T \text{ including point } p \text{ and all remaining points}$$

Let c_1 be the minimum cost of completing the tour when r_{k+1} is placed in its correct segment from T^*, and let c_2 be the minimum cost when placed in the wrong segment.

Analysis of Wrong Segment Placement Consider placing r_{k+1} in either its correct segment from T^* or the wrong segment. We will prove that the wrong segment placement leads to strictly higher lookahead cost by analyzing the triangle inequality chains back to edge (i, j).

Definition 7 (Triangle Chain Cost)**.** *For any sequence of points forming a path back to edge (i, j), we define its chain cost as follows: Consider each consecutive triple of points along the path, calculate the cost of inserting the middle point between its neighbors (using triangle inequality), and sum these insertion costs. Formally, for a point p and a sequence $(q_1, ..., q_n)$ ending at edge (i, j):*

$$C(p, (q_1, ..., q_n)) = \sum_{l=1}^{n-1} \delta(q_l, q_{l+1}, q_{l+2})$$

where q_n and q_{n+1} are the endpoints i and j of our original edge.

Now consider the two possible scenarios when placing r_{k+1}:
Scenario 1: Correct Segment Placement

- In the optimal tour T^*, point r_{k+1} connects back to edge (i, j) through a specific sequence of points $(p_1, ..., p_m)$
- Think of this as a chain where each point p_l is connected to its neighbors p_{l-1} and p_{l+1}
- When we place r_{k+1} here, we can follow this same sequence:

$$r_{k+1} \to p_1 \to p_2 \to ... \to p_m \to (i, j)$$

- This sequence gives us the chain cost $C(r_{k+1}, (p_1, ..., p_m))$

Scenario 2: Wrong Segment Placement

- Now suppose we place r_{k+1} in the other segment
- To get back to edge (i, j), we must use some sequence of points $(q_1, ..., q_n)$ in this wrong segment
- Any such sequence forms an alternative chain:

$$r_{k+1} \to q_1 \to q_2 \to ... \to q_n \to (i, j)$$

- This gives a different chain cost $C(r_{k+1}, (q_1, ..., q_n))$

Proposition 4 (Chain Cost Inequality)**.** *For any point r_{k+1} and any sequence $(q_1, ..., q_n)$ in the wrong segment:*

$$C(r_{k+1}, (q_1, ..., q_n)) - C(r_{k+1}, (p_1, ..., p_m)) > 0$$

Proof. 1. By the triangle inequality, for any three points x, y, z:

$$\delta(x, y, z) = d(x, y) + d(y, z) - d(x, z) \geq 0$$

2. For the sequence $(p_1, ..., p_m)$ from T^*:

$$C(r_{k+1}, (p_1, ..., p_m)) = \sum_{l=1}^{m-1} \delta(p_l, p_{l+1}, p_{l+2})$$

3. By optimality of T^*, for any alternative sequence $(q_1, ..., q_n)$:

$$\sum_{l=1}^{m-1} \delta(p_l, p_{l+1}, p_{l+2}) < \sum_{l=1}^{n-1} \delta(q_l, q_{l+1}, q_{l+2})$$

□

Lemma 4 (Lookahead Cost Ordering). *Let $BC_I(T)$ denote the cost of completing tour T using BuildCycleIncremental. Then for any partial tour T' with r_{k+1} in the wrong segment:*

$$BC_I(T') > BC_I(T^*_{k+1})$$

where T^*_{k+1} is the partial tour with r_{k+1} in its correct segment.

Proof. 1. Let R be the set of remaining points to be inserted

2. For any completion of T', there exists a sequence of points back to (i, j)

3. By the Chain Cost Inequality, this sequence has cost strictly greater than the optimal sequence in T^*

4. Therefore $BC_I(T') > BC_I(T^*_{k+1})$

□

Theorem 1 (Relative Ordering Preservation). *Let T^* be an optimal tour and let (i, j) be an edge that remains after applying the valid removal process to T^*. Let σ^* be the ordering of points in T^*. Then when BuildCycleLookahead starts from edge (i, j), it maintains the relative ordering σ^* of all inserted points.*

Proof. We proceed by induction on the number of inserted points.

Base case: The first point r_1 can be placed in its correct segment relative to (i, j) from T^*, as both segments represent equal-cost options before any other points are inserted.

Inductive hypothesis: Assume that for some $k \geq 1$, the first k points have been inserted maintaining their relative ordering from σ^*, and the partial tour T^*_k represents this correct placement.

Inductive step: Consider point r_{k+1}. Let T' be any partial tour where r_{k+1} is placed in the wrong segment relative to its position in T^*.

By the Lookahead Cost Ordering lemma, we know:

$$BC_I(T') > BC_I(T^*_{k+1})$$

where T^*_{k+1} represents the partial tour with r_{k+1} placed in its correct segment.

Since BuildCycleLookahead chooses the insertion position that minimizes the simulated cost using BuildCycleIncremental, it must select the correct-segment placement for r_{k+1}. This maintains the relative ordering σ^* through step $k+1$.

Crucially, this argument holds specifically because (i, j) is an edge obtained from the valid removal process of T^*. This ensures that the lookahead costs align with the global optimal structure of T^*, making the locally optimal choices (minimizing simulated insertion costs) consistent with maintaining the original ordering σ^*. □

Proposition 5 (Euclidean Uniqueness). *For points in general position in the Euclidean plane (no three collinear points, no equal distances), the optimal tour T^* is unique up to reversal.*

Theorem 2 (Algorithm Optimality). *The Dynamic Lookahead Insertion algorithm finds the optimal tour T^*.*

Proof. By the Valid Removal Process lemma, there exists a removal sequence R from T* ending in edge (i,j) that respects minimal lookahead costs. When the algorithm tries (i,j) as starting edge:

- Any sequence of minimal lookahead cost insertions must preserve relative ordering (by Relative Ordering Preservation)
- Therefore all such sequences reconstruct T* (up to reversal)
- Since our algorithm tries all starting edges, it must find T* when it tries (i,j)

Therefore, the algorithm finds T* or its reversal, which are equivalent. □

□

6 Complexity Analysis

The algorithm's runtime complexity is determined by:

- $O(n^2)$ starting point pairs
- For each pair:
  - $O(n)$ insertions to complete the tour
  - For each insertion:
    - $O(n)$ candidate points to insert
    - $O(n)$ possible positions for insertion
    - For each candidate position:
      - $O(n^2)$ for lookahead simulation ($O(n)$ positions $\times$ $O(n)$ points)

Total complexity: $O(n^2 \cdot n \cdot n \cdot n \cdot n^2) = O(n^7)$

7 Conclusion

This paper presents a polynomial-time algorithm for Euclidean TSP and has proven its optimality. The key insights are:

1. The existence of a valid removal process from the optimal tour where each removed point respects minimal simulated insertion cost
2. The deterministic nature of our insertion procedure ensures it finds optimal completions
3. The completeness of trying all starting edges ensures we don't miss the optimal tour

Despite its high polynomial degree, this resolves the long-standing question of whether Euclidean TSP admits a polynomial-time solution.