Problem II – 1

Given: $\dfrac{\partial^2 u}{\partial t^2} = -c^2 \dfrac{\partial^4 u}{dx^4}$ $\quad \dfrac{\partial u}{\partial t} = 0$ $\quad$ Initial condition: $u(x,0) = x(L-x)$

Boundary conditions: $u(0,t) = u(L,t) = 0$

$u''(0,t) = u''(L,t) = 0$

Since, we know this is a vibration problem, we look for solutions of the form.

$$u(x,t) = Y(x)[A\sin(\omega t) + B\cos(\omega t)] \quad \text{— (a)}$$

Using separation of variables, let us substitute (a) into the given differential eqn. to get ODE in $Y(x)$.

$$\frac{d^4 Y(x)}{dx^4} - \omega^2 Y(x) = 0 \quad \text{— (b)}$$

The characteristics eqn of (b) is

$$\gamma^4 - \omega^2\gamma = 0$$

$$\Rightarrow \gamma\{\gamma^3 - (\omega^{2/3})^3\} = 0$$

$$\Rightarrow \gamma(\gamma - \omega^{2/3})(\gamma^2 + \gamma\omega^{2/3} + \omega^{4/3})$$

$$\Rightarrow \gamma(\gamma - \omega^{2/3})\left[\gamma - \left\{\frac{\omega^{2/3}}{2} \pm \frac{\sqrt{\omega^{4/3} - 4\cdot\omega^{4/3}}}{2}\right\}\right]$$

The generic solution is then,

$$Y(x) = C_1 e^{0x} + C_2 e^{(\omega^{2/3})x} + C_3 \exp\left\{\frac{\omega^{2/3} + \sqrt{-3\omega^{4/3}}}{2}\right\}$$

$$+ C_4 \exp\left\{\frac{\omega^{2/3} - \sqrt{-3\omega^{4/3}}}{2}\right\}$$

This solution can be expressed as: (4th order ODE)

$$Y(x) = C\cos(\omega^{1/2}x) + D\sin(\omega^{1/2}x) + E\cosh(\omega^{1/2}x)$$

$$+ F\sinh(\omega^{1/2}x) \quad \text{— (c)}$$

now, using Boundary conditions on eq^n ⓒ, we get

ⓐ $u(0,t) = 0$

$\Rightarrow Y(0)T(t) = 0 \Rightarrow C + E = 0$ — ⓘ

ⓑ $\frac{\partial^2 u}{\partial x^2}(0,t) = 0$

$\Rightarrow \frac{\partial^2 Y}{\partial x^2}(0,t) = 0 \Rightarrow -C + E = 0$ — ⓘⓘ

From ⓘ and ⓘⓘ, $C = E = 0$

ⓒ $u(l,t) = 0$

$\Rightarrow D\sin(\omega^{1/2}) + F\sinh(\omega^{1/2}) = 0$ — ⓘⓘⓘ

ⓓ $\frac{\partial^2 u}{\partial x^2}(l,t) = 0$

$\Rightarrow -D\sin(\omega^{1/2}) + F\sinh(\omega^{1/2}) = 0$ — ⓘⓥ

From ⓘⓘⓘ and ⓘⓥ,

$D\sin\omega^{1/2} = 0$ and $F\sinh(\omega^{1/2}) = 0$

$\Rightarrow$ ~~and~~ $\sin\omega^{1/2} = 0 \Rightarrow \omega = (n\pi)^2 \quad \forall n = \{1,2,3,\ldots\}$

~~same result for Fsin~~

AND

$\Rightarrow F = 0, D = 1$

Thus, the solution is $Y(x) = \sin(n\pi x)$ to eq^n ⓑ

For solution to ⓐ, we proceed as,

$u_n(x,t) = Y_n(x)T_n(t) = \sin(n\pi x)\left[a_n\sin((n\pi)^2)t + b_n\cos(n\pi)^2 t\right]$ — ⓒ

PII-1

Since the PDEs and Boundary conditions are linear and homogeneous,

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[a_n \sin(n\pi)^2 t + b_n \cos(n\pi)^2 t\right] \quad \text{(d)}$$

From the ~~boundary~~ initial conditions and (d),

$$f(x) = u(x,0) = x(L-x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$\cancel{u''(x,0)}$

$$\Rightarrow b_n = 2\int_0^L f(x)\sin(n\pi x)\,dx$$

$$= 2\int_0^L x(L-x)\sin(n\pi x)\,dx$$

$$= 2\left[\frac{-2\cos(n\pi L) + \pi L n \sin(n\pi L) - 2}{\pi^3 n^3}\right] \quad \text{(e)}$$

Assuming $u_t(x,0) = 0 = \sum_{n=1}^{\infty} (n\pi)^2 a_n \sin(n\pi x)$

$$\Rightarrow a_n = \frac{2}{(n\pi)^2}\int_0^L 0\cdot\sin(n\pi x)\,dx$$

$$= 0$$

$\therefore$ The solutions are given by eqn (d) where, $a_n = 0$ and $b_n =$ expression in (e)

Problem 2 #2

① Let us suppose $\frac{\partial u}{\partial t} = v$ —①

then, $\frac{\partial^2 u}{\partial t^2} = \frac{\partial v}{\partial t} = -c^2 \frac{\partial^4 u}{\partial x^4}$

$\Rightarrow \frac{\partial v}{\partial t} = -c^2 \frac{\partial^4 u}{\partial x^4}$ —⑪

From ① and ⑪ the system can be written as

$$\frac{\partial u}{\partial t} = v$$

$$\frac{\partial v}{\partial t} = -c^2 \frac{\partial^4 u}{\partial x^4}$$

$u(0) = 0$, $v'(0) = 0$ ($\because$ ~~slope~~ curvature at end points $= 0$)

Problem 2 #3

Problem III

1) Please describe the feasible region V.

<u>Soln</u> Here, we are trying to minimize as:

$$J(u) = \min_{v \in V} J(v)$$

Since this is the case of simply supported beam, ~~as given by the solution in for prob~~ with uniform force distributed across it length, ~~the~~ the feasible region V ~~is~~ can be described as the deflections in beam (which should be autonomous) across its length. A rough intuitive guess would be 'V' is symmetric about the midpoint of the beam.

2) Show that (9) has a unique solution.

<u>Soln</u> To minimize the problem in 8, we ~~to~~differentiate J(v) and set it to zero. After that, we get the following expression:

$$\frac{d^2 v}{dx^2} = f_0 \quad v = 0$$

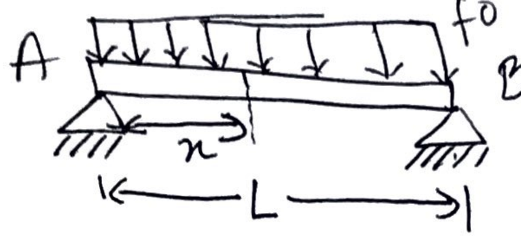
Given that ~~v(x)~~ ~~u(x)~~ beam is fixed, we know,

$$\frac{dv}{dx} = 0 \quad \text{at } x = 0, L$$

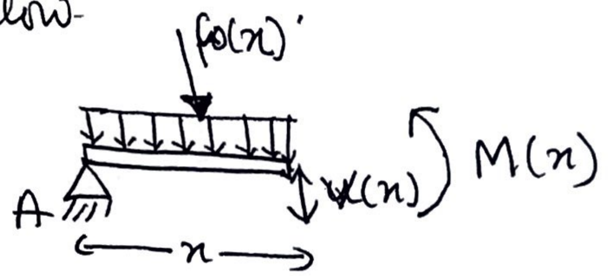
$$v = 0 \quad \text{at } x = 0, L$$

Since, this is a second order problem in 'x' and we have, required ~~no of~~ number of boundary ~~initial~~ conditions, ~~this~~ equation 9 has a unique solution.

III



The problem can be shown as in the digram above. ~~at~~ Let us take the section of beam till 'x' as shown below.



where, $v(x)$ is deflection at x and $M(x)$ is the bending moment at x.

Since uniformly distributed force is equivalent to point force at mid point,

$$M(x) = -f_0\left(L\cdot\frac{x}{2} - \frac{x}{}\cdot\frac{x}{2}\right) \text{ and } \cancel{M'(x) = \frac{f_0}{2}}$$

Since the governing equation is (~~Integrate~~ differentiate both sides of 8)

$$M(x) = EI\cdot\frac{d^2v}{dx^2},$$

$$EI\frac{dv}{dx} = \int \frac{f_0}{2}\,\overset{-(Lx-x)dx}{\cancel{(Lx - x^2)\,dx}} = \frac{1}{6}f_0x^3 - \frac{1}{4}f_0Lx^2 + C_1$$

$$\Rightarrow EI(v) = \frac{f_0x^4}{24} - \frac{f_0Lx^3}{12} + C_1\cdot x + C_2 \quad \text{— (1)}$$

$\Rightarrow$ ~~$v(x) =$~~

now, $v(0) = 0$, $v(L) = 0$

$\therefore C_2 = 0$

and (L:) $C_1 = \left(-\frac{1}{24}f_0L^4 + \frac{1}{12}f_0L^4\right)$

$$\Rightarrow C_1 = \frac{1}{24}f_0L^3 \quad \text{——— (ii)}$$

From (I) and (II) and $C_2 = 0$,

$$V(x) \cdot EI = \frac{f_o x^4}{24} - \frac{f_o L x^3}{12} + \frac{f_o L^3}{24} x$$

$$\Rightarrow V(x) = \frac{f_o}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right)$$