This is an incomplete collection of solutions to exercises from *Topology*, 2nd edition, by James Munkres.

Chapter 2

Topological Spaces and Continuous Functions

§17 Closed Sets and Limit Points

Problem 6

Problem

Let A, B, and A_α denote subsets of a space X. Prove the following:

1. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3. $\overline{\bigcup A_\alpha} \supseteq \bigcup \overline{A_\alpha}$. Give an example where equality fails.

Solution

1. We have $A \subseteq B \subseteq \bar{B}$, and $\bar{B}$ is a closed set. Hence $\bar{A} \subseteq \bar{B}$.
2. Clearly $A \cup B \subseteq \bar{A} \cup \bar{B}$, and the latter is closed, hence $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Conversely, clearly $\bar{A} \subseteq \overline{A \cup B}$ and $\bar{B} \subseteq \overline{A \cup B}$. So $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.
3. Let $Y = \bigcup A_\alpha$. Then for each α, $A_\alpha \subseteq Y$, and so $\overline{A_\alpha} \subseteq \bar{Y}$. So $\bigcup \overline{A_\alpha} \subseteq \bar{Y}$.

   For an example where equality fails, let $X = \mathbb{R}$ with the standard topology. Treating $\mathbb{Q}$ as an index set, let $A_\alpha = \{\alpha\}$ for $\alpha \in \mathbb{Q}$. Then $\overline{\bigcup A_\alpha} = \bar{\mathbb{Q}} = \mathbb{R}$. And $\bigcup \overline{A_\alpha} = \mathbb{Q}$.

Problem 8

Problem

Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\subseteq$ or $\supseteq$ holds.

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
2. $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$
3. $\overline{A \setminus B} = \bar{A} \setminus \bar{B}$

Solution

1. We have $\overline{A \cap B} \subseteq \bar{A}$ and $\overline{A \cap B} \subseteq \bar{B}$, hence $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. The reverse inclusion does not hold, for example let $X = \mathbb{R}$ with the standard topology, $A = (0, 1)$, and $B = (1, 2)$.
2. The same as the above, taking an index set of size 2.
3. We show that $\bar{A} \setminus \bar{B} \subseteq \overline{A \setminus B}$. Let $x \in \bar{A} \setminus \bar{B}$. We show that every neighbourhood of x intersects $A \setminus B$. Fix a neighbourhood U_0 of x such that $U_0 \subseteq X \setminus \bar{B}$. Let U be an arbitrary neighbourhood of x. Then since $x \in \bar{A}$, $U \cap U_0$, which is also a neighbourhood of x, intersects A. Let $y \in U \cap U_0 \cap A$. Then clearly $y \in U \cap (A \setminus B)$, which shows that every neighbourhood of x intersects $A \setminus B$.

   The converse, $\overline{A \setminus B} \subseteq \bar{A} \setminus \bar{B}$, does not hold. For example, let $X = \mathbb{R}$ with the standard topology, $A = [0, 1]$, and $B = (0, 1)$.

Problem 9

Problem

Let $A \subseteq X$ and $B \subseteq Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}$$

Solution

$A \times B$ is a subset of the closed set $\bar{A} \times \bar{B}$, and so $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$. Conversely, let $x \times y \in \bar{A} \times \bar{B}$. Consider any basis neighbourhood of it in $X \times Y$, $U \times V$. Then U intersects A and V intersects B, so $U \times V$ intersects $A \times B$. So $x \times y \in \overline{A \times B}$.

Problem 13

Problem

Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution

Let $\mathcal{T}$ be the toplogy on X and let $\Omega = (X \times X) \setminus \Delta$.

$$
\begin{aligned}
& X \text{ is Hausdorff} \\
\iff & \forall x, y \in X \ (x \neq y \implies \exists U, V \in \mathcal{T}(x \in U, y \in V, U \cap V = \emptyset)) \\
\iff & \forall (x \times y) \in \Omega \ (\exists U, V \in \mathcal{T}(x \in U, y \in V, U \cap V = \emptyset)) \\
\iff & \forall (x \times y) \in \Omega \ (\exists U, V \in \mathcal{T}(x \times y \in U \times V \subseteq \Omega)) \\
\iff & \Omega \text{ is open} \\
\iff & \Delta \text{ is closed}
\end{aligned}
$$

Problem 19

Problem

If $A \subseteq X$, we define the boundary of A by the equation

$$\text{Bd } A = \bar{A} \cap \overline{X \setminus A}$$

1. Show that Int A and Bd A are disjoint, and $\bar{A} = \text{Int } A \cup \text{Bd } A$.
2. Show that Bd $A = \emptyset \iff A$ is both open and closed.
3. Show that U is open $\iff$ Bd $U = \bar{U} \setminus U$.
4. If U is open, is it true that $U = \text{Int } (\bar{U})$? Justify your answer.

Solution

1. $X \setminus A$ is a subset of the closed set $X \setminus \text{Int } A$, hence $\overline{X \setminus A} \subseteq X \setminus \text{Int } A$. Thus $\text{Bd } A \subseteq \overline{X \setminus A} \subseteq X \setminus \text{Int } A$. $\text{Bd } A$ and $\text{Int } A$ are disjoint.

   Let $x \in \bar{A}$. If every neighbourhood of x intersects $X \setminus A$, then $x \in \overline{X \setminus A}$, and hence $x \in \text{Bd } A$. Otherwise, there is some neighbourhood of x which does not intersects $X \setminus A$, that is, it is a subset of A, and so $x \in \text{Int } A$. Thus $\bar{A} = \text{Int } A \cup \text{Bd } A$.

2. If $\text{Bd } A = \emptyset$, then $\bar{A}$ and $\overline{X \setminus A}$ form a partition of X. Thus $A = \bar{A}$ and $X \setminus A = \overline{X \setminus A}$. Both A and $X \setminus A$ are closed.

   Conversely, if A is both open and closed, then $X \setminus A$ is closed as well, and so $\text{Bd } A = A \cap (X \setminus A) = \emptyset$.

3. For all $A \subseteq X$, we know that $\text{Int } A$ and $\text{Bd } A$ form a partition of $\bar{A}$, and hence $\text{Bd } A = \bar{A} \setminus \text{Int } A$. Then, noting that A and $\text{Int } A$ are always subsets of $\bar{A}$,

$$\begin{aligned} & U \text{ is open} \\ \iff & \text{Int } U = U \\ \iff & \bar{U} \setminus \text{Int } U = \bar{U} \setminus U \\ \iff & \text{Bd } U = \bar{U} \setminus U \end{aligned}$$

4. No. Let $X = \mathbb{R}$ with the standard topology, and let $U = (0, 1) \cup (1, 2)$.

§18 Continuous functions

Problem 13

Problem

Let $A \subseteq X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

Solution

Let $g_1, g_2 : \bar{A} \to Y$, which agree on A. We have to show that $g_1 = g_2$. Let

$$h : \bar{A} \to Y \times Y,\ h(x) = g_1(x) \times g_2(x)$$

Consider the diagonal $\Delta = \{y \times y : y \in Y\}$. Since Y is Hausdorff, Δ is closed. As h is continuous, $h^{-1}(\Delta)$ is closed. It includes A, and hence includes $\bar{A}$.