

This is an incomplete collection of solutions to exercises from *Topology*,
2nd edition, by James Munkres.

Chapter 2

Topological Spaces and Continuous Functions

§17 Closed Sets and Limit Points

Problem 6

Problem

Let A , B , and A_α denote subsets of a space X . Prove the following:

1. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3. $\overline{\bigcup A_\alpha} \supseteq \bigcup \bar{A}_\alpha$. Give an example where equality fails.

Solution

1. We have $A \subseteq B \subseteq \bar{B}$, and \bar{B} is a closed set. Hence $\bar{A} \subseteq \bar{B}$.
2. Clearly $A \cup B \subseteq \bar{A} \cup \bar{B}$, and the latter is closed, hence $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Conversely, clearly $\bar{A} \subseteq \overline{A \cup B}$ and $\bar{B} \subseteq \overline{A \cup B}$. So $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.
3. Let $Y = \bigcup A_\alpha$. Then for each α , $A_\alpha \subseteq Y$, and so $\bar{A}_\alpha \subseteq \bar{Y}$. So $\bigcup \bar{A}_\alpha \subseteq \bar{Y}$.

For an example where equality fails, let $X = \mathbb{R}$ with the standard topology. Treating \mathbb{Q} as an index set, let $A_\alpha = \{\alpha\}$ for $\alpha \in \mathbb{Q}$. Then $\bigcup A_\alpha = \mathbb{Q} = \mathbb{R}$. And $\bigcup \bar{A}_\alpha = \mathbb{Q}$.

Problem 8

Problem

Let A , B , and A_α denote subsets of a space X . Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions \subseteq or \supseteq holds.

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
2. $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$
3. $\overline{A \setminus B} = \bar{A} \setminus \bar{B}$

Solution

1. We have $\overline{A \cap B} \subseteq \bar{A}$ and $\overline{A \cap B} \subseteq \bar{B}$, hence $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. The reverse inclusion does not hold, for example let $X = \mathbb{R}$ with the standard topology, $A = (0, 1)$, and $B = (1, 2)$.
2. The same as the above, taking an index set of size 2.
3. We show that $\bar{A} \setminus \bar{B} \subseteq \overline{A \setminus B}$. Let $x \in \bar{A} \setminus \bar{B}$. We show that every neighbourhood of x intersects $A \setminus B$. Fix a neighbourhood U_0 of x such that $U_0 \subseteq X \setminus \bar{B}$. Let U be an arbitrary neighbourhood of x . Then since $x \in \bar{A}$, $U \cap U_0$, which is also a neighbourhood of x , intersects A . Let $y \in U \cap U_0 \cap A$. Then clearly $y \in U \cap (A \setminus B)$, which shows that every neighbourhood of x intersects $A \setminus B$.

The converse, $\overline{A \setminus B} \subseteq \bar{A} \setminus \bar{B}$, does not hold. For example, let $X = \mathbb{R}$ with the standard topology, $A = [0, 1]$, and $B = (0, 1)$.

Problem 9

Problem

Let $A \subseteq X$ and $B \subseteq Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}$$

Solution

$A \times B$ is a subset of the closed set $\bar{A} \times \bar{B}$, and so $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$. Conversely, let $x \times y \in \bar{A} \times \bar{B}$. Consider any basis neighbourhood of it in $X \times Y$, $U \times V$. Then U intersects A and V intersects B , so $U \times V$ intersects $A \times B$. So $x \times y \in \overline{A \times B}$.

Problem 13

Problem

Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution

Let \mathcal{T} be the topology on X and let $\Omega = (X \times X) \setminus \Delta$.

X is Hausdorff

$$\iff \forall x, y \in X (x \neq y \implies \exists U, V \in \mathcal{T} (x \in U, y \in V, U \cap V = \emptyset))$$

$$\iff \forall (x \times y) \in \Omega (\exists U, V \in \mathcal{T} (x \in U, y \in V, U \cap V = \emptyset))$$

$$\iff \forall (x \times y) \in \Omega (\exists U, V \in \mathcal{T} (x \times y \in U \times V \subseteq \Omega))$$

$$\iff \Omega \text{ is open}$$

$$\iff \Delta \text{ is closed}$$

Problem 19

Problem

If $A \subseteq X$, we define the boundary of A by the equation

$$\text{Bd } A = \bar{A} \cap \overline{X \setminus A}$$

1. Show that $\text{Int } A$ and $\text{Bd } A$ are disjoint, and $\bar{A} = \text{Int } A \cup \text{Bd } A$.
2. Show that $\text{Bd } A = \emptyset \iff A$ is both open and closed.
3. Show that U is open $\iff \text{Bd } U = \bar{U} \setminus U$.
4. If U is open, is it true that $U = \text{Int } (\bar{U})$? Justify your answer.

Solution

1. $X \setminus A$ is a subset of the closed set $X \setminus \text{Int } A$, hence $\overline{X \setminus A} \subseteq X \setminus \text{Int } A$. Thus $\text{Bd } A \subseteq \overline{X \setminus A} \subseteq X \setminus \text{Int } A$. $\text{Bd } A$ and $\text{Int } A$ are disjoint.

Let $x \in \bar{A}$. If every neighbourhood of x intersects $X \setminus A$, then $x \in \overline{X \setminus A}$, and hence $x \in \text{Bd } A$. Otherwise, there is some neighbourhood of x which does not intersect $X \setminus A$, that is, it is a subset of A , and so $x \in \text{Int } A$. Thus $\bar{A} = \text{Int } A \cup \text{Bd } A$.

2. If $\text{Bd } A = \emptyset$, then \bar{A} and $\overline{X \setminus A}$ form a partition of X . Thus $A = \bar{A}$ and $X \setminus A = \overline{X \setminus A}$. Both A and $X \setminus A$ are closed.

Conversely, if A is both open and closed, then $X \setminus A$ is closed as well, and so $\text{Bd } A = A \cap (X \setminus A) = \emptyset$.

3. For all $A \subseteq X$, we know that $\text{Int } A$ and $\text{Bd } A$ form a partition of \bar{A} , and hence $\text{Bd } A = \bar{A} \setminus \text{Int } A$. Then, noting that A and $\text{Int } A$ are always subsets of \bar{A} ,

$$\begin{aligned} U \text{ is open} \\ \iff \text{Int } U = U \\ \iff \bar{U} \setminus \text{Int } U = \bar{U} \setminus U \\ \iff \text{Bd } U = \bar{U} \setminus U \end{aligned}$$

4. No. Let $X = \mathbb{R}$ with the standard topology, and let $U = (0, 1) \cup (1, 2)$.