PARTIAL DIFFERENTIATION

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** If $u = \log(x^3+y^3+z^3-3xyz)$, then prove that :-

① $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

② $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = \dfrac{-9}{(x+y+z)^2}$

Solve ① :- Given, $u = \log(x^3+y^3+z^3-3xyz)$ ...(i)

Partially diff w.r.t (x) in eqⁿ (i),

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \text{...(ii)}$$

Again, Partially diff eqⁿ (i) w.r.t (y),

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad \text{...(iii)}$$

Again, Partially diff eqⁿ (i) w.r.t (z),

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad \text{...(iv)}$$

Adding eqⁿ (ii), (iii) & (iv), we get,

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$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= 3\left(\frac{x^2+y^2+z^2-xy-yz-xz}{x^3+y^3+z^3-3xyz}\right)$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$$

$$= \frac{3}{x+y+z}$$

Hence Proved

Solve ② :- Taking L.H.S,

$\Rightarrow \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u$

$\Rightarrow \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)u$

$\Rightarrow \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}\right)$

$\Rightarrow \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\left(\dfrac{3}{x+y+z}\right)$

$\Rightarrow \dfrac{\partial}{\partial x}\left(\dfrac{3}{x+y+z}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{3}{x+y+z}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{3}{x+y+z}\right)$

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$\Rightarrow \dfrac{-3}{(x+y+z)^2} - \dfrac{3}{(x+y+z)^2} \dfrac{-3}{(x+y+z)^2}$

$\Rightarrow LHS = \dfrac{-9}{(x+y+z)^2}$ Hence proved

*2 If $x^x y^y z^z = c$, then show that $\boxed{\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log_e x)^{-1}}$,

here z is a function of x & y and $x = y = z$.

*3 If $u = (x^2+y^2+z^2)^{-1/2}$, $x^2+y^2+z^2 \neq 0$, then prove that :-

(i) $\left(x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z}\right) = -u$

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

*4 If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$, then prove that

$\sin 2x \cdot \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2$

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*5 If $u = e^{xyz}$, then show that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

Solve 2* Given $x^x y^y z^z = c$

Taking log both side,

$\log(x^x y^y z^z) = \log c$

$x \log x + y \cdot \log y + z \log z = \log c$ ... (i)

Partially diff w.r.t x in eq^n (i),

$\left(\dfrac{x}{x} + \log x\right) + 0 + \left(\dfrac{z}{z} + \log z\right) \dfrac{\partial z}{\partial x} = 0$

$\dfrac{\partial z}{\partial x} = -\left(\dfrac{1 + \log x}{1 + \log z}\right)$ ... (ii)

Again, partially diff. eq^n (i) w.r.t y,

$\left(\dfrac{y}{y} + \log y\right) + \left(\dfrac{z}{z} + \log z\right) \dfrac{\partial z}{\partial y} = 0$

$\dfrac{\partial z}{\partial y} = -\left(\dfrac{1 + \log y}{1 + \log z}\right)$ ... (iii)

~~Multiplying eqⁿ (ii) & eq (iii), we get~~

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{1+\log x}{1+\log z}\right)\left(\frac{1+\log y}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log x+\log y+\log x\cdot\log y}{1+(\log z)^2+2\log z}$$

Again partially diff eqⁿ (iii) w r t (x),

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -(1+\log y)\left\{\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]\right\}$$

$$\frac{\partial^2 z}{\partial x \partial y} = (1+\log y)\,\frac{1}{(1+\log z)^2}\left(0+\frac{1}{z}\right)\frac{\partial z}{\partial x}$$

$$= \frac{1+\log y}{z(1+\log z)^2}\cdot\left[-\left(\frac{1+\log x}{1+\log z}\right)\right]$$

$$= \frac{-(1+\log x)(1+\log x)}{x(1+\log x)^2(1+\log x)}$$

$\{\because x = y = z\}$

$$= \frac{-1}{x(1+\log x)} = \frac{-1}{x\log(xe)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = [x\log(xe)]^{-1}$$

Hence Proved

Solⁿ 3* (i) Given : $u = (x^2+y^2+z^2)^{-1/2}$ ... (i)

Partially diff eqⁿ (i) w r t (x),

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial u}{\partial x} = \left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x)$$

$$x\frac{\partial u}{\partial x} = \frac{-x^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{... (ii)}$$

Similarly on diff eqⁿ (i) w r t y & z, we get,

$$y\frac{\partial u}{\partial y} = \frac{-y^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{... (iii)}$$

$$z\frac{\partial u}{\partial z} = \frac{-z^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{... (iv)}$$

Adding eqⁿ (ii), (iii) & (iv), we have,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{-x^2-y^2-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = u$$

Hence proved

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(ii) From eq^n (ii),

$$\frac{\delta u}{\delta x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

Partially diff. eq^n w.r.t (x),

$$\frac{\delta^2 u}{\delta x^2} = -\left[\frac{(x^2+y^2+z^2)^{3/2}(1) - \frac{3x}{2} \cdot 2x\,(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$$

$$\frac{\delta^2 u}{\delta x^2} = -\left[\frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{-(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2-3x^2)}{(x^2+y^2+z^2)^3}$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{-(y^2+z^2-2x^2)}{(x^2+y^2+z^2)^{5/2}} \quad \ldots (v)$$

Similarly, on again partially diff. eq^n (iii) and (iv) w.r.t y & z respectively,

$$\frac{\delta^2 u}{\delta y^2} = -\frac{(x^2+z^2-2y^2)}{(x^2+y^2+z^2)^{5/2}} \quad \ldots (vi)$$

$$\frac{\delta^2 u}{\delta z^2} = -\frac{(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{5/2}} \quad \ldots (vii)$$

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Adding eq^n (v), (vi) & (vii),

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = -\left[\frac{y^2+z^2-2x^2+x^2+y^2-2z^2+x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = 0$$

Hence Proved

Solve 4 *

$$u(x,y,z) = \log(\tan x + \tan y + \tan z) \quad \text{—(1)}$$

Partially diff. w.r.t (x) in eq^n (1),

$$\frac{\delta u}{\delta x} = \frac{1}{\tan x+\tan y+\tan z} \cdot (\sec^2 x)$$

$$\sin 2x \, \frac{\delta u}{\delta x} = \frac{2\sin x \cos x}{(\tan x+\tan y+\tan z)} \cdot \frac{1}{(\cos^2 x)}$$

$$\sin 2x \, \frac{\delta u}{\delta x} = \frac{2\tan x}{\tan x+\tan y+\tan z} \quad \text{—(2)}$$

Similarly, on partially diff. eq^n (1) w.r.t y & z, we get,

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z} \quad \text{— (3)}$$

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} \quad \text{— (4)}$$

Adding eqn (2), (3) & (4),

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2\tan x}{\tan x + \tan y + \tan z} + \frac{2\tan y}{\tan x + \tan y + \tan z} + \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Hence Proved

Solve 5* $\quad u = e^{xyz} \quad \ldots (i)$

Partially diff eqn (i) w.r.t (x),

$$\frac{\partial u}{\partial x} = e^{xyz}(yz) \quad \ldots (ii)$$

Similarly, $\dfrac{\partial u}{\partial y} = e^{xyz}(xz) \quad \ldots (iii)$

And, $\dfrac{\partial u}{\partial z} = e^{xyz}(xy) \quad \ldots (iv)$

Partially diff eqn (ii) w.r.t (y), we get,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = e^{xyz}(xz)(yz) + e^{xyz}(z)$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{xyz}(xyz^2 + z)$$

Again partially diff eqn w.r.t (z), we get,

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(xy)(xyz^2+z) + e^{xyz}(2xyz+1)$$

$$= e^{xyz}[(xy)(xyz^2+z) + 2xyz+1]$$

$$= e^{xyz}[x^2y^2z^2 + xyz + 2xyz + 1]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2+y^2z^2)\, e^{xyz}$$

Hence proved