Taylor's Theorem

If $f(a+x)$ be a function of the variable (x) such that it can be expanded in assending powers of (x) and this expansion be differentiable any no. of time then theorem state that

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \dots \frac{x^n}{n!} f^n(a), \dots$$

Proof :- Let, $f(a+x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots$ — (1)

$$f'(a+x) = A_1 + 2x A_2 + 3A_3 x^2 + \dots$$
$$f''(a+x) = 2A_2 + 3\cdot 2 A_3 x$$

Substituting $x = 0$ in each above eqⁿ,

$$\cancel{f(a) = A_0 + aA_1 + a^2 A_2 + a^3 A_3 + \dots}$$
$$\cancel{f'(a) = A_1 + 2a A_2 + 3a^2 A_3 + \dots}$$
$$\cancel{f''(a) = 2A_2 + 3\cdot 2 a A_3}$$

$$f(a) = A_0, \quad f'(a) = A_1, \quad \frac{f''(a)}{2!} = A_2$$

From eqⁿ (1),

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

Hence proved

1 * Expand $\tan\left(x+\frac{\pi}{4}\right)$ as for the term x^3 & evaluate $\tan 46.5°$.

Given, $f(a+x) = \tan\left(x+\frac{\pi}{4}\right) \Rightarrow \boxed{a=\frac{\pi}{4}}$

Let, $f(x) = \tan x$

$f'(x) = \sec^2 x = 1+\tan^2 x$

$f''(x) = 2\sec^2 x \tan x$

$f'''(x) = 2[\tan x \, 2\sec x \cdot \sec x \tan x + \sec^2 x \cdot \sec^2 x]$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

At $x = \pi/4$, $f(\pi/4) = 1$, $f''(\pi/4) = 4$

$f'(\pi/4) = 2$, $f'''(\pi/4) = 16$

By Taylor's Theorem,

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \ldots$$

$$\tan\left(x+\frac{\pi}{4}\right) = 1 + 2x + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \ldots$$

Put $x = 1.5°$ or 0.02618

$$\tan(46.5°)° = 1 + 2\times(0.02618) + 2(0.02618)^2 + \frac{8}{3}(0.02618)^3 + \ldots$$

$$\tan(46.5)° = 1.05378$$

2 * Expand $\sin x$ in power of $(x-\pi/2)$. Hence find the value of $\sin 91°$ correct to 4 decimal places. (0.9998)

3 * Expand $\cos x$ in power of $(x-\pi/4)$ by Taylor's theorem.

4 * Prove that using by Taylor's theorem
$$\log(x+h) = \log h + \left(\frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots\right)$$

Solve 2 * Let $f(x) = \sin(x) = \sin\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$

Successive diff. both side w.r.t (x),

At $x = \pi/2$

$f'(x) = \cos x$, $f'(\pi/2) = 0$

$f''(x) = -\sin x$, $f''(\pi/2) = -1$

$f'''(x) = -\cos x$, $f'''(\pi/2) = 0$

$f^{iv}(x) = \sin x$, $f^{iv}(\pi/2) = 1$

$f(\pi/2) = 1$

By Taylor's Theorem,

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \frac{x^4}{4!} f^{iv}(a) + \ldots$$

$$\sin x = \sin\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = 1 + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^4}{4!}(1) + \ldots$$

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$$\sin x = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24} - \cdots$$

Put $x = 91^\circ = 1.58825$

$$\sin 91^\circ = 1 - \frac{(1.58825 - \pi/2)^2}{2} + 1.588 + \cdots$$

$$= 1 - 0.0001 - 0.0001$$

$$\sin 91^\circ = 0.9998 \quad \text{Ans}$$

Solve 3* Let, $f(x) = \cos x = \cos\left[\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\right]$

Successive diff both side w.r.t (x)

$f(x) = \cos x$, $f(\pi/4) = 1/\sqrt{2}$

$f'(x) = -\sin x$, $f'(\pi/4) = -1/\sqrt{2}$

$f''(x) = -\cos x$, $f''(\pi/4) = -1/\sqrt{2}$

$f'''(x) = \sin x$, $f'''(\pi/4) = 1/\sqrt{2}$

$f^{iv}(x) = -\cos x$, $f^{iv}(\pi/4) = 1/\sqrt{2}$

By Taylor's theorem,

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \frac{x^4}{4!} f^{iv}(a) + \cdots$$

$$\cos x = \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{x-\pi/4}{\sqrt{2}}\right) - \frac{(x-\pi/4)^2}{\sqrt{2}\cdot 2!} + \frac{(x-\pi/4)^3}{\sqrt{2}\cdot 3!}$$

Ans

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Solve 4* Let, $f(x) = \log(x)$

$f(x+h) = \log(x+h)$

Successive diff w.r.t x both side,

$$f'(x) = \frac{1}{x}$$

$$f''(x) = \frac{-1}{x^2}$$

$$f'''(x) = \frac{2}{x^3}$$

Substituting $x = h$ in all above eq^n, we get,

$$f(h) = \log h, \quad f'(h) = \frac{1}{h}, \quad f''(h) = \frac{-1}{h^2},$$

$$f'''(h) = \frac{2}{x^3}, \ldots$$

By Taylor's theorem,

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots$$

$$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots$$

Hence proved

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$f(x) = f(a) + (x-a) f'(a) + (x-a)^2 f''(a) + (x-a)^3 f'''(a) + \dots$$

(8) x^n

* Prove that : $\tan^{-1}(x+h) = \tan^{-1}x + h\sin z \frac{\sin z}{1} - h(\sin z)^2 \frac{\sin 2z}{2} + (h\sin z)^3 \frac{\sin 3z}{3} + \dots$

where $z = \cot^{-1} x$

$z = \cot^{-1} x$

$\boxed{x = \cot z}$

$\frac{dx}{dz} = -\csc^2 z$

$\frac{dz}{dx} = -\sin^2 z$

Soln: Let, $f(x) = \tan^{-1} x$

$\therefore f(x+h) = \tan^{-1}(x+h)$

Now, • $f(x) = \tan^{-1} x \quad \dots (i)$

$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z}$

$= \frac{1}{\csc^2 z} = \sin^2 z$

$\therefore \cancel{f'(x) = \sin^2 x} \quad \dots$

$\cancel{f''(x) = 2\sin x \cos x} \quad \dots (iii)$

• $f'(x) = \sin z \cdot \sin z \quad \dots (ii)$

$f''(x) = (2\sin z \cos z) \frac{dz}{dx} = \sin 2z (-\sin^2 z)$

$\cancel{f''(x) = -2\sin^2 z \frac{dz}{dx}}$

• $f''(x) = -\sin^2 z \sin 2z \quad \dots (iii)$

$f'''(x) = -\left(\sin 2z \, 2\sin z \frac{dz}{dx} + 2\sin^2 z \cos 2z \frac{dz}{dx}\right)$

$f'''(x) = -2\sin z (\sin 2z \cos z + \cos 2z \sin z) \frac{dz}{dx}$

$f'''(x) = -2\sin z \sin(2z+z)(-\sin^2 z)$

• $f'''(x) = 2\sin^3 z \sin 3z \quad \dots (iv)$

By Taylor's theorem,

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$

$\tan^{-1}(x+h) = \tan^{-1} x + h \sin^2 z + \frac{h^2}{2!}(-\sin^2 z \sin 2z) + \frac{h^3}{3!}(2\sin^3 z \sin 3z) + \dots$

$\tan^{-1}(x+h) = \tan^{-1} x + (h\sin z)\sin z - (h\sin z)^2 \frac{\sin 2z}{2} + (h\sin z)^3 \frac{\sin 3z}{3} + \dots$

Hence proved

* Expand $f(x) = \sin x$ in ascending powers of $\left(x - \frac{\pi}{2}\right)$ using Taylor's theorem and find $\sin 91°$ correct to 4 dec.