"UNIT - 2"

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"DIFFERENTIAL CALCULUS"

Maclaurin's Theorem

If $f(x)$ be a function of variable x such that it can be expanded in ascending powers of x & expansion be differentiable any number of ~~time~~ times, then the theorem states that

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0)$$

Proof :- let, $f(x) = A_0 + A_1 x^1 + A_2 x^2 + A_3 x^3 + \dots + A_n x^n$ —(1)

$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^4 + \dots + (A_n) n x^{n-1}$

$f''(x) = 2A_2 + 3 \times 2 A_3 x + \dots$

$f'''(x) = 3 \times 2 \times 1 A_3 + 4 \times 3 \times 2 A_4 x + \dots$

Putting $x = 0$ in all above eq^n, we get,

$f(0) = A_0$, $f'(0) = A_1$, $\frac{f''(0)}{2!} = A_2$, $\frac{f'''(0)}{3!} = A_3$

From eq^n (1),

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

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1* Expand $\sin x$ by Maclaurin's Theorem, hence proved that $\sin(x) = \cancel{x - \frac{x^3}{3!} + x^5 - x \dots}$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \cancel{(-1)^n \frac{x^{2n+1}}{(2n+1)!}} + \dots$$

Let,
$$f(x) = \sin x$$
$$f'(x) = \cos x \quad \{\text{Successive diff. w.r.t } (x)\}$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{iv}(x) = \sin x$$
$$f^{v}(x) = \cos x$$
$$f^{vi}(x) = -\sin x$$
$$f^{vii}(x) = -\cos x \dots$$

Substituting $x = 0$ in eq^n & each of above eq^n, we get,

$f(0) = 0$ $\qquad$ $f^{iv}(0) = 0$

$f'(0) = 1$ $\qquad$ $f^{(v)}(0) = 1$

$f''(0) = 0$ $\qquad$ $f^{vi}(0) = 0$

$f'''(0) = -1$ $\qquad$ $f^{vii}(0) = -1 \dots$

By Maclaurin's Theorem,

$$\Rightarrow f(x) = f(0) + x' f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\Rightarrow \sin(x) = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \frac{x^6}{6!}(0) + \frac{x^7}{7!}(-1) + \dots$$

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$$\Rightarrow \boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$ Hence proved

*2) Expand $\cos(x)$ by Maclaurin's theorem, hence proved that

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

~~Let $\cos(x) = f(x)$~~

*3) Find the value of $\log(1+x)$ by Maclaurin's theorem.

Ans. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + \dots$

4* Apply maclaurin's theorem to prove that

$$\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$$

$$\boxed{\begin{aligned}(1-x)^{-1} &= 1 + x + x^2 + x^3 + \dots \\ (1+x)^{-1} &= 1 - x + x^2 - x^3 + x^4 + \dots\end{aligned}}$$

~~Let $f(x) = \tan$~~

5* Find the ~~fa~~ value of $\tan^{-1} x$ by using Maclaurin's theorem

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Solve 2 * Let $f(x) = \cos x$

$f'(x) = -\sin x$ (successive diff. w.r.t x)

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{iv}(x) = \cos x$

$f^{v}(x) = -\sin x$

$f^{vi}(x) = -\cos x$

Substituting $x = 0$ in each above eq^n, we get,

$f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$

$f^{iv}(0) = 1$, $f^{v}(0) = 0$, $f^{vi}(0) = -1$

By maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$$

$$+ \frac{x^5}{5!} f^{v}(0) + \frac{x^6}{6!} f^{vi}(0) + \ldots$$

$$\cos(x) = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1)$$

$$+ \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \ldots$$

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots}$$

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Solve 3 * Let, $f(x) = \log(1+x)$

Successive diff w.r.t x,

$f'(x) = \frac{1}{1+x}$ $\quad$ $f'''(x) = \frac{2}{(1+x)^3}$

$f''(x) = \frac{-1}{(1+x)^2}$ $\quad$ $f^{iv}(x) = \frac{-6}{(1+x)^4}$

By ~~m~~ substituting $x=0$ in each above eq^n,

$f(0) = 0$

$f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{iv} = -6$

By Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$$

$$f(x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

~~**Solve 4 *** Let, $f(x) = \log(\sec x)$~~

~~Successive diff w.r.t x,~~

~~$f'(x) = \frac{\tan x \sec x}{\sec x} = \tan x$, $f''(x) = \frac{1}{1+x^2}$~~

~~$f'''(x) = \frac{-2x}{(1+x^2)^2}$, $f^{iv}(x) = \frac{2}{(1+x^2)^3}$, $f^{v}(x) = \frac{-6}{(1+x^2)^4}$~~

$f^{vi}(x) = \frac{+6x^4}{(1+x^2)^5}$

Solve 4 * Let, $f(x) = \log(\sec x)$

By successive differentiation w r t (x), we get,

$f'(x) = \tan x$

$f''(x) = \sec^2 x$

$f'''(x) = 2\sec^2 x \tan x$

$\cancel{f^{iv}(x) = 2\sec^3 x \tan x + 4\sec^2 x \tan^2 x}$

$\cancel{f^{v}(x) = 2\{\sec^5 x \tan x + 2\sec^4 x \tan x + 3\sec^3 x \tan^2 x + 4\sec^2 x \tan x\}}$

$f^{iv}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f^{v}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$f^{vi}(x) = 16\sec^6 x + 64\sec^4 x \tan^2 x + \ldots$

Substituting $x = 0$ in each above eq^n,

$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 1$

$f'''(0) = 0, \quad f^{iv}(0) = 2, \quad f^{v}(0) = 0, \quad f^{vi}(0) = 16$

By Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + f^{v}(0)\frac{x^5}{5!} + f^{vi}(0)\frac{x^6}{6!} + \ldots$$

$$\log(\sec x) = 0 + 0 + \frac{x^2}{2} + 0 + \frac{x^4}{4\times 3\times 2!} \times 2 + 0 + \frac{x^6}{6\times 5\times 4\times 3\times 2\times 1}(16) + \ldots$$

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$$\boxed{\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots}$$

Solve 5 * Let, $f(x) = \tan^{-1} x$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$$\therefore \quad f'(x) = 1 - x^2 + x^4 - x^6 + \dots$$

$$f''(x) = -2x + 4x^3 - 6x^5 + \dots$$

$$f'''(x) = -2 + 8x^2 - 30x^4 + \dots$$

$$f^{iv}(x) = 16x - 120x^3 + \dots$$

$$f^{v}(x) = 16 - 360x^2 + \dots$$

$$f^{vi}(x) = -720x + \dots$$

Substituting $x = 0$ in each ~~of~~ above eq^n,

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -2$

$f^{iv}(0) = 0$, $f^{v}(0) = 16$, $f^{vi}(0) = -720$

By Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \frac{x^6}{6!} f^{vi}(0)$$

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$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

6 * Expand $\sin^{-1}x$ by Maclaurin's theorem.

- $(1+x)^n = 1 - nx + n(n-1)\frac{x^2}{2!} - n(n-1)(n-2)\frac{x^3}{3!} + n(n-1)(n-2)(n-3)\frac{x^4}{4!} + \dots$

- $(1-x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \dots$

Solve 6 * Let, $f(x) = y = \sin^{-1}x$

$$y_1 = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{2}\left(\frac{-1}{2}-1\right)\frac{x^4}{2!} - \frac{1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\frac{x^6}{3!} - \frac{1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\left(\frac{-1}{2}-3\right)\frac{x^8}{4!} + \dots$$

$$y_1 = 1 - \frac{x^2}{2} + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \frac{35}{128}x^8 - \dots$$

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$$y_2 = -x + \frac{3}{2}x^3 - \frac{15}{8}x^5 + \frac{35}{16}x^7 - \dots$$

$$y_3 = -1 + \frac{9}{2}x^2 - \frac{75}{8}x^4 + \frac{245}{16}x^6 - \dots$$

$$y_4 = 9x - \frac{75}{2}x^3 + \frac{735}{8}x^5 - \dots$$

$$y_5 = 9 - \frac{225}{2}x^2 + \frac{735\times5}{8}x^4 - \dots$$

$$y_6 = 225x + \frac{735\times5}{2}x^3 - \dots$$

$$y_7 = 225 + \frac{735\times5\times3}{2}x^2 - \dots$$

$$y_8 = 735\times5\times3\times2\,x - \dots$$

Substituting $x=0$ in each above eq^n,

$$y = 0,\ y_1 = 1,\ y_2 = 0,\ y_3 = -1,\ y_4 = 0$$
$$y_5 = 9,\ y_6 = 0$$

By Maclaurin's theorem,

Leibnitz's theorem :-

$D^n(uv) = {}^nC_0 (D^n u)v + {}^nC_1 (D^{n-1}u)Dv + {}^nC_2 (D^{n-2}u)D^2 v + \dots$

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$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0)$$

$$+ \frac{x^5}{5!} f^{v}(0) + \frac{x^6}{6!} f^{vi}(0) + \dots$$

$$\sin^{-1} x = x - \frac{x^3}{3!} + \frac{9x^5}{5!} - \frac{15x^7}{7!} + \dots$$

Imp

7* Expand $e^{a \sin^{-1} x}$ hence so that

$e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2}{3!}\sin^3\theta + \dots$

Let, $f(x) = e^{a\sin^{-1}x}$

$$f'(x) = \frac{a e^{a \sin^{-1} x}}{\sqrt{1-x^2}}$$

$$f''(x) = a \left[\frac{\sqrt{1-x^2}\, e^{a\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} + e^{a\sin^{-1}x} \frac{(2x)}{2\sqrt{1-x^2}}}{1-x^2} \right]$$

$$f''(x) = a \left(\frac{\sqrt{1-x^2}\, e^{a\sin^{-1}x} + x e^{a\sin^{-1}x}}{1-x^2 \sqrt{1-x^2}} \right)$$

$$f''(x) = a e^{a\sin^{-1}x} \left(\frac{x + \sqrt{1-x^2}\, e^{a\sin^{-1}x}}{(1-x^2)(\sqrt{1-x^2})} \right)$$

Substituting $x = 0$ in each above eq^n,

$f(0) = 1$, $f'(0) = a$, $f''(0) = 2a$

By Maclaurin's theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

$$e^{a\sin^{-1}x} = 1 + ax + ax^2 + \ldots$$

(OR)

Let, $y = e^{a\sin^{-1}x}$ — (1)

$$y_1 = e^{a\sin^{-1}x} \frac{a}{\sqrt{1-x^2}} \quad -(2)$$

$$(1-x^2)\,y_1^2 = \left(a e^{a\sin^{-1}x}\right)^2$$

$$(1-x^2)\,y_1^2 = y^2 a^2$$

$$-2xy_1^2 + (1-x^2)\,2y_1y_2 = a^2\, 2y\,y_1$$

$$(1-x^2)\,y_2 - xy_1 = a^2 y$$

$$(1-x^2)\,y_2 - xy_1 - a^2 y = 0 \quad -(3)$$

By Lebinitz's theorem,

$$D^n(uv) = {}^nC_0 (D^n u) v + {}^nC_1 (D^{n-1} u) Dv + {}^nC_2 (D^{n-2} u) D^2 v + \ldots$$

$$\Big[(1)(D^n y_2)(1-x^2) + n (D^{n-1} y_2) D(1-x^2) + \frac{n(n-1)}{2!} (D^{n-2} y_2) D^2(1-x^2) + 0\Big]$$

$$-\left[(D^n y_1)x + n(D^{n-1} y_1) D(x)\right] + \cancel{\frac{n(n-1)}{2!}} - a^2 y_n = 0$$

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$$\left[y_{n+2}(1-x^2)+n(y_{n+1})(-2x)+\frac{n(n-1)}{2}(y_n)(-2)\right]$$

$$-\left[xy_{n+1}+ny_n\right]-a^2y_n=0$$

$$(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-ny_n-xy_{n+1}-a^2y_n=0$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2+ny_n-ny_n-a^2y_n$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-a^2y_n-n^2=0 \quad -(4)$$

Substituting $x=0$ in eq^n (1), (2) & (3) & (4),

$$y_0=1,\ (y_1)_0=a,\ (y_2)_0=a^2$$

$$(y_{n+2})_0=n^2+a^2 \quad ; \text{ where } n=1,2,3\ldots$$

$$(y_3)_0=1^2+a^2\ (n=1),\ (y_4)_0=2^2+a^2\ (n=2)$$

By Maclaurin's theorem,

$$f(x)=f(0)+xf'(0)+\frac{x}{2!}f''(0)+\ldots$$

$$e^{a\sin^{-1}x}=1+xa+\frac{x^2}{2!}a^2+\frac{x^3}{3!}(1^2+a^2)+\ldots$$

Putting $a=1$ & $x=\sin\theta \Rightarrow \sin^{-1}x=\theta$, we get

$$e^{\theta}=1+\sin\theta+\frac{\sin^2\theta}{2!}+\frac{2\sin^3\theta}{3!}+\ldots$$

Hence Proved ||

8* Prove t

Let

$(1-x^2$

$(1-x$

$(1-x^2)\,2$

$(1-x^2$

$(1-x^2)y$

By Lebinit

$[{}^nC_0(D^n y_2)$

$D^2(1-x^2)+$

$(y_{n+2})(1$

$-(y_{n+1})$

$(1-x^2)(y_{n+}$

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$(y_n)(-2)$

$n y_n$

$n - a^2 y_n = 0$

$- (4)$

$(4),$

8* Prove that $(\sin^{-1}x)^2 = \frac{2}{2!}x^2 + \frac{2\cdot 2^2 x^4}{4!} + \frac{2\cdot 2^2 4^2 x^6}{6!} + \ldots$

Let, $y = (\sin^{-1}x)^2$ — (1)

$y_1 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$ — (2)

$(1-x^2)\,y_1^2 = (\sin^{-1}x)^2\cdot 4$

$(1-x^2)\,y_1^2 = 4y$

$(1-x^2)\,2y_1y_2 + y_1^2(-2x) = 4y_1$

$(1-x^2)\,y_2 - xy_1 = 2$

$(1-x^2)y_2 - xy_1 - 2 = 0$ — (3)

By Lebinitz's theorem,

$[{}^nC_0(D^n y_2)(1-x^2) + {}^nC_1(D^{n-1}y_2)D(1-x^2) + {}^nC_2(D^{n-2}y_2)D^2(1-x^2) + 0] - [{}^nC_0(D^n y_1)x + {}^nC_1(D^{n-1}y_1)Dx] - = 0$

$(y_{n+2})(1-x^2) + n(y_{n+1})(-2x) + \frac{n(n-1)}{2}(y_n)(-2) - (y_{n+1})x - n\,y_n - = 0$

$(1-x^2)(y_{n+2}) - 2nx\{y_{n+1}\} - y_n n^2 + ny_n - xy_{n+1} - ny_n - = 0$

get

Proved

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$(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - \cancel{2y_n} - n^2 y_n = 0$ — (4)

Substituting $x=0$ in eq^n (1), (2), (3) & (4),

$(y_0) = 0$, $(y_1)_0 = 0$, $(y_2)_0 = 2$,

$(y_{n+2})_0 = n^2 (y_n)_0 \cancel{+2}$, where $n = 1, 2, 3 \ldots$

$(y_3)_0 = 0 \cancel{+2}$, $n = 1$
$(y_4)_0 = 2^2 \cancel{+2}\, 2$, $n = 2$
$(y_5)_0 = 0 \cancel{+2}$, $n = 3$
$(y_6)_0 = 4^2 \cancel{+2}\, 2^2\, 2$, $n = 4$

~~$(y_3)_0 = 1$~~
~~$(y_4)_0 = 6$~~
~~$(y_5)_0 = 11$~~
~~$(y_6)_0 = 18$~~

By Maclaurin's theorem,

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$

$\cancel{(\sin^{-1}x)^2 = 0 + 0 + \frac{x^2}{2!}(2) + \frac{x^3}{3!}\,3 + \frac{x^4}{4!}\cdot 6 + \frac{x^5}{5!}\,11}$

$(\sin^{-1}x)^2 = \cancel{\frac{x^2}{2!}}\ \frac{2}{2!}x^2 + \frac{2\cdot 2^2}{4!}x^4 + \frac{2\cdot 2^2\cdot 4^2}{6!}x^6 + \ldots$