BETA & GAMMA FUNCTION

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Beta Function

Beta function, denoted by $B(m,n)$, is defined by the definite integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Some basic integral formulaes :-

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$,

   where $a < c < b$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$,

   If $f(x)$ is an even function

   $= 0$, If $f(x)$ is an odd func^n

6. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, If $f(2a-x) = f(x)$

   $= 0$, If $f(2a-x) = -f(x)$

Property 1 : Symmetrical Property

$$B(m,n) = B(n,m)$$

By definition of Beta function,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$$B(m,n) = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1} \, dx$$

$$\left\{ \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right\}$$

$$B(m,n) = \int_0^1 x^{n-1} (1-x)^{m-1} \, dx$$

$$\therefore \quad B(m,n) = B(n,m)$$

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Property 2: $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, dx$

By definition of Beta function, we have,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

Putting $x = \frac{1}{1+y} \Rightarrow dx = \frac{-1}{(1+y)^2} dy$,

at $x = 0$, $y = \infty$ & at $x = 1$, $y = 0$, then

$$B(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \left(\frac{-dy}{(1+y)^2}\right)$$

$$B(m,n) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \left(\frac{y}{1+y}\right)^{n-1} \frac{dy}{(1+y)^2}$$

$$B(m,n) = \int_0^\infty \left(\frac{1}{1+y}\right)^{m-1+n-1+2} (y)^{n-1} \, dy$$

$\left\{\because \int_b^a f(x)dx = -\int_a^b f(x)\, dx\right\}$

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} \, dy$$

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, dx$$

$\left\{\because \int_0^a f(x)\, dx = \int_0^a f(t)\, dt\right\}$

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Property 3: $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

By definition of Beta function,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

Putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

at $x = 0$, $\theta = 0$ & at $x = 1$, $\theta = \pi/2$

Now,

$$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \left[1 - \sin^2\theta\right]^{n-1} 2\sin\theta\cos\theta \cdot d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1} \cdot d\theta$$

Gamma Function

Gamma function, denoted by Γn is defined by the definite integral

$$\Gamma n = \int_0^\infty x^{n-1} \cdot e^{-x}\, dx$$

Fundamental Properties of Gamma Function

1. $\Gamma(1) = 1$
2. $\Gamma(n+1) = n\Gamma n$, $n > 0$
3. $\Gamma(n+1) = n!$, $n = 1, 2, 3 \ldots$

$\int u \cdot v \, dx = u \int v \, dx - \int \left\{ \frac{d}{dx} u \left(\int v \, dx \right) \right\} dx$

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Property 1 : $\Gamma(1) = 1$

By definition of Gamma function,

$\Gamma n = \int_0^\infty x^{n-1} e^{-x} \, dx$

For $n = 1$,

$\Gamma(1) = \int_0^\infty x^{1-1} (e^{-x}) \, dx$

$\Gamma(1) = \int_0^\infty e^{-x} \, dx$

$\Gamma(1) = -\left[e^{-x} \right]_0^\infty$

$\Gamma(1) = -\left[e^{-\infty} - e^{0} \right]$

$\Gamma(1) = -\left[0 - 1 \right]$ $\{ \because e^{-\infty} = 0 \}$

$\Gamma 1 = 1$

Hence proved

Property 2 : $\Gamma(n+1) = n \Gamma n$

By definition of Gamma function,

$\Gamma(n+1) = \int_0^\infty x^n e^{-x} \, dx$

$\Gamma(n+1) = \left[x^n e^{-x} \right]_0^\infty + \int_0^\infty (n x^{n-1} e^{-x}) \, dx$

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$$\Gamma(n+1) = 0 + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$\Gamma(n+1) = n\,\Gamma n \qquad \left\{\because \Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx\right\}$$

Hence proved

Property 3 : $\Gamma(n+1) = n!$, $n = 1, 2, 3,$

From above property, we have

$$\Gamma(n+1) = n\,\Gamma n = n\left[(n-1)\,\Gamma(n-1)\right]$$

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$$\Gamma(n+1) = n(n-1)(n-2)(n-3)\ldots\ldots 3\cdot 2\cdot 1\ \Gamma(1)$$

$$\Gamma(n+1) = n! \quad , n = 1, 2, 3$$

Hence proved

$\left\{\because \Gamma(1) = 1 \ \&\ \text{~~Hence pro~~} \atop n(n-1)(n-2)(n-3)\ldots = n!\right\}$

Relation b/w Beta & Gama Function

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad (m, n > 0)$$

Proof :-

Application of Beta & Gamma Function

To prove that : $\Gamma(1/2) = \sqrt{\pi}$

We know that

$$B(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}, \quad (m,n>0) \ldots (i)$$

Putting $m = n = 1/2$ in eq^n (i), we get,

$$B(1/2, 1/2) = \frac{\Gamma 1/2 \; \Gamma 1/2}{\Gamma(1/2 + 1/2)}$$

$$B(1/2, 1/2) = \frac{[\Gamma(1/2)]^2}{\Gamma(1)}$$

$$\cancel{B(1/2, 1/2)} \cancel{=} \cancel{\int_0^1} \quad \left\{ \begin{array}{l} B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ \Gamma(1) = 1 \end{array} \right\}$$

$$\int_0^1 (x)^{-1/2} \cdot (1-x)^{-1/2} dx = \Gamma(1/2)^2$$

$$\Gamma(1/2)^2 = \int_0^1 \frac{1}{\sqrt{x} \; \sqrt{1-x}} dx$$

Putting $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta \, d\theta$

$\{$at $x = 0, \theta = 0$ & $x = 1, \theta = \pi/2\}$

$$(\Gamma 1/2)^2 = \int_0^{\pi/2} \frac{1}{(\sin\theta)(\cos\theta)} (2\sin\theta\cos\theta \, d\theta)$$

$$\left(\Gamma \tfrac{1}{2}\right)^2 = 2\int_0^{\pi/2} 1 \, d\theta$$

$$\left(\Gamma \tfrac{1}{2}\right)^2 = 2\left[\theta\right]_0^{\pi/2}$$

$$\left(\Gamma \tfrac{1}{2}\right)^2 = 2\left(\frac{\pi}{2} - 0\right)$$

$$\left(\Gamma \tfrac{1}{2}\right)^2 = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence proved

* Evaluate :- ① $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

② $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$

Solve ① : Let, $\displaystyle I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

$$I = \int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}\,dx$$

$$I = \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}$$

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$I = B(9,15) - B(15,9)$ $\{\because$

$\left\{\begin{matrix} B(m,n) = \\ B(n,m) \end{matrix}\right\} \left\{\because \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = B(m,n)\right\}$

$I = B(9,15) - B(9,15) = 0$ Ans

Solve (2): Let, $I = \int_0^\infty \frac{x^4(1+x^5)\,dx}{(1+x)^{15}}$

$I = \int_0^\infty \frac{x^4 + x^9}{(1+x)^{15}}\,dx$

$I = \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$

$I = B(5,10) + B(10,5)$

$I = 2\,[B(5,10)]$

$I = 2\left[\frac{\Gamma 5 \cdot \Gamma 10}{\Gamma(5+10)}\right]$

$I = 2 \times \frac{4! \cdot 9!}{14!} = \frac{2 \times 4 \times 3 \times 2 \times 1}{14 \times 13 \times 12 \times 11 \times 10}$

$I = \frac{48}{240{,}240} = \frac{1}{5{,}005}$ Ans

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* Prove that :- $B(m,n) = B(m+1, n) + B(m, n+1)$

$\{(m, n > 0)\}$

Proof :-

We have,

$$B(m+1, n) = \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)} \quad \ldots (i)$$

And,

$$B(m, n+1) = \frac{\Gamma m \cdot \Gamma(n+1)}{\Gamma(m+n+1)} \quad \ldots (ii)$$

Adding eqⁿ (i) & (ii), we have,

$$B(m+1, n) + B(m, n+1) = \frac{\Gamma(m+1)\cdot\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m \cdot \Gamma n+1}{\Gamma(m+n+1)}$$

$$= \frac{\Gamma m+1\, \Gamma n + \Gamma m\, \Gamma n+1}{\Gamma(m+n+1)} \qquad \cancel{\Gamma m \cdot \Gamma n} \quad \cancel{\Gamma m+1 \quad \Gamma n+1}$$

$\{\because \Gamma n+1 = n\Gamma n\}$

$$= \frac{m\Gamma m\ \Gamma n + \Gamma m \cdot n\Gamma n}{\Gamma(m+n+1)}$$

$$= \frac{\Gamma m \cdot \Gamma n\,(m+n)}{(m+n)\,\Gamma m+n}$$

$$= \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

$$B(m+1, n) + B(m, n+1) = B(m, n)$$

Hence Ans