DOUBLE & TRIPLE INTEGRALS

Date

Page No:

Evaluation of Double Integrals :-

The double integral over a region (R) is evaluated by two successive integrations. Let the region R be bounded by the curve $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = a$, $x = b$, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$

$$= \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx$$

Date:

Page No:

* Evaluate :- (1) $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1+x^2+y^2}$

Solu (1) :- Let, $I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2} \right] dx$

$$I = \int_0^1 \left[\frac{1}{\sqrt{(1+x^2)}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx$$

$$I = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left\{ \tan^{-1}(1) - \tan^{-1}(0) \right\} dx$$

$$I = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$I = \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2}\right) \right]_0^1$$

$$I = \frac{\pi}{4} \left\{ \log\left[1 + \sqrt{1+1^2}\right] - \log\left[0 + \sqrt{1+0^2}\right] \right\}$$

$$I = \frac{\pi}{4} \left\{ \log\left[1 + \sqrt{2}\right] - \log 1 \right\}$$

$\cancel{I = \frac{\pi}{4} \log \sqrt{2}}$? $\quad I = \frac{\pi}{4} \log\left(1+\sqrt{2}\right)$

Ans

Date

Page No:

② $\int_0^1 \int_0^{x^2} e^{y/x} \, dy \, dx$

Solve ② :- Let, $I = \int_0^1 \left[\int_0^{x^2} e^{y/x} \, dy \right] dx$

$I = \int_0^1 \left[x \cdot e^{y/x} \right]_0^{x^2} dx$

$I = \int_0^1 x \cdot (e^{x} - e^{0}) \, dx$

$I = \int_0^1 x (e^{x} - 1) \, dx$

$I = x \left\{ \int_0^1 (e^{x} - 1) \right\} - \int_0^1 1 \cdot (e^{x} - x) \, dx$

$\cancel{I = x \left\{ [e^{x}]_0^1 - [x]_0^1 \right\} + \left[e^{x} - \frac{x^2}{2} \right]_0^1}$

$\cancel{I = x \{ (e - 1) - 1 \} - \left[e^1 - \frac{1}{2} - 1 + 0 \right]}$

$\cancel{I = xe - 2x - e - \frac{3}{2}}$

$I = \left[x (e^{x} - x) \right]_0^1 - \left[e^{x} - \frac{x^2}{2} \right]_0^1$

$I = e^{1} - 1 - \left[e - \frac{1}{2} - 1 \right] = e - 1 - e + \frac{1}{2} + 1$

$\boxed{I = \frac{1}{2}}$ Ans

Date:
Page No:

(3) Evaluate :- $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)}\,\sqrt{(1-y^2)}}$

Solve (3) :- Let, $I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\int_0^1 \frac{dy}{\sqrt{1-y^2}}\right] dx$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left[\sin^{-1} y\right]_0^1 dx$$

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\sin^{-1} 1 - \sin^{-1} 0\right) dx$$

$$I = \frac{\pi}{2} \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$I = \frac{\pi}{2} \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2} \left(\sin^{-1} 1 - \sin^{-1} 0\right)$$

$$I = \frac{\pi}{2} \left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} \quad \text{Ans}$$

Date:
Page No:

(4) Evaluate :- $\int_1^2 \int_0^x \left(\frac{dx\,dy}{x^2+y^2}\right)$

Solve (4) :-

Let, $I = \int_1^2 \left(\int_0^x \frac{dy}{x^2+y^2}\right) dx$

$$I = \int_1^2 \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$I = \int_1^2 \frac{1}{x} \left[\tan^{-1} 1 - \tan^{-1} 0\right] dx$$

$$I = \int_1^2 \frac{1}{x} \left(\frac{\pi}{4}\right) dx$$

$$I = \frac{\pi}{4} \left[\log x\right]_1^2$$

$$I = \frac{\pi}{4} (\log 2 - \log 1)$$

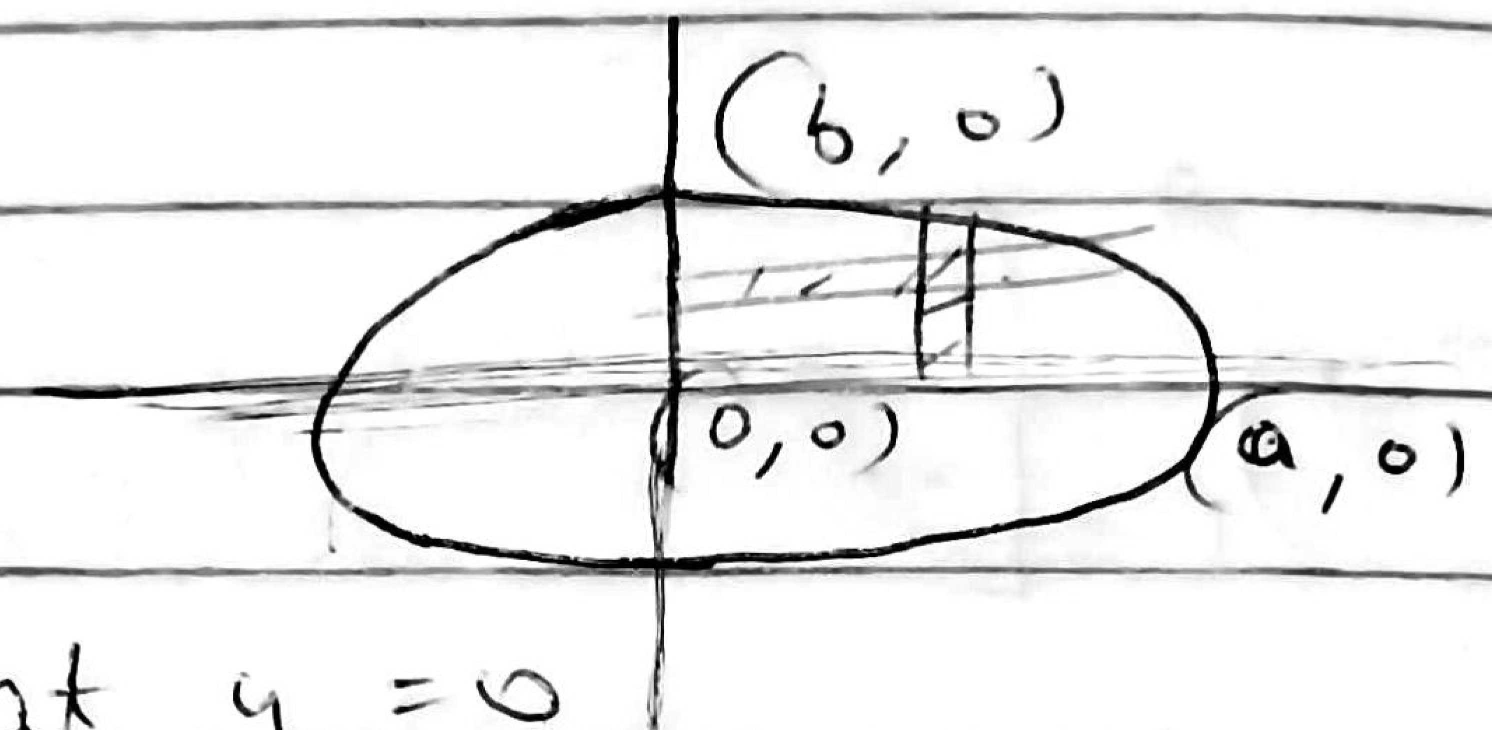
$$\boxed{I = \frac{\pi}{4} \log 2}$$

Ans

Date:
Page No:

(5) Compute the value of $\iint_R dx\,dy$ when R is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of positive quadrant.

Solu(5):-



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at $x = 0$, $y = b$, at $y = 0$, $x = a$

$$\therefore \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2} = f(x)$$

Now,

Required value $= \iint_R dx\,dy$

$$= \int_{x=0}^{a} \int_{y=0}^{f(x)} dx\,dy$$

$$= \int_0^a \left[\int_0^{f(x)} 1\,dy \right] dx$$

$$= \int_0^a [y]_0^{f(x)}\,dx = \int_0^a (f(x) - 0)$$

$$= \int_0^a f(x)\,dx$$

$$= \int_0^a \left(\frac{b}{a}\sqrt{a^2 - x^2} \right) dx$$

$$= \frac{b}{a} \int_0^a \left(\sqrt{a^2 - x^2} \right) dx$$

Date

Page No:

Putting $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

when, $x = 0$, $\theta = 0$ & $x = a$, $\theta = \pi/2$, then

$$= \frac{b}{a}\int_0^{\pi/2}\left(a^2 - a^2\sin^2\theta\right)^{1/2} a\cos\theta\, d\theta$$

$$= ab\int_0^{\pi/2}\sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= ab\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2}(\cos 2\theta + 1)\, d\theta \quad \{\because \cos 2\theta = 2\cos^2\theta - 1\}$$

$$= \frac{ab}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/2}$$

$$= \frac{ab}{2}\left(0 + \frac{\pi}{2} - 0 - 0\right)$$

$$= \frac{\pi ab}{4}$$ Ans