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Homogenous Function

A function is called homogenous function if the degree of each ~~tem~~ term is cont.

Let, $f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \dots + a_n y^n$

be a function of x & y. $x^n f(y/x) = f(x,y)$

Euler's Theorem on Homogenous Func^n

If $f(x,y)$ is a homogenous function of x & y of degree (n) then theorem states that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

Proof :- $\because f(x,y) = x^n F(y/x)$ ... (i)

Partially Diff w r t (x),

$$\cancel{\frac{\partial f}{\partial x} = x^n \left(\frac{-y}{x^2}\right) \frac{\partial F}{\partial x} + f(n x^{n-1})}$$

$$\cancel{\frac{\partial f}{\partial x} = x^n \frac{\partial F}{\partial x} +}$$

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$$\frac{\partial f}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) n x^{n-1}$$

$$\frac{\partial f}{\partial x} = -x^{n-2} y f'\left(\frac{y}{x}\right) + n x^{n-1} f\left(\frac{y}{x}\right)$$

$$x\left(\frac{\partial f}{\partial x}\right) = -x^{n-1} y f'\left(\frac{y}{x}\right) + n x^n f\left(\frac{y}{x}\right) \quad \dots (ii)$$

Now, Partially diff eq^n (i) w.r.t (y),

$$\frac{\partial f}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y \frac{\partial f}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right) \quad \dots (iii)$$

By adding eq^n (ii) & (iii), we get,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n\left[x^n f\left(\frac{y}{x}\right)\right]$$

$$\boxed{x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf}$$

Hence proved

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(1) (+)

*1 If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

Solve 1* $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\sin u = \frac{x^2+y^2}{x+y}$$

$f\left(\frac{y}{x}\right)$

$$\sin u = \frac{x^2}{x}\left(\frac{1+y^2/x^2}{1+y/x}\right)$$

$$\sin u = x^1 f\left[\frac{1+(y/x)^2}{1+(y/x)}\right]$$

$$\sin u = f(x,y)$$

By Euler's theorem,

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = 1(\sin u)$$

$$\Rightarrow x(\cos u)\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$F(x,y) = x^n f(y/x)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

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$$\Rightarrow \cancel{x\cos u\,(x+y)\frac{\partial u}{\partial x} = \sin u}$$

$$\Rightarrow \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \sin u$$

$$\Rightarrow \boxed{\tan u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}}$$

Hence Proved

2* If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

3* If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = \sin 4u - \sin 2u$

$= 2\cos 3u \sin u$

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Solve 2 * $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$

$$e^u = \frac{x^4+y^4}{x+y}$$

$$e^u = \frac{x^4}{x}\left[\frac{1+(y/x)^4}{1+(y/x)}\right]$$

$$e^u = x^3 f\left(\frac{1+(y/x)^4}{1+(y/x)}\right)$$

$$e^u = \cancel{x^3} f(x,y)$$

By Euler's theorem,

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 3(e^u)$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

Hence proved

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Solve 3 * $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$\tan u = x^2\left[\frac{1+(y/x)^3}{1-(y/x)}\right]$$

$$\tan u = \cancel{x^2}\, f(x,y)$$

i) By Euler's theorem,

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2(\tan u)$$

$$\Rightarrow x\sec^2 u\cdot\frac{\partial u}{\partial x} + y\sec^2 u\cdot\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(\frac{\tan u}{\sec^2 u}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \quad \text{— (1)}$$

Hence proved

$p = \frac{\partial f}{\partial x}$	$q = \frac{\partial f}{\partial y}$	$\frac{\partial^2 f}{\partial x^2}$
$s = \frac{\partial^2 f}{\partial x \partial y}$	$t = \frac{\partial^2 f}{\partial y^2}$	

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(ii) Now, ~~Σ~~ Partially diff. eqⁿ (1) w.r.t x,

$$\Rightarrow x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u \frac{\partial u}{\partial x} \quad (2)$$

{Multiplying by x both side}

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1) x \frac{\partial u}{\partial x} \quad \cdots (2)$$

~~Again~~

Similarly, on partially diff. eqⁿ (1) w.r.t y,

$$\Rightarrow y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1) y \frac{\partial u}{\partial y} \quad \cdots (3)$$

Adding eqⁿ (2) & (3),

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1) \sin 2u$$

$$\Rightarrow \quad = 2\cos 2u \sin 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2 \sin u \cdot \cos 3u$$

Hence proved