Maxima & Minima of Two Variables

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(i) Find $u = f(x, y)$.

(ii) Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$, then solve the eq^n $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$. Let, one set of roots be $x = a$ & $y = b$ i.e. pair of (a, b) this part is called stationary points.

(iii) Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ &

$t = \frac{\partial^2 f}{\partial y^2}$ at (a, b)

(iv) Calculate $rt - s^2$ for this pair.

- If $rt - s^2 > 0$ and $r > 0$, then $f(x, y)$ is a minimum for pair (a, b).
- If $rt - s^2 > 0$ and $r < 0$, then $f(x, y)$ is a maximum for pair (a, b).
- If $rt - s^2 < 0$, then the function f is neither maxima nor minima.

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- If $rt - s^2 = 0$, the case is doubtful and further investigation is required.
So, we ~~say~~ shall leave this case.

1* Discuss the maxima and minima of the function x^3+y^3-3axy

Solu 1* Let, $u = x^3+y^3-3axy = f(x,y)$

Partially diff w r t (x),

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay$$

Similarly,

$$\frac{\partial u}{\partial y} = 3y^2 - 3ax$$

Now,

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay = 0 \quad \& \quad \frac{\partial u}{\partial y} = 3y^2 - 3ax = 0$$

$3x^2 = 3ay$ $\quad\quad$ $3y^2 = 3ax$

$x^2 = ay$ — (1) $\quad\quad$ $y^2 = ax$ — (2)

$x^2 - ay = 0$ — (1) $\quad\quad$ $y^2 - ax = 0$ — (2)

Squaring both side in eq^n (1),

$\cancel{(x^2 - ay)} = \cancel{0}$ $\quad$ $(x^2)^2 = (ay)^2$

$\cancel{x^2 + a^2y^2 - 2ayx^2} = \cancel{0}$

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$\cancel{x^4 + a^2y^2 - 2a}$ $\quad$ $x^4 = a^2y^2$

$x^4 = a^2(ax)$

$x^4 = a^3x$

$x^4 - a^3x = 0$

$x(x^3 - a^3) = 0$

$x = 0, \boxed{(x-a)(x^2+a^2+ax)} = 0$

$x = 0, \quad x = a$

and, $y = 0, \quad y = a$

$(0,0)$ & (a,a)

Now,

$$r = \frac{\partial^2 u}{\partial x^2} = 6x$$

$r_{(a,a)} = 6a$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -3a$$

$s_{(a,a)} = -3a$

$$t = \frac{\partial^2 u}{\partial y^2} = 6y$$

$t_{(a,a)} = 6a$

Now,

$rt - s^2 = (6a)(6a) - (3a)^2$

$rt - s^2 = 27a^2$

$rt - s^2 > 0$

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Since $rt - s^2$ is positive & (r) is positive or negative according as (a) is +ve or -ve, we have maxima or minima according as (a) is (-ve) or (+ve) at $x = y = a$

2* Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ & determine their nature.

Solu 2* Let, $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ — ①

Partially diff w.r.t (x) ~~w.r.t~~ in eqⁿ ①

$$\frac{\delta u}{\delta x} = 4x^3 - 4x + 4y$$

At $\frac{\delta u}{\delta x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0$

$\Rightarrow x^3 - x + y = 0 \; \ldots (i)$

$x(x^2 - 1) + y = 0$ — ②

Also,
Partially diff w.r.t (y) in eqⁿ ①,

$$\frac{\delta u}{\delta y} = 4y^3 - 4y + 4x$$

At $\frac{\delta u}{\delta y} = 0$

$y^3 - y + x = 0 \; \ldots (ii)$

$y(y^2 - 1) + x = 0$ — ③ $\Rightarrow$ $\boxed{x = -y(y^2-1)}$

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~~From eqⁿ ②,~~

~~$[-y(y^2-1)]\,[\{-y(y^2-1)\}^2 + y] = 0$~~

From eqⁿ (i) & (ii),

$x^3 - x + y = 0$
$y^3 + x - y = 0$
$x^3 + y^3 = 0$

$\boxed{x^3 = -y^3}$

$\boxed{(x+y)(x^2+y^2-xy) = 0 \quad \therefore x + y = 0 \quad x = -y}$

$-y(y^2 - 1) = -y$

$y^2 - 1 = 1$

$y^2 = 2$

$\boxed{y = \pm\sqrt{2}}$ then, $\boxed{x = \pm\sqrt{2}}$

Hence, stationary points are ~~$\pm\sqrt{2}$ and~~ ~~$(\pm\sqrt{2}, \mp\sqrt{2})$~~.

$(0,0)$, $(\sqrt{2}, -\sqrt{2})$ & $(-\sqrt{2}, \sqrt{2})$ Ans

Case ①:- At $(0,0)$.

Now, $\frac{\delta^2 u}{\delta x^2} = 12x^2 - 4 = r$ | $r_{(0,0)} = -4$

$\frac{\delta^2 u}{\delta y^2} = 12y^2 - 4 = t$ | $t_{(0,0)} = -4$

$\frac{\delta^2 u}{\delta x\, \delta y} = 4 = s$ | $s_{(0,0)} = 4$

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Now, $rt - s^2 = (-4)(4) - (4)^2$
$= 16 - 16$

$rt - s^2 = 0$

Hence, the case is doubtful and further investigation is required. So, we shall leave this case

Case ② :- At $(\sqrt{2}, -\sqrt{2})$

$r_{(\sqrt{2}, -\sqrt{2})} = 20$, $t_{(\sqrt{2}, -\sqrt{2})} = 20$ & $s = 4$

Note, $rt - s^2 = (20)(20) - (4)^2$
$rt - s^2 = 384$
$\therefore rt - s^2 > 0$

Hence, $rt - s^2 > 0$ & $r > 0$, then given function (u) is minimum for $(\sqrt{2}, -\sqrt{2})$.

Now,
$u = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2 \times (\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$

$u = 4 + 4 - 4 - 8 - 4$
$u = -8$

Case ③ :- At $(-\sqrt{2}, \sqrt{2})$

$r_{(\sqrt{2}, \sqrt{2})} = 20$, $t_{(\sqrt{2}, \sqrt{2})} = 20$, $s = 4$
$rt - s^2 = 384 > 0$ & $r > 0$

Hence function u is minimum for $(\sqrt{2}, \sqrt{2})$ also. Ans

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3* Discuss the maxima & minima of function $u = \sin x \sin y \sin(x+y)$
or
If x, y & z are the angles of ΔABC, then find the maxima & minima of the function $u = \sin x \sin y \sin z$

Solu 3* Given: $u = \sin x \sin y \sin(x+y)$ ... (i)

Partially diff eqⁿ (i) w.r.t (x), we get,

$$\frac{\partial u}{\partial x} = \sin y [\sin x \cos(x+y) + \sin(x+y)\cos x]$$

$$\frac{\partial u}{\partial x} = \sin y \cdot \sin(2x+y)$$

Similarly,
$$\frac{\partial u}{\partial y} = \sin x \sin(2y+x)$$

Now,
$\frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$

$\sin y \sin(2x+y) = 0$ | $\sin x \sin(2y+x) = 0$
$\sin y = 0$ & $\sin(2x+y) = 0$ | $\sin x = 0$ & $\sin(2y+x) = 0$
$\therefore y = 0$ & $2x+y = 0$ or π or 2π | $x = 0$ & $2y+x = 0$ or π or 2π

$y = 0$ or $2x+y = 0, \pi, 2\pi$ ... (iii)
$x = 0$ or $2y+x = 0, \pi, 2\pi$ ... (iv)

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From eq^n (iv),

$$2y + x = \pi$$

Putting $y = \pi - 2x$ from eq^n (iii), we get,

$$2(\pi - 2x) + x = \pi$$
$$-3x = -\pi$$
$$\boxed{x = \pi/3}$$

Then, from eq^n (iii)

$$y = \pi - 2\left(\frac{\pi}{3}\right)$$

$$\boxed{y = \pi/3}$$

Similarly, on putting $y = 2\pi - 2x$ from eq^n (iii), in

$$2y + x = 2\pi \quad \text{(from (iv))}, \text{ we get}$$

$$2(2\pi - 2x) + x = 2\pi$$
$$-3x = -2\pi$$
$$\boxed{x = 2\pi/3}$$

Then, from eq^n (iii),

$$y = 2\pi - 2\left(\frac{2\pi}{3}\right)$$

$$\boxed{y = \frac{2\pi}{3}}$$

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Hence, ~~stat~~ critical points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $(2\pi/3, 2\pi/3)$.

Now,

$$r = \frac{\partial^2 u}{\partial x^2} = 2\sin y\cos(2x+y)$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2\sin x\cos(2y+x)$$

$$s = \frac{\partial^2 u}{\partial x\,\partial y} = \sin x\cos(2y+x) + \sin(2y+x)\cos x$$

$$s = \sin(2x+2y) = \sin\boxed{2(x+y)}$$

~~Then~~,

Case (1) :- At $(\pi/3, \pi/3)$

$$r = -\sqrt{3}, \quad t = -\sqrt{3} \quad \& \quad s = -(\sqrt{3}/2)$$

then,

$$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - (-\sqrt{3}/2)^2$$
$$= 3 - 3/4 = 9/4 > 0$$

Since, ~~But~~ $rt - s^2 > 0$. But, $r < -\sqrt{3}$

Hence, given function is maximum for $(\pi/3, \pi/3)$.

Now,

$$u = \sin(\pi/3)\sin(\pi/3)\sin(\pi/3 + \pi/3)$$

$$u = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$

$$\boxed{u_{max} = \frac{3\sqrt{3}}{8}}$$

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Case ②: at $(2\pi/3, 2\pi/3)$

$r = \sqrt{3}$, $t = \sqrt{3}$, $s = \sqrt{3}/2$

$$rt - s^2 = (\sqrt{3})(\sqrt{3}) - (\sqrt{3}/2)^2$$
$$= 3 - 3/4 = 9/4 > 0$$

Hence, $rt - s^2 > 0$ & $r > 0$

Since, given function is minimum for $(2\pi/3, 2\pi/3)$

Now,

$$u = \sin(2\pi/3) \sin(2\pi/3) \sin(2\pi/3 + 2\pi/3)$$

$$u = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{u_{min} = \frac{-3\sqrt{3}}{8}}$$