Disruption Optimization

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In this article, we look at an optimization model for reconfiguration. We assume that initially our system is in state S_0. From S_0, it can move into any state S_i with probability p_i. For each state S_i, we have a vector X_i of decision variables, which has a feasible region of F_i and determines the cost $C_i(X_i)$ of being in state S_i. We further assume that transition from S_0 to any state S_i results in reconfiguration expenses given by the function $L(X_0, X_i)$. The objective of our model is to minimize the expected reconfiguration cost $E_{\hat{X}}[L(X_0, \hat{X})]$, while remaining within ϵ_i of the optimal cost θ_i^* associated with each state S_i. Mathematically, the formulation is as follows:

$$\begin{aligned} \min \quad & E[L(X_0, \hat{X})] & \\ \text{s.t.} \quad & X_i \in F_i, & \forall i \in I \\ & C_i(X_i) \leq \theta_i^* + \epsilon_i, & \forall i \in I \end{aligned}$$

where the set I represents the index set of all possible system transition states, $E[L(X_0, \hat{X})] = \sum_i p_i L(X_0, X_i)$ and $\theta_i^* = \min_{X_i \in F_i} C_i(X_i)$.

We now illustrate the above formulation in terms of a shortest path problem. We assume that the initial state S_0 of our network is as shown in Fig. 1. The network experiences a disruption due to which the link $(2, 5)$ gets broken and the new state changes to S_1 (shown in Fig. 2). Clearly, with source as node 1 and destination as node 6, the shortest path costs θ_0^* and θ_1^* in networks 1 and 2 are respectively 2.5 (for path 1-2-5-6) and 3 (for path 1-3-4-6). We now define the reconfiguration cost function as $L(X_0, X_1) = \sum_{e \in E(S_0)} |X_{0,e} - X_{1,e}|$. In this expression, each component $X_{i,e}$ of the decision vector $X_i \in \{0, 1\}^{|E(S_i)|}$ takes on values 1 or 0 depending on whether edge $e \in E(S_i)$ belongs to the optimal path P_i chosen for state S_i or not. Summarizing the description above, we can generalize the formulation as follows:

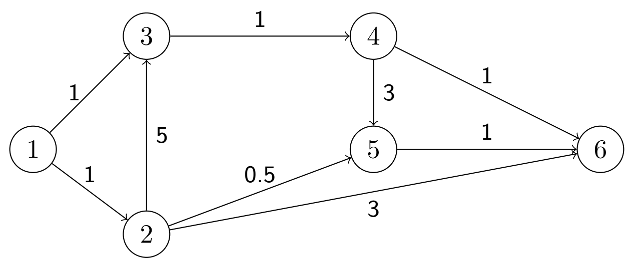

Figure 1: Initial network state, S_0

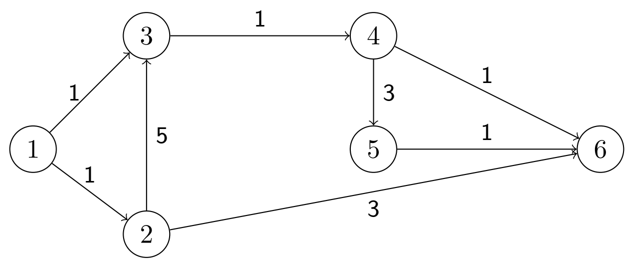

Figure 2: Disrupted network state, S_1

$$
\begin{aligned}
\min\ & \sum_{i \in I} p_i \left(\sum_{e \in E(S_0)} |X_{0,e} - X_{i,e}| \right) & \\
\text{s.t.}\ & \sum_{e \in E(Out(s))} X_{i,e} - \sum_{e \in E(In(s))} X_{i,e} = 1, & \forall i \in I \\
& \sum_{e \in E(Out(j))} X_{i,e} - \sum_{e \in E(In(j))} X_{i,e} = 0, & \forall i \in I, \forall j \in N - \{s,t\} \\
& \sum_{e \in E(Out(t))} X_{t,e} - \sum_{e \in E(In(t))} X_{t,e} = -1, & \forall i \in I \\
& \sum_{e \in E(S_i)} C_{i,e} X_{i,e} \leq \theta_i^* + \epsilon_i, & \forall i \in I \\
& X_{i,e} \in \{0,1\}, & \forall i \in I, \forall e \in E(S_i)
\end{aligned}
$$

In the above formulation, the set I represents the index set of possible transition states. $E(In(j))$ and $E(Out(j))$ denote respectively the sets of incoming and outgoing edges from node j. s and t represent the source and destination nodes respectively. Thus, for our simple shortest path example, $I = \{1\}$, p_1=1, $s = 1$, $t = 6$, $\theta_0^* = 2.5$ and $\theta_1^* = 3$.

We now consider the following cases in our example.
CASE 1: $\epsilon_0 < 0.5$ and $\epsilon_1 < 1$
In this case, the solution to the problem is P_0 = 1-2-5-6 and P_1 = 1-3-4-6. Since P_0 and P_1 do not have any edges in common, therefore, the total reconfiguration cost $L(X_0, X_1)$ for this case will be 6 .

CASE 2: $\epsilon_0 \geq 0.5$ and $\epsilon_1 < 1$
In this case, both paths P_0 and P_1 are exactly the same, namely 1-3-4-6. Since, there are no uncommon edges between the two paths, we get the total reconfiguration cost $L(X_0, X_1)$ for this case as equal to 0.

CASE 3: $\epsilon_0 < 0.5$ and $\epsilon_1 \geq 1$
In this case, the solution to the problem is P_0 = 1-2-5-6 (with path length equal to 2.5) and P_1 = 1-2-6 (with path length equal to 4). The edges that are uncommon between the paths are (2,5), (5,6) and (2,6). Hence, $L(X_0, X_1)$ for this case equals 3.

Indices and Sets

$t \in \mathscr{T}$	set of discrete time intervals ($\mathscr{T} = \{1, 2, ..., T\}$),
$i \in N$	set of all nodes ($N = \{1, 2, ..., n\}$),
$\omega \in \Omega$	set of all scenarios,
$Sink$	Sink node for all unmet demand
E	Set of all edges

Data

c_{ij}	per unit flow cost on edge (i,j)
$d_i^{\omega_t}$	equals 1 if node i is not disrupted at time t under scenario ω and 0 otherwise
M	a very big number

Decision Variables

$X_{i,j}^{k,\omega_t}$	flow for source node k on edge (i, j) during time period t under scenario ω
$y_i^{\omega_t}$	equals 1 if node i is active at time t under scenario ω and 0 otherwise
$z_i^{\omega_t}$	equals 1 if spare is deployed at node i under scenario ω at time t and 0 otherwise
$s_{k,Sink}^{\omega_t}$	flow from k to Sink representing unmet demand

$$\min \sum_{\omega \in \Omega} p^{\omega} \left(\sum_{t \in \mathscr{T}} \sum_{k \in N} \sum_{(i,j) \in E} c_{ij} X_{ij}^{k,\omega_t} + M \sum_{t \in \mathscr{T}} \sum_{k \in N} s_{k,Sink}^{\omega_t} \right)$$

$$\text{s.t.} \sum_{j \in V^+(i)} X_{i,j}^{k,\omega_t} - \sum_{j \in V^-(i)} X_{i,j}^{k,\omega_t} = -y_k^{\omega_t}, \qquad \forall k \in N, \forall i \in N, \forall t \in T, \forall \omega \in \Omega$$

$$\sum_{j \in V^+(k)} X_{k,j}^{k,\omega_t} + s_{k,Sink}^{\omega_t} = n y_k^{\omega_t}, \qquad \forall k \in N, \forall t \in T, \forall \omega \in \Omega$$

$$X_{i,j}^{k,\omega_t} \leq M y_i^{\omega_t}, \qquad \forall i \in N, k \in N, \forall (i,j) \in E, \forall t \in T, \forall \omega \in \Omega$$

$$y_i^{\omega_t} \leq y_i^{\omega_{t-1}} + z_i^{\omega_t}, \qquad \forall i \in N, \forall t \in T, \forall \omega \in \Omega$$

$$y_i^{\omega_t} \leq d_i^{\omega_t} + z_i^{\omega_t}, \qquad \forall i \in N, \forall t \in T, \forall \omega \in \Omega$$

$$\sum_{i \in N} \sum_{t \in \mathscr{T}} z_i^{\omega_t} \leq T, \qquad \forall \omega \in \Omega$$

Indices and Sets

$s \in S$	Set of all scenario supernodes
$u \in \mathscr{U}$	Set of reconstruction resources ($\mathscr{U} = \{1, 2, ..., U\}$),
$a \in N$	Set of all nodes ($N = \{1, 2, ..., n\}$),
$e \in E$	Set of all edges

Data

d_{ij}	equals 1 if edge (i, j) is not disrupted and 0 if it is disrupted

Decision Variables (all decision variables super-scripted with s)

$Y_a^{u,in}$	binary variable which equals 1 if resource u is at node a at the start and 0 otherwise
$Y_a^{u,out}$	binary variable which equals 1 if resource u is at node a at the end and 0 otherwise
R_e^u	binary variable which equals 1 if resource u is used to reconstruct edge e and 0 otherwise
X_{ij}^k	flow on edge (i, j) for node k commodity
Z_{ka}	binary variable which equals 1 if there exists a directed path from node k to node a and 0 otherwise

$$
\begin{aligned}
\min \sum_{s \in S} & p_s Cost(Y^{in,s}, Y^{out,s}) \\
\text{s.t.} \quad & \sum_{i \in V^-(a)} X_{i,a}^k - \sum_{i \in V^+(a)} X_{a,i}^k = Z_{ka}, && \forall a, k \in N \\
& \sum_{i \in V^-(a)} X_{i,a}^k - \sum_{i \in V^+(a)} X_{a,i}^k = -\sum_a Z_{ka}, && a \neq k, \forall k \in N \\
& \sum_{k \in N} \sum_{a \in N - \{k\}} Z_{ka} \geq 0.25n(n-1) \\
& X_{ij}^k \leq n \Bigg(\sum_{u \in \mathscr{U}} R_{ij}^u + d_{ij} \Bigg), && \forall k \in N, \forall (i,j) \in E \\
& \sum_{e \in E} R_e^u \leq 1, && \forall u \in \mathscr{U} \\
& R_e^u \leq Y_a^{u,in}, && \forall e \in Edges(a), \forall u \in \mathscr{U} \\
& \sum_{a \in N} Y_a^{u,in} \leq 1 && \forall u \in \mathscr{U} \\
& \sum_{a \in N} Y_a^{u,out} \leq 1 && \forall u \in \mathscr{U} \\
& \sum_{a \in N} Y_a^{u,out} \leq 1 - \sum_{e \in E} R_e^u && \forall u \in \mathscr{U} \\
& Y^{in,s} = Y^{out,parent(s)} && \forall s \in S
\end{aligned}
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