Classical Optimization Techniques

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Outline

Basics of Optimization

Un-constrained Problem: Minimize/ Maximize $f(x)$, x$\in\Omega$
$f(x)$: Objective function
x: [x_1, x_2, ..., x_n] is a decision variable
in Ω decision space.

Constrained Problem: Minimize/ Maximize $f(x)$, x$\in\Omega$
Such that g_i(x) $\leq$ 0, i=1, 2, ..., m (Inequality constraint)
h_j(x) = 0, j=1, 2, ..., p (Equality constraint)
x $\in \Omega$

Linear Programming Problems

Linear Programming problems are special case of optimization problems. Find $x_1, x_2, ...x_n$ which optimize the linear function $z = c_1x_1 + c_2x_2 + ...c_nx_n$
Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + ...a_{1j}x_j + ...a_{1n}x_n \ (\leq = \geq) \ b_1$$
$$a_{21}x_1 + a_{22}x_2 + ...a_{2j}x_j + ...a_{2n}x_n \ (\leq = \geq) \ b_2$$
$$\vdots$$
$$a_{i1}x_1 + a_{i2}x_2 + ...a_{ij}x_j + ...a_{in}x_n \ (\leq = \geq) \ b_i$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + ...a_{mj}x_j + ...a_{mn}x_n \ (\leq = \geq) \ b_m$$

and non-negative restrictions

$$x_j \geq 0, \quad j = 1, 2, ...n$$

where all a_{ij},b_{ij} and c_{ij} are constants and x_j are variables.

Practical Scenario: Example-1

Suppose a company manufactures two types electric vehicles say A and B.
Both vehicles go through two technical processes for manufacturing: First a technical manufacturing, second a finishing.

- Vehicle A requires 2 hours of manufacturing and 1 hour finishing.
- Vehicle B requires 1 hours of manufacturing and 2 hour finishing.

The manufacturing unit runs for **104 hours per month** and finishing unit runs for **76 hours per month**.
Profit on selling one vehicle of A is Rs. 6.00 lakhs and one vehicle B is Rs.11.00 lakhs.
The company can sale all that it produces.

How many of each type of electric vehicles should be manufactured to obtain best return?

Solution-1: Formulation as an optimization problem

Let the manufacturing company manufacture x_1 and x_2 vehicles of type A and B respectively.

$$Total\ \ profit : Z = 6x_1 + 11x_2 \quad (Rs)$$

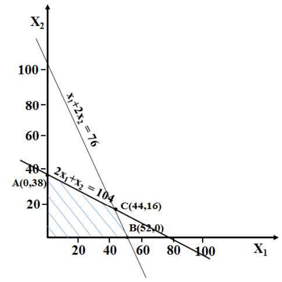
Total time to manufacture x_1 vehicles of type A and x_2 of type B is $2x_1 + x_2$.
Similarly, total time in finishing required is: $x_1 + 2x_2$

Since manufacturing has 104 hours: $2x_1 + x_2 \leq 104$.
Similarly finishing has 76 hours: $x_1 + 2x_2 \leq 76$.

So the company's manufacturing problem is expressed as,

$$\begin{aligned} Maximize\ \ Z &= 6x_1 + 11x_2 \\ Subject\ to\ constraints:\ \ 2x_1 + x_2 &\leq 104 \\ x_1 + 2x_2 &\leq 76 \\ x_1 \geq 0\ \ and\ \ x_2 &\geq 0 \end{aligned}$$

Solution-1: Geometrical (or Graphical Method)



Consider the constraints as equalities $2x_1 + x_2 = 104$ and $x_1 + 2x_2 = 76$

For $2x_1 + x_2 = 104$: x_1=0 $\Rightarrow$ x_2=104 and x_2=0 $\Rightarrow$ x_1=52

For $x_1 + 2x_2 = 76$: x_1=0 $\Rightarrow$ x_2=38 and x_2=0 $\Rightarrow$ x_1=76

$(2x_1 + x_2 = 104) \bigcap (x_1 + 2x_2 = 76)$: x_1=44 and x_2=16

Therefore, A(0,38), B(52,0) and C(44,16) are the feasible solutions.

For $Z = 6x_1 + 11x_2 = 0$, Z_A=(6x0)+(11x38) = 418

Z_B=(6x52)+(11x0) = 312

Z_C=(6x44)+(11x16) = 440 (Maximum value)

Therefore, C(44,16) is the optimal point and Z_c is the optimal value.

Example-2: Minimization Problem

Minimize $Z = 20x_1 + 10x_2$

Such that $x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$

$x_1, x_2 \geq 0$

Solution-2 by Graphical method:

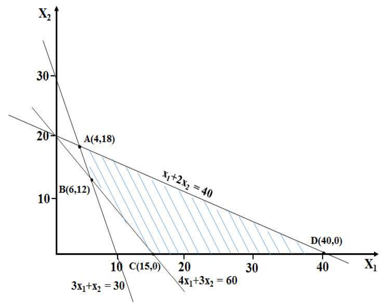
Consider the constraints as equalities $x_1 + 2x_2$ = 40, $3x_1 + x_2$ = 30 and $4x_1 + 3x_2$ = 60.

For $x_1 + 2x_2$ = 40: x_1=0 $\Rightarrow$ x_2=20 and x_2=0 $\Rightarrow$ x_1=40

For $3x_1 + x_2$ = 30: x_1=0 $\Rightarrow$ x_2=30 and x_2=0 $\Rightarrow$ x_1=10

For $4x_1 + 3x_2$ = 60: x_1=0 $\Rightarrow$ x_2=20 and x_2=0 $\Rightarrow$ x_1=15

Solution-2 by Graphical method



$(x_1 + 2x_2 = 40) \bigcap (3x_1 + x_2 = 30)$: x_1=4 and x_2=18
$(3x_1 + x_2 = 30) \bigcap (4x_1 + 3x_2 = 60)$: x_1=6 and x_2=12

Therefore, A(4,18), B(6,12), C(15,0), D(40,0) are feasible solutions and the region is feasible region.

For $Z = 20x_1 + 10x_2$, Z_A=(20x4)+(10x18) = 260
Z_B=(20x6)+(10x12) = 240 (minimum value)
Z_C=(20x15)+(10x0) = 300
Z_D=(20x40)+(10x0) = 800

Example-3 (Unbounded solution case)

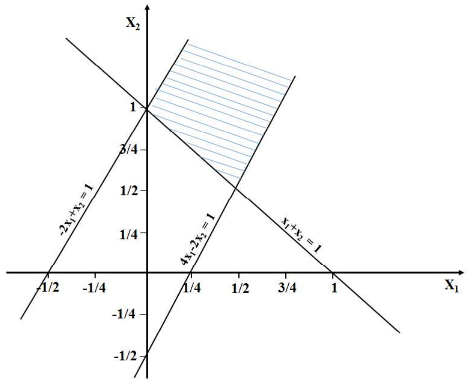
Maximize $Z = 4x_1 + 5x_2$
Such that $x_1 + x_2 \geq 1$
$-2x_1 + x_2 \leq 1$
$4x_1 - 2x_2 \leq 1$
$x_1, x_2 \geq 0$

Solution-3 by Graphical method:

Consider the constraints as equalities $x_1 + x_2 = 1$, $-2x_1 + x_2 = 1$ and $4x_1 - 2x_2 = 1$.

For $x_1 + x_2 = 1$: x_1=0 $\Rightarrow$ x_2=1 and x_2=0 $\Rightarrow$ x_1=1
For $-2x_1 + x_2 = 1$: x_1=0 $\Rightarrow$ x_2=1 and x_2=0 $\Rightarrow$ x_1=$\frac{-1}{2}$
For $4x_1 - 2x_2 = 1$: x_1=0 $\Rightarrow$ x_2=$\frac{-1}{2}$ and x_2=0 $\Rightarrow$ x_1=$\frac{1}{4}$

Solution-3 by Graphical method



From figure, the solutions are observed to be unbounded.

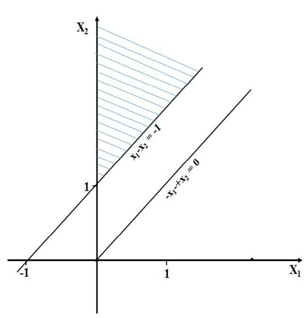
Example-4 (No solution case)

Maximize $Z = 3x_1 + 4x_2$

Such that $x_1 - x_2 \leq$ -1

$-x_1 + x_2 \leq 0$

$x_1, x_2 \geq 0$



For $x_1 - x_2$ = -1: x_1=0 $\Rightarrow$ x_2=1 and x_2=0 $\Rightarrow$ x_1=-1

For $-x_1 + x_2$ = 0: x_1=0 $\Rightarrow$ x_2=0 and x_2=0 $\Rightarrow$ x_1=0

Solution-4: No common point is observed and hence no feasible solution is obtained. Therefore, the solution is no solution.

Symmetric Dual Problems

Example-1: Find $x_1, x_2, ..., x_n$ which maximize $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\geq b_m \\ and\ x_1, x_2, ... + x_n &\geq 0 \end{aligned}$$

Solution-1: Find $w_1, w_2, ..., w_m$ which minimize $Z_D = b_1w_1 + b_2w_2 + ... + b_mw_m$ such that

$$\begin{aligned} a_{11}w_1 + a_{21}w_2 + ...a_{m1}w_m &\leq c_1 \\ a_{12}w_1 + a_{22}w_2 + ...a_{m2}w_m &\leq c_2 \\ &\vdots \\ a_{1n}w_1 + a_{2n}w_2 + ...a_{mn}w_m &\leq c_n \\ and\ w_1, w_2, ..., w_m &\geq 0 \end{aligned}$$

Symmetric Dual Problems

Find $x_1, x_2, ...x_n$ which maximize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$ such that

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\geq b_m \\ and\ x_1, x_2, ... + x_n &\geq 0 \end{aligned}$$

The dual problem of the linear programming is obtained by,

1. Transposing the coefficient matrix.
2. Interchanging the role of constant terms and coefficients of the objective function.
3. Reverting the inequalities.
4. Minimizing the objective function instead of maximizing it.

Symmetric Dual Problems

Example-2: Write the dual of the problem Minimize Z=3x_1+x_2
Such that 2x_1+3x_2 $\geq$ 2, x_1+x_2 $\geq$ 1 and x_1, x_2 $\geq$ 0.

Solution-2: Minimize Z = (3,1)[x_1, x_2] = cx
Such that $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, Ax $\geq$ b

Dual of the problem is Max Z = b'w = (2,1)[w_1 w_2] = 2w_1+w_2
Such that A'w $\leq$ c'

$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$2w_1 + w_2 \leq 3$$
$$3w_1 + w_2 \leq 1 \textit{ and } w_1, w_2 \geq 0$$

Symmetric Dual Problems

Example-3: Write the dual of the problem Maximize Z=x_1+2x_2
Such that 2x_1-3x_2 $\leq$ 3, 4x_1+x_2 $\leq$ -4 and x_1, x_2 $\geq$ 0.

Solution-3: Dual of the problem is Min Z_D = 3w_1-4w_2
Such that

$$2w_1 + 4w_2 \geq 1$$
$$-3w_1 + w_2 \geq 2 \textit{ and } w_1, w_2 \geq 0$$