Some additional developments and equations

Each of the chirps to be transmitted is a linear modulated continuous waveform, with sawtooth modulation, and has the form:

$$s_l^{TX}(t) = \cos\left(2\pi\left(f_{carrier}t + \int_0^t \frac{BW}{T_{chirp}}\tau d\tau\right)\right) = \cos\left(2\pi\left(f_{carrier}t + \frac{BW}{T_{chirp}}\frac{t^2}{2}\right)\right) \quad 0 < t < T_{chirp}, 1 \le l \le N_{chirp}$$

Tchirp is the sweep time for each chip (how long does one chirp last)

L is the nth number of chirp

M is the mth number of targets

Dm is the initial distance of target at the beginning of the first chirp (distance is constant Within one chirp)

(note: inst. frequency is derivative of phase, so phase is integral of frequency, and linear frequency changes thus lead to quadratic phase changes)

For each chirp, the received signal at a reference sensor/antenna will include a delay caused by the round trip to each target, with an initial radial distance (range) d_m at the beginning of the first chirp and, if the target is moving, an updated distance $d_m + v_m(l-1)T_{chirp}$ at the beginning of each chirp (this will create a Doppler effect).

$$s_l^{RX}(t) = \sum_{m=1}^{N_{targets}} s_l^{TX}\left(t - \frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right) = \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(f_{carrier}\left(t - \frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right) + \frac{BW}{2T_{chirp}}\left(t - \frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right)^2\right)\right)$$

(assume constant target speeds over all chirps during a scan, step-wise distance increment at each chirp (distance is constant within each chirp), same amplitude for each received target echo)

(neglect reflection coefficients, target size&orientation, dispersion, propagation, additive noise)

Note: if we consider only the carrier frequency component (i.e., no linear modulated continuous waveform), for moving objects the term observed over different chirps (along "slow time" $(l-1)T_{chirp}$, as opposed to "fast time" t), corresponds to the commonly found expression $\frac{-2v_m}{\lambda_{carrier}}$ for the Doppler shift or Doppler frequency in the literature:

$$s_l^{RX}(t) = \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(f_{carrier}\left(t - \frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right)\right)\right) = \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(f_{carrier}t - f_{carrier}\frac{2v_m}{c}(l-1)T_{chirp} - f_{carrier}\frac{2d_m}{c}\right)\right)$$

$$= \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(f_{carrier}t - \frac{2v_m}{\lambda_{carrier}}(l-1)T_{chirp} - \frac{2d_m}{\lambda_{carrier}}\right)\right) = \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(f_{carrier}t + f_{m,shift}(l-1)T_{chirp} - \frac{2d_m}{\lambda_{carrier}}\right)\right)$$

Back to the previous case including linear modulated continuous waveform, if we now consider all the receiver sensors (antennas) instead of only a reference sensor, then additional delays occur between the different sensor signals, depending on the azimuth angle of arrival (or direction of arrival, DOA) of each target:

$$s_{l,k}^{RX}(t) = s_l^{RX}\left(t - \frac{d_s(k-1)\sin(\theta_m)}{c}\right) \quad 1 \le k \le N_{antennas}$$

Ds is the distance between receiver antennas

Here, to simplify the derivation, instead of using the above delay, we use instead a resulting global phase shift applied only at the carrier frequency in the cosine signals (from $x(t-\tau) \xleftrightarrow{F.T.} X(\omega)e^{-j\omega\tau}$ and at $\omega_{carrier}$ this becomes $X(\omega_{carrier})e^{-j2\pi f_{carrier}\tau}$, i.e., phase shift of $-2\pi f_{carrier}\tau$).

$$s_{l,k}^{RX}(t) = \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(f_{carrier}\left(t - \frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right) + \frac{BW}{2T_{chirp}}\left(t - \frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right)^2 - f_{carrier}\frac{d_s(k-1)\sin(\theta_m)}{c}\right)\right) \quad 1 \le k \le N_{antennas}$$

The above equations (and polarity for the phase) assumes that the reference sensor (antenna) is at the rightmost location on the axis of sensors, which is perpendicular relative to the broadside (frontal) direction with azimuth $\theta = 0$. For positive azimuth angles, this leads to delays and negative phase shifts in sensor signals, relative to rightmost sensor signal.

Mixing of the transmitted signal $s_l^{TX}(t)$ and received signals $s_{l,k}^{RX}(t)$ is performed to produce a "beat signal", which can be sampled at a much lower frequency than the carrier frequency (terms in red are for successive simplifications):

(note: $\cos(a)\cos(b) = \frac{\cos(a-b)}{2} + \frac{\cos(a+b)}{2} = \frac{\cos(b-a)}{2} + \frac{\cos(a+b)}{2}$, and only the term $\frac{\cos(a-b)}{2} = \frac{\cos(b-a)}{2}$ will remain after low-pass filtering)

$$\begin{aligned}
y_{l,k}(t) &= \left(s_l^{TX}(t) \times s_{l,k}^{RX}(t)\right) * h_{lp}(t) \\
&= \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(\frac{2f_{carrier}(d_m + v_m(l-1)T_{chirp})}{c} + \frac{BW}{2T_{chirp}}\left(2\frac{2(d_m + v_m(l-1)T_{chirp})}{c}t - \left(\frac{2(d_m + v_m(l-1)T_{chirp})}{c}\right)^2\right) + f_{carrier}\frac{d_s(k-1)\sin(\theta_m)}{c}\right)\right) \\
&\approx \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(\frac{2f_{carrier}(d_m + v_m(l-1)T_{chirp})}{c} + \frac{BW}{T_{chirp}}\left(\frac{2(d_m + v_m(l-1)T_{chirp})}{c}t\right) + \frac{d_s(k-1)\sin(\theta_m)}{\lambda_{carrier}}\right)\right) \\
&\approx \sum_{m=1}^{N_{targets}} \cos\left(2\pi\left(\frac{2f_{carrier}(d_m + v_m(l-1)T_{chirp})}{c} + 2\frac{BW}{T_{chirp}}\frac{d_m}{c}t + \frac{d_s(k-1)\sin(\theta_m)}{\lambda_{carrier}}\right)\right) \quad 0 < t < T_{chirp}, 1 \le l \le N_{chirp}, 1 \le k \le N_{antennas}
\end{aligned}$$

(no scaling factor $\frac{1}{2}$ from the multiplication or overall gain of low-pass filter is considered above)

$y_{l,k}(t)$ are the low-pass filtered “beat” signals at the output of the mixers.

Although $y_{l,k}(t)$ is measured over $0 < t < T_{chirp}$ in practice, the equation above is only true for the part of $0 < t < T_{chirp}$ where $s_{l,k}^{RX}(t)$ includes the echo from $s_l^{TX}(t)$, i.e., there is a small portion at the beginning where $s_{l,k}^{RX}(t)$ does yet include the echo from $s_l^{TX}(t)$ and where a residual signal from the previous chirp may also be present. But this only represents a small portion of the interval $0 < t < T_{chirp}$ and it can be neglected.

Since the term $\frac{2f_{carrier}d_m}{c}$ does not change within a full radar scan (including all the chirps), as d_m is the initial range or radial distance at the beginning of the first chirp, it is a constant phase offset which can be dropped in the analysis (or in simulations):

$$y_{l,k}(t) \approx \cos\left(2\pi\left(2\frac{BWd_m}{T_{chirp}c}t + 2\frac{f_{carrier}v_m}{c}(l-1)T_{chirp} + \frac{d_s\sin(\theta_m)}{\lambda_{carrier}}(k-1)\right)\right)$$
$$= \cos\left(2\pi\left(2\frac{BWd_m}{T_{chirp}c}t + 2\frac{v_m}{\lambda_{carrier}}(l-1)T_{chirp} + \frac{d_s\sin(\theta_m)}{\lambda_{carrier}}(k-1)\right)\right) \quad 0 < t < T_{chirp}, 1 \le l \le N_{chirp}, 1 \le k \le N_{antennas}$$

We can observe the fluctuations in “fast time” (within a chirp) $\cos\left(2\pi\left(\frac{2BWd_m}{T_{chirp}c}t\right)\right)$, the fluctuations in “slow time” (across different chirps) $\cos\left(2\pi\left(\frac{2v_m}{\lambda_{carrier}}(l-1)T_{chirp}\right)\right)$, and the fluctuations across sensors $\cos\left(2\pi\left(\frac{d_s\sin(\theta_m)}{\lambda_{carrier}}(k-1)\right)\right)$. A 3-D Fourier transform can thus show the oscillations in those 3 axis, in particular the locations where the 3 oscillations occur jointly. Each of those oscillations has information about a different quantity (range, velocity, azimuth).

In digital systems or in simulations, the signal is discretized with a sampling rate $f_s = 1/T_s$ higher than twice the maximum “beat frequency”, i.e, twice the maximum frequency found in the “beat signal”, which is approximately $2\frac{BWd_m}{T_{chirp}c}$.

$$y_{l,k}[n] = \cos\left(2\pi\left(2\frac{BWd_m}{T_{chirp}c}nT_s + 2\frac{v_m}{\lambda_{carrier}}(l-1)T_{chirp} + \frac{d_s\sin(\theta_m)}{\lambda_{carrier}}(k-1)\right)\right) \quad 0 \le n \le \lfloor T_{chirp}/T_s \rfloor - 1, 1 \le l \le N_{chirp}, 1 \le k \le N_{antennas}$$

The different resulting discrete time frequencies involved ($2\frac{BWd_m}{T_{chirp}c}\frac{1}{f_s}$, $2\frac{v_m}{\lambda_{carrier}}T_{chirp}$, $\frac{d_s\sin(\theta_m)}{\lambda_{carrier}}$) can be used to obtain scaling factors required to map the default scales of the different discrete time Fourier transform frequencies (-0.5 to 0.5 cycle/sample), to obtain measurements of the initial range d_m, the radial speed v_m and the sine of azimuth angle $\sin(\theta_m)$.

Also, it is easy to show that the 3D Fourier transform is a separable transform, and can be computed with successive 1-D Fourier transforms (FFTs) applied separately to each dimension:

$$Y(\omega_r,\omega_v,\omega_{\sin\theta}) = \sum_{n=0}^{\lfloor T_{chirp}/T_s\rfloor-1}\sum_{l=1}^{N_{chirps}}\sum_{k=1}^{N_{antennas}} y_{l,k}[n]e^{-j(\omega_r n+\omega_v(l-1)+\omega_{\sin\theta}(k-1))} = \sum_{k=1}^{N_{antennas}}\left(\sum_{l=1}^{N_{chirps}}\left(\sum_{n=0}^{\lfloor T_{chirp}/T_s\rfloor-1} y_{l,k}[n]e^{-j\omega_r n}\right)e^{-j\omega_v(l-1)}\right)e^{-j\omega_{\sin\theta}(k-1)}.$$

Moreover, in practice the discrete time angular frequencies $\omega_r,\omega_v,\omega_{\sin\theta}$ are also discretized, and the Fourier transforms are computed using FFTs.

Some other relations for linear modulated continuous waveform radars, found from references:

```
BW_chirp = c/(2*Dres); % Hz, bandwidth (eq. 1, "Radar Sensor Signal Acquisition and Multidimensional FFT Processing for Surveillance Applications in
                       % Transport Systems")
                       % confirmed for LMCW radar https://dsp.stackexchange.com/questions/50431/deriving-the-resolution-equation-of-an-fmcw-radar)

f_beat_max = abs(2*Vmax/lambda_carrier) + K_chirp * 2 * Dmax/c;  % equation fbeat = -2*V/lambda_carrier + K_chirp * 2 * D/c resulting from linearly
                  % increasing inst. frequency, valid in steady state section when both emitter and receiver signals observed in beat signal
                  %(eq. 4, "Radar Sensor Signal Acquisition and Multidimensional FFT Processing for Surveillance Applications in Transport Systems")

Vres = lambda_carrier/(2*nb_chirps*T_chirp);% (?) eq.3.3 "LMCW-radar. Signal processing and parameter estimation". Here will give same result as
               % initial specification if lambda_carrier=d (with spatial_aliasing)
               % but factor of 2 difference if deduced from data in p.3 left column with T_chirp replaced by
              "nb_chirp_samples / fs", "Radar Image Reconstruction from Raw ADC Data using Parametric Variational Autoencoder with Domain Adaptation"

Vmax = Vres * (nb_chirps/2) % from previously used nb_chirps = 2* round(Vmax_kmh / Vres_kmh); % p.609 1st par 2nd column, "Radar Sensor Signal
                            % Acquisition and Multidimensional FFT Processing for Surveillance Applications in Transport Systems"

Dmax = Dres * (chirp_nb_samples/2) % similar form to above, deduced from p.3 left column,
                                   %"Radar_Image_Reconstruction_from_Raw_ADC_Data_using_Parametric_Variational_Autoencoder_with_Domain_Adaptation"
                                   % Dres depends only on BW and c, so it will be valid and Dmax can be computed from it.

theta_res_worst_case = lambda_carrier / (2*(nb_sensors-1)*d); % rad. (angle dependent, worst case of
                                                               % theta_res = lambda_carrier / (2*(nb_sensors-1)*d*cos(theta)),
                   % eq. 2, "Radar Sensor Signal Acquisition and Multidimensional FFT Processing for Surveillance Applications in Transport Systems")
theta_res_degrees_worst_case = theta_res_worst_case /pi*180 % degrees

target_cross_range_res_worst_case = theta_res_worst_case * target_D
              % m, eq. 3, "Radar Sensor Signal Acquisition and Multidimensional FFT Processing for Surveillance Applications in Transport Systems")
```