Module-1 (Complex Numbers)

*) $z = x+iy = re^{i\theta}$ ($r=\sqrt{x^2+y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$).

$\Rightarrow x = r\cos\theta$, $y = ir\sin\theta$.

* De moivre's theorem.

$\rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

* Binomial theorem

$\rightarrow (\cos\theta + i\sin\theta)^n = {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta(\sin\theta i) + {}^nC_2\cos^{n-2}\theta\sin^2\theta i^2 - - -$

$= \cos n\theta + i\sin n\theta \quad - - - + {}^nC_n(i\sin\theta)^n$.

* $x^n + \frac{1}{x^n} = 2\cos n\theta$: $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

* use for combining homogeneous terms. to convert θ to $n\theta$/ $n\theta$ to θ.

$(x+\frac{1}{x})^n = (2\cos\theta)^n$; $(x-\frac{1}{x})^n = (2i\sin\theta)^n$.

* Roots

i) $x^3=1$, $\Rightarrow$ Cube roots of unity $\Rightarrow 1(e^{i2k\pi}), \omega(e^{i\frac{2k\pi}{3}}), \omega^2(e^{\frac{i2k\pi}{6}})$

$\Rightarrow n^{th}$ roots of unity $\Rightarrow e^{i2k\pi/n}$, put $n=0,1,2,--,n$ for roots.

* Circular functions (Period $\rightarrow 2\pi$)

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$; $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$; $\tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$

* Hyperbolic function (Period $\rightarrow 2\pi i$)

$\rightarrow \sin ix = i\sinh x$; $\sinh ix = i\sin x$

$\cos ix = \cosh x$; $\cosh ix = \cos x$

$\tan ix = i\tanh x$; $\tanh ix = i\tan x$

$\rightarrow \sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$, $\tanh x = \frac{e^x-e^{-x}}{e^x+e^{-x}}$

$\rightarrow e^x = \cosh x + \sinh x$; $e^{-x} = \cosh x - \sinh x$

$\rightarrow \sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$; $\tanh(-x) = -\tanh x$.

$\rightarrow \cosh^2 x - \sinh^2 x = 1$;

$\text{sech}^2 x + \tanh^2 x = 1$;

$\coth^2 x - \text{cosech}^2 x = 1$.

$\rightarrow \sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

$\rightarrow \sinh 3x = 3\sinh x + 4\sinh^3 x$

$\cosh 3x = 4\cosh^3 x - 3\cosh x$

$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$.

*

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

$\rightarrow \sinh x \xrightarrow{D} \cosh x$

$\cosh x \xrightarrow{D} \sinh x$

$\tanh x \xrightarrow{D} \text{sech}^2 x$

$\rightarrow \cosh x + \sinh x = \cosh nx + \sinh nx$

Inverse hyperbolic function.

i) $\sinh^{-1}x = \log(x+\sqrt{x^2+1})$ $\rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$.

ii) $\cosh^{-1}x = \log(x+\sqrt{x^2-1})$ $\rightarrow \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$.

iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ $\rightarrow \int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$.

$\Rightarrow \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$; $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$; $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$.

Separation of Real & Imaginary.

1) $\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$.

2) $\cos(x+iy) = \cos x\cosh y - i\sin x\sinh y$.

3) $\tan(x+iy) = \frac{\sin 2x}{\cos 2x+\cosh 2y} + \frac{i\sinh 2y}{\cos 2x+\cosh 2y}$.

4) $\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$.

5) $\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$.

6) $\tanh(x+iy) = \frac{\sinh 2x}{\cosh 2x+\cos 2y} + \frac{i\sin 2y}{\cosh 2y+\cos 2y}$.

$\rightarrow \tan(\alpha+i\beta) = x+iy$; $\tan(\alpha-i\beta) = x-iy$

$\Rightarrow \tan 2\alpha = \frac{2x}{1-(x^2+y^2)} = \tan[(\alpha+i\beta)+(\alpha-i\beta)]$

Logarithm of Complex Numbers.

$x+iy = re^{i\theta}$; $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

$\Rightarrow \log(x+iy) = \log r + i\theta = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$= \log r + i(2k\pi+\theta)$.

Module-2 (Diffrential Calculus).

* Derivative of n^{th} order.

1) $y = (ax+b)^m$

$y_n = (m)(m-1)(m-2) \ldots (m-n+1)a^n(ax+b)^{m-n}$: $n<m$

$= m!\,a^n$: $n=m$

$= 0$: $n>m$.

i) $y = \frac{1}{ax+b}$; $y_n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$ ii) $y = \frac{1}{x+b}$; $y_n = \frac{(-1)^n n!}{(x+b)^{n+1}}$

2) $y = \log(ax+b)$: $y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

3) $y = a^{mx}$; $y_n = m^n a^{mx}(\log a)^n$. $y = e^{mx}$; $y_n = m^n e^{mx}$

4) $y = \sin(ax+b)$; $y_n = a^n \sin(ax+b+\frac{n\pi}{2})$.

$y = \cos(ax+b)$; $y_n = a^n\cos(ax+b+\frac{n\pi}{2})$

5) $y = e^{ax}\sin(bx+c)$: $y_n = (a^2+b^2)^{n/2} e^{ax}\sin(bx+c+n\tan^{-1}(\frac{b}{a}))$

$y = e^{ax}\cos(bx+c)$: $y_n = (a^2+b^2)^{n/2} e^{ax}\cos(bx+c+n\tan^{-1}(\frac{b}{a}))$

* Leibnitz's theorem (Convert in terms of y).

if $y = uv$:

$\Rightarrow y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots + {}^nC_n u v_n$.

$\Rightarrow y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2}v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3}v_3 + \ldots$

* Maclaurin's Series.

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^{n}(0) + \ldots$

★ Expansion

i) $e^x = 1 + x + \frac{x^2}{2!} + \ldots$

ii) $e^{-x} = 1 - x + \frac{x^2}{2!} - \ldots$

iii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$; $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

iv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$; $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

v) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$; $\tanh x = x - \frac{x^3}{3!} + \frac{2x^5}{15} - \ldots$

vi) $\log(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots$

$\log(1-x) = -x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} \ldots$

vii) $(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots$

★ Method of diffrentiation & Integration.

↳ used for expansion.

viii) $\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

Taylor Series ⇒ Expand in powers of $(x-a)$...

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

Rolle's The^m ($f(x)$)

i) C in $[a,b]$, ii) D in (a,b), iii) $f(a) = f(b)$

then there exists atleast one c such that $f'(c) = 0$

LMVT ($f(x)$)

i) C in $[a,b]$, ii) D in (a,b),

then there exist a c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

CMVT ($f(x)$ & $g(x)$)

i) C in $[a,b]$ ii) D in (a,b)

then $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Infinite series

* limit is finite ⇒ convergent

limit is infinite ⇒ divergent

limit does not ~~exist~~ unique ⇒ oscillatory.

P-test

$\sum_{n=1}^{\infty} \frac{1}{n^P}$ $\quad P>1$ ⇒ convergent.

$\quad P \leq 1$ ⇒ divergent.

Comparison test

if $\sum U_n$ & $\sum V_n$ are series of +ve terms.

⇒ $\lim_{n\to\infty}\left(\frac{U_n}{V_n}\right) = l$ (finite & non-zero)

⇒ U_n & V_n both converge/diverge.

D'Alembarts Ratio test.

$\sum U_n$ is +ve series.

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$

i) $\sum U_n$ is convergent $l<1$

ii) $\sum U_n$ is divergent if $l>1$.

iii) Ratio test fails if $l=1$.

Module-3 (Partial Differention).

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ & $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$ & $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$.

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right)$.

→ Commutative property.

$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$.

→ $\left(\frac{\partial u}{\partial x}\right)_y$ ⇒ Keep y (constant).

★ Chain rule.

$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t} \cdot \frac{\partial t}{\partial x}$; $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial t} \times \frac{\partial t}{\partial y}$.

★ Composite function (one variable).

$\Rightarrow u = f(x,y)$; $x = \phi(t)$; $y = \psi(t)$

$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial t}$.

$\Rightarrow u = f(x,y,z)$; $x = \phi(t)$; $y = \varphi(t)$; $z = \delta(t)$

$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial t}$.

★ Composite function (two Variable)

$\Rightarrow z = f(x,y)$; $x = \phi(u,v)$; $y = \varphi(u,v)$

$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$ & $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$.

★ Implicit function.

⇒ if $f(x,y) = c$

$\Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$

★ Euler's theorem [Homogeneous functions] (2 Variables)

$u = f(x,y)$, $u = f(X,Y) = t^n f(x,y)$

where, $X = xt$, $Y = yt$ & (n) is deegre of ~~fun~~ Hom. function

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

★ Euler's theorem [Homogeneous function] (3 variables)

$u = f(x,y,z)$, $u = f(X,Y,Z) = t^n f(x,y,z)$

where, $X = xt$, $Y = yt$ & (n) is deegre of Hom. function

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$.

→ Deduction's from Euler's Theorem.

1) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

2) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$,

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ $\rightarrow z = f(u)$ & $z = f(x,y)$

Similarly

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$

$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = nP$ $\rightarrow P = f(u)$ & $P = f(x,y,z)$

3) $z = f(u)$, $g(u) = n\frac{f(u)}{f'(u)}$
↳ degree (n).

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$.

★ Error's & Approximation.

$u = f(x,y,z) \Rightarrow \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\times\delta y + \frac{\partial u}{\partial z}\delta z$.

$\therefore \delta x \Rightarrow$ Absolute error.

$\frac{\delta x}{x} \Rightarrow$ Relative error.

$\frac{\delta x}{x}\times 100 \Rightarrow$ Percentage error.

$\rightarrow$ Approx $(x) = x + \delta x$.

★ Maxima & Minima $(f(x,y))$

1) Solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ simultaneously for x & y.

2) obtain values, $r = \frac{\partial^2 f}{\partial x^2}$; $s = \frac{\partial^2 f}{\partial x\partial y}$; $t = \frac{\partial^2 f}{\partial y^2}$.

3) If i) $rt - s^2 > 0$ $\xrightarrow{\&}$ $r < 0$ (or $t < 0$) $\Rightarrow$ Maximum.
$\xrightarrow{\&}$ $r > 0$ (or $t > 0$) $\Rightarrow$ Minimum.

ii) $rt - s^2 < 0 \Rightarrow$ Neither maximum nor minimum.
i.e saddle point.

iii) $rt - s^2 = 0 \Rightarrow$ failure point (Non-Conclusion).

*Langrange's method of Undetermined Multipliers.

→ $\phi(x,y,z)=0$ ① → condition for $f(x,y,z)$[stationory]
[Maximum & Minimum]

→ $F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$.

⇒ $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}$ ② ; $\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}$ ③ ; $\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}$ ④.

→ find λ using ①, ②, ③, ④.

Module-4 (Matrices & Vectors).

→ diagonal matrix ⇒ All non-diagonal = 0

→ Trace = Sum of Principal diagonal elements.

→ Transpose ⇒ Interchange row by col & col by row.

→ Symmetric matrix ⇒ $a_{ij} = a_{ji}$ ⇒ $A = A^T$

→ Skew Symmetric matrix ⇒ $a_{ij} = -a_{ji}$; $a_{ii} = 0$ ⇒ $A = -A^T$

→ Transposed Conjugate ⇒ $A^\theta = \overline{A^T}$

→ Hermitian matrix ⇒ $A = A^\theta$ $(a_{ij} = \overline{a_{ji}})$

→ Skew Hermitian matrix ⇒ $A = -A^\theta$ $(a_{ij} = -\overline{a_{ji}})$ & diagonal elements are either ~~purely real~~ ~~or~~ purely imagenery or zero

→ Orthogonal matrix

i) for real matrix ⇒ $A_1^T A_2 = A_1 A_2^T = 0$

ii) for ~~the~~ complex matrix ⇒ $A_1^\theta A_2 = A_1 A_2^\theta = 0$

→ Unitory matrix ⇒ $AA^\theta = I$

→ Singular ⇒ $|A| = 0$ &

Non-Singulorr ⇒ $|A| \neq 0$.

→ Minor ⇒ $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ & Cofactor ⇒ $a_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

→ Adjoint is transpose of Cofactors (Adj A).

→ Inverce ⇒ $A^{-1} = \frac{1}{|A|}\text{Adj}A$ & $AA^{-1} = I_n$.

→ Solution of non homogeneous by inverce.

⇒ $AX = B$ ⇒ $X = A^{-1}B$.

→ Equilvance ⇒ $A \sim B$

if B is obtained by elementorry transformation of A.

$\underset{(a)}{A} = \underset{(b)}{I_n}\,\underset{(c)}{A}$ ⇒ $\underset{(a)}{I_n} = \underset{(b)}{B}\,\underset{(c)}{A}$ ⇒ $B = A^{-1}$

Reduce from row echlon ⇒ $I_n = A^{-1}A$

★ Rank of A Matrix (if ($\rho(A) = r$))

→ at least one minor of order (r) which is non-zero.

→ Every minor of order greater than r is zero.

i) Row-echlon form [Upper triangular matrix]

$\Rightarrow \rho(A) =$ No. of Non zero rows in row echlon form

ii) Reduced normal form / Canonical form.

Conversion of matrix $A = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = r$

iii) PAQ form

$\Rightarrow$ if A is matrix of order $m \times n$

$\Rightarrow [A] = PAQ = [I_m]A[I_n]$ ($\because P = I_m$ & $Q = I_n$) & $\boxed{A^{-1} = Q \cdot P}$

★ Non homogeneous linear Equations ($AX = B$)

$[A|B] = [A:B] = [A \,\vdots\, B]$ → Augmented matrix.

a) consistent $\rho(A) = \rho(A:B) \Rightarrow$ One or more solution

i) If $\rho(A) = \rho(A:B) = n \Rightarrow$ Unique solutions.

ii) If $\rho(A) = \rho(A:B) < n \Rightarrow$ Infinite Solutions.

b) Inconsistent $\rho(A) \neq \rho(A:B) \Rightarrow$ No solution

★ homogeneous linear equations ($AX = 0$)

↳ System is consistent when eq^n are homogeneous.

a) Non trivial (Non-zero Sol^n) $\Rightarrow \rho(A) < n$ & $|A| = 0$

b) Trivial (All zero) $\Rightarrow (x_1 = x_2 = \cdots = x_n = 0) \Rightarrow \rho(A) = n$ & $|A| \neq 0$

★ linear Dependence & Independence of Vectors.

$X_1, X_2, X_3, \ldots, X_r \Rightarrow$ Set of r Vectors.

$K_1, K_2, K_3, \ldots, K_r \Rightarrow$ Set of r Scalars.

such that $K_1X_1 + K_2X_2 + K_3X_3 + \cdots + K_rX_r = 0$ → solve like non-homogeneous

a) linear dependence ($\rho(A) < n$ & $|A| = 0$) if $K_1, K_2, K_3, \ldots K_r$ are not all zero.

b) linearly independence ($\rho(A) = n$ & $|A| \neq 0$) if $K_1 = K_2 = K_3 = \cdots = K_r = 0$

★ for dependent vectors

No. of parameters = No. of Unknowns (n) − Rank of Coeficient Matrix ($\rho(A)$).