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Module-1

Gamma function

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

iv) $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

i) $\Gamma(n+1) = n\Gamma n = n!$ / $\Gamma n = (n-1)!$

ii) Not defined for $x = 0, -1, -2, \ldots$

iii) if n is negative use

$\Gamma n = \Gamma(n+1)/n$

Type : 01 $\Rightarrow \int_0^\infty e^{-ax^n} dx = \frac{1}{n(a)^{1/n}} \Gamma\frac{1}{n}$, put $ax^n = t$

Type : 02 $\Rightarrow \int_0^\infty x^m e^{-ax^n} dx = \frac{1}{n \times (a)^{m+1/n}} \Gamma\frac{m+1}{n}$, put $ax^n = t$

Type 03 : $\Rightarrow \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$, put $\log x = -t$

Type 04 : $\Rightarrow \int_0^\infty \frac{x^a dx}{a^x} = \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$, put $a^x = e^t$

Beta function

(i) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

(ii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x \, dx$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Properties

(i) $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

ii) $\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\, \Gamma 2m}{2^{2m-1}}$

iii) $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Type 01 : $\Rightarrow \int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \times \beta(m+1, n+1)$

put $(x-a) = (b-a)t$

Type 02 : $\Rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{\beta(m,n)}{(a+b)^m \cdot a^n}$

put $\frac{x}{a+bx} = \frac{t}{a+b}$.

Type 03 : $\Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m,n)$, put $x = \tan^2\theta$.

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Type 04: $\Rightarrow \int_0^{\pi} \frac{\sin^{n-1}x\, dx}{(a+b\cos x)^n} = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} \beta(n/2, n/2)$

put $\tan\frac{x}{2} = t$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

Type 05: $\Rightarrow \int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^{2m}b^{2m}}$

Divide by $\cos^2\theta$ to convert into tan & sec form

Type 06: $\Rightarrow \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$

⊛ DUIS (Diffrentiation under integral sign).

1. Constant limits.

$I = \int_{\alpha}^{\beta} f(x,a)dx$, then $\frac{dI}{da} \int_{\alpha}^{\beta} \left(\frac{\partial f}{\partial a}\right) dx$

where α, β are constants & does not depend on a

2. When limits are a function of (a).

$I = \int_{\phi(a)}^{\psi(a)} f(x,a)\, dx$

then $\frac{dI}{da} = \int_{\phi(a)}^{\psi(a)} \left(\frac{\partial f}{\partial a}\right) dx + \frac{\partial \psi}{\partial a}[f(\psi(a), a)] - \frac{\partial \phi}{\partial a}[f(\phi(a), a)]$

Module - 3.

D.E of 1st order & 1st Deegre.

1. Variable Seprable $\Rightarrow \frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow \int g(y)\,dy = \int f(x)\,dx + C$

2. Homogeneous Equations.
   → Deegre of every term is same.
   → put $y = Vx$, then apply Variable seprable

3. Non-Homogeneous Equations.
   if $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$

   put $x = x' + h$ & $y = y' + k$ (when $\frac{a}{a'} \neq \frac{b}{b'}$)

   $$\frac{dy'}{dx'} = \frac{a(x'+h) + b(y'+k) + c}{a'(x'+h) + b'(y'+k) + c'}$$

   Such that $ah + bk + c = 0$, $a'h + b'k + c' = 0$ → obtain (h, k)

   then equation is converted into homogeneous
   the substitute values of x & y in place of x' & y'.

4. Exact Diffrential Equations.
   $$Mdx + Ndy = 0$$
   Condition $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

   Solution $\Rightarrow \int Mdx$ (treating y constant) $+ \int$ (Terms free from x in N) $dy = C$
   → Alternatively.
   $\int Ndy$ (treating x constant) $+ \int$ (Terms free from y in M) $dx = C$

5. Non-Exact diffrential Equations
   → $Mdx + Ndy = 0$ where $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

   Case i:- M & N are homogeneous eq^n in x and y
   then $I.F = \frac{1}{Mx + Ny}$

   Case ii:- if equation is form $f(xy)\,y\,dx + g(xy)\,x\,dy = 0$
   then $I.F = \frac{1}{Mx - Ny}$ here $Mx - Ny \neq 0$

Case iii :- if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ ↳ function of y-only.

then $I.F = e^{\int f(y)dy}$

Case iv :- if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ ↳ function of x-only.

then $I.F = e^{\int f(x)dx}$

Then Next Step $\Rightarrow M^* = M \times (IF)$; $N^* = N \times (IF)$.

∴ General Solution is

$$\int M^* dx \left(\text{treating } y \text{ Constant}\right) + \int \left(\text{Terms free from } x \text{ in } N^*\right) dy = c$$

Case V :- Eqⁿ $x^{m_1}y^{m_2}(a_1 y dx + b_1 y dy) + x^{n_1}y^{n_2}(a_2 y dx + b_2 x dy) = 0$

then $I.F = x^h y^k$

where $\frac{m_1+h+1}{a_1} = \frac{m_2+k+1}{b_1}$ & $\frac{n_1+h+1}{a_2} = \frac{n_2+k+1}{b_2}$

6. Linear Diffrential Equations

→ $\frac{dy}{dx} + P(x)y = \phi(x)$ / $\frac{dx}{dy} + P(y)x = \phi(y)$

→ $I.F = e^{\int P(x)dx}$ / $I.F = e^{\int P(y)dy}$.

→ Solⁿ ⇒ $y \times (I.F) = \int \phi(x) \times (I.F).dx + c$ / Solⁿ ⇒ $x \times (I.F) = \int \phi(y) \times (I.F) dy + c$

7. Non-linear Diffrential Equation

→ $\frac{dy}{dx} + P(x)y = \phi(x)y^n$

$$\frac{1}{y^n}\left(\frac{dy}{dx}\right) + P(x) \times \left(\frac{1}{y^{n-1}}\right) = \phi(x)$$

put $y^{1-n} = t$, then solve as linear D.E & then substitute $t = y^{1-n}$

→ RC-circuit ⇒ $\frac{dq}{dt} + \left(\frac{1}{RC}\right) q = \frac{V}{R}$ / $\frac{di}{dt} + \left(\frac{1}{RC}\right) i = \frac{1}{R}\frac{dv}{dt}$

→ R-L Circuit ⇒ $\frac{di}{dt} + \left(\frac{R}{L}\right) i = \frac{V}{L}$

→ Mechanical systems ⇒

$v = \frac{dx}{dt}$; $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}$

→ Growth of y w.r.t time ⇒ $\frac{dy}{dt} = ky$

→ Decay of y w.r.t time ⇒ $\frac{dy}{dt} = -ky$

→ Newton's law of cooling ⇒ $\frac{dT}{dt} = -K(T - T_0)$

dT → change in tempreture.

T → Tempreture of body.

T_0 → Tempreture of Surrounding.

Module-4.

HODE

★ for C.F

if roots of Auxillary equation are

i) $m_1 \neq m_2$ ⇒ C.F = $c_1 e^{m_1 x} + c_2 e^{m_2 x}$

ii) $m_1 = m_2$ ⇒ C.F = $(c_1 + c_2 x) e^{m_1 x}$

iii) $m_1 = \alpha + ib$, $m_2 = \alpha - ib$ ⇒ C.F = $e^{\alpha x}(c_1 \cos\beta x + c_2 \cancel{\cos} \sin\beta x)$

iv) $m_1 = m_3 = \alpha + ib$, $m_2 = m_4 = \alpha - ib$

⇒ C.F = $e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

★ for P.I

$f(D)\, y = X$ ⇒ P.I ; $y = \frac{1}{f(D)} X$.

i) $\frac{1}{D} X = \int X dx$ ii) $\frac{1}{D-a} X = e^{ax}\int e^{-ax} \cdot X\, dx$ iii) Partial fractions.

$\frac{1}{D+a} X = e^{-ax}\int e^{ax} \cdot X\, dx$

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i) General Method $\Rightarrow y(x) = y_{C.F} + y_{P.I}$

ii) Short-Cut Method $(f(D)\cdot y = \varphi(x))$ if $f(a)=0$

a) $\varphi(x) = e^{ax+b} \Rightarrow P.I = \frac{1}{f(a)} \cdot e^{ax+b}$; $\Rightarrow P.I = \frac{x^r}{f^r(a)} e^{ax+b}$.

b) $\varphi(x) = \sin(ax+b) / \cos(ax+b)$, $f(D) = \phi(D^2)$

$P.I = \frac{\varphi(x)}{\phi(-a^2)}$; if $\phi(-a^2) = 0$ $P.I = \frac{x^{r+1}}{\phi^{(r+1)}(-a^2)} \varphi(x)$

c) $\varphi(x) = x^m$ $f(D) = 1+\phi(D)$

$P.I = \frac{\varphi(x)}{f(D)} = \frac{x^m}{(1+\phi(D))} = (1+\phi(D))^{-1} x^m$

now by expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

d) $\varphi(x) = e^{ax}.V$

$P.I = e^{ax}\left[\frac{1}{f(D+a)} \cdot V\right] \longrightarrow$ then apply above cases.

e) $\varphi(x) = xV$

$P.I = \left\{x - \frac{f'(D)}{f(D)}\right\} \frac{1}{f(D)} . V$. $\longrightarrow$ then apply above cases

iii) Method of Variation of parameters.

$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = \varphi(x)$ if $C.F = C_1y_1 + C_2y_2$

let $P.I = V_1(x)y_1 + V_2(x)y_2$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$; $V_1(x) = -\int \frac{y_2\varphi(x)dx}{W}$; $V_2(x) = \int \frac{y_1 \varphi(x) dx}{W}$.

iv) Cauchy Euler LDE's

$z = \log x$, $x = e^z$, $z = \log x$

$\cancel{\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}}$, $\cancel{\Rightarrow x^2\frac{d^2y}{dx^2}}$

let $\frac{dz}{dx} = \frac{1}{x}$, $Dy = \frac{dy}{dx}$, $\mathcal{D}y = \frac{dy}{dz}$.

i) $xDy = \mathcal{D}y$

ii) $x^2D^2y = \mathcal{D}(\mathcal{D}-1)y$

iii) $x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$

~~Sto~~ Solve in z & at last substitute $z = \log x$

v) Legendre's Linear Equations

$$(a_0(ax+b)^n D^n + a_1(ax+b)^{n-1} D^{n-1} + \cdots + a_n)y = \varphi(x)$$

In this case put $ax+b = e^z$

$$\Rightarrow x = \frac{e^z - b}{a} \;;\; z = \log(ax+b)$$

$\rightarrow D \equiv \frac{d}{dx}$; $\mathcal{D} = \frac{d}{dz}$

~~i) $Dy = \frac{b}{(ax+b)} \mathcal{D}y$~~

i) $(ax+b)Dy = b\mathcal{D}y$

ii) $(ax+b)^2 D^2 y = b^2 \mathcal{D}(\mathcal{D}-1)y$

iii) $(ax+b)^3 D^3 y = b^3 \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$

→ Solve the HODE & then substitute $z = \log(ax+b)$

Application of HODE

$$\left.\begin{aligned} &\rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = v \\ &\rightarrow L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = \frac{dv}{dt} \end{aligned}\right\} \text{LCR Circuit.}$$