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Module - 1

* Vectors

↳ $\mathbb{R}^n$-space : set of all ordered n-tuples.

↳ $|u+v| \leq |u| + |v|$, Triangle inequality

~~as also known as Cauchy-Schwarz inequality~~

↳ norm / magnitude

$$|u| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

↳ Inner product (dot)

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

↳ Distance (d) = $|u - v|$

↳ Angle $(\theta) = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$

↳ Orthogonal $\Rightarrow \theta = 90^\circ \Rightarrow u \cdot v = 0$

also $|u+v|^2 = |u|^2 + |v|^2$

↳ Cauchy-schwarz Inequality , $|u \cdot v| \leq |u||v|$

* Vector-spaces

If all of the following ~~te~~ axioms are satisfied for every element u, v, w and scalar c & d . V is ~~a~~ called a vector space.

i) if u & v are vectors ⇒ u+v is a vector (closure under vector addition)

~~ii) $(u+v) + w = u + (v+w)$~~

ii) $u+v = v+u$ (commutative)

iii) $u + (v+w) = (u+v) + w$ (associative)

iv) $u + 0 = 0 + u = u$ (additive identity)

v) $u + (-u) = 0$ (additive inverse)

vi) cu is vector (closure under scalar multiplication)

vii) $c(u+v) = cu + cv$ (distributive)

viii) $(c+d)u = cu + du$ (distributive)

ix) $c(du) = (cd)u$ (associative)

x) $I \cdot u = u$ (multiplicative Identity).

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★ Subspaces

↳ A non-empty subset W of a vector space V is called a subspace of V if W itself a vector space ~~defined~~ under operations defined on V.

↳ Conditions for W to be subspace of V

Closed under add^n i) if u & v are vectors in W, then u+v is in W

Closed under scalar multi^{n} ii) if u is vector in W, then ku is in W. where k is a scalar constant.

★ Linear Combination

↳ a vector V is called linear combination of $V_1, V_2, \dots, V_n$ if

$$V = K_1V_1 + K_2V_2 + \dots + K_nV_n$$

where $K_1, K_2, \dots, K_n$ are scalars.

★ Span

↳ The set of all vectors that are linear combination of vectors in the set is called span of S and denoted as

span S / span $\{V_1, V_2, \dots, V_n\}$

↳ solve like linear combination and if system is consistent then V will be in span S.

★ Linear Dependence & Linear Independence

$$K_1V_1 + K_2V_2 + \dots + K_nV_n = 0 \quad \text{—(i)}$$

↳ if determinant of coefficent's of eq^n (i) is zero then vectors are linearly dependent, otherwise they are linearly independent.

↳ $S = \{V_1, V_2, \dots, V_r\}$ in R^n is linearly dependent if $\boxed{r > n}$, no. of Unknowns > no. of equations (no. of non-zero rows)

* linear dependence & Independence of functions

↪ If $f_1(x), f_2(x), \ldots, f_n(x)$

$$W(\text{Wronskian}) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \cdots & f_n^{n-1}(x) \end{vmatrix}$$

$W \neq 0 \Rightarrow$ linear Independent.

$W = 0 \Rightarrow$ No conclusion can be made

* Basis

↪ The set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is called a basis for V if

i) linearly indepent (S)

ii) S spans V.

* Dimension

↪ Number of elements in the finite basis of a vector space.

Ex:- $C(R) \rightarrow$ complex $\Rightarrow$ Dim = 2

$$\left.\begin{array}{l} \dim(R^n) = n \\ \dim(P_n) = n+1 \\ \dim(M_{nm}) = nm \\ \dim\{0\} = 0 \end{array}\right\} \text{Can be found from their Standard basis}$$

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★ Basis & Dimension for solution page of homo. system

let $AX=0$, m equations & n unknowns

1) Solve ~~...~~ using gauss elimination method. If system has only a trivial solⁿ the solⁿ space is $\{0\}$ hence no basis and no dimension

2) if solution vector x contain arbitrary constants (parameters) $t_1, t_2, t_3, \ldots, t_p$ express x as a linear combination of vectors $x_1, x_2, x_3, \ldots, x_p$ with $t_1, t_2, t_3, \ldots, t_n$ as coefficients

i.e $x = t_1x_1 + t_2x_2 + \ldots + t_px_p$

3) The set of vectors $\{x_1, x_2, \ldots, x_n\}$ form a basis for the solution space of $AX=0$ and hence solution space is P.

$\Rightarrow$ $\underset{(p)}{\text{dimension}} = \underset{\left(\substack{\text{No. of}\\ \text{Unknowns}}\right)}{n} - \underset{\left(\substack{\text{no. of non-zero rows}\\ \text{in row echelon form}}\right)}{r}$

★ Reduction to basis

$V = \text{span } S$ and $\dim(V) < n$

1) Consider $k_1v_1 + k_2v_2 + k_3v_3 + \ldots + k_nv_n = 0$

2) Construct augmented matrix & reduce to row-echelon form.

3) The vectors corresponding to colums containing leading 1's form a basis for V.

$\longrightarrow$ By changing order of vectors in S, other possible bases can be found.

★ Extension to Basis

LI $\Rightarrow S = \{v_1, v_2, \ldots, v_n\}$, if $\dim(V) = n > m$

i) $S' = \{v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_n\}$ are standard vectors for R^n.

ii) follow all steps of reduction of basis.

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Module 2 (Inner product space)

★ Inner product space

let V be a real vector space.

$\underbrace{\langle u, v \rangle}_{\text{function from } V\times V \to R} \to$ real inner product space

for each ordered pair of vectors u & v.

In such a way that for all u, v, w in V and scalar k, following axioms are satisfied.

(a) $\langle u, v \rangle = \langle v, u \rangle$ (Symmetry)

(b) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (Additivity)

(c) $\langle ku, v \rangle = k\langle u, v \rangle$ (Homogeneity)

(d) $\langle u, u \rangle \geq 0 \quad \forall u$ in V (Non-negativity)

→ A) Euclidiom Inner Product

$u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$ are in R^n.

$$u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

hence any vector space w.r.t euclidian inner product is an inner product space.

→ B) Weighted Euclidian Inner Product

$u = (u_1, u_2, \ldots, u_n)$ & $v = (v_1, v_2, \ldots, v_n)$ are in R^n, and $(w_1, w_2, \ldots, w_n)$ are real numbers called weights.

$$\langle u, v \rangle = w_1u_1v_1 + w_2u_2v_2 + \ldots + w_nu_nv_n.$$

→ C) Inner product generated by matrix

u & v are in R^n and expressed as $N\times 1$ matrices and A is an $n\times n$ invertible matrix then

$$\langle u, v \rangle = (A\times u) \underset{\text{euclidian inner product.}}{\cdot} (A\times v)$$

if $A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\Rightarrow \langle u, v \rangle = \begin{bmatrix} xu_1 \\ yu_2 \end{bmatrix} \cdot \begin{bmatrix} xv_1 \\ yv_2 \end{bmatrix} = \underset{w_1}{x^2}u_1v_1 + \underset{w_2}{y^2}u_2v_2.$$

$$\Rightarrow x = \sqrt{w_1} \text{ \& } y = \sqrt{w_2}.$$

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→ D) Norm or length in inner product space $(||u||)$

$\Rightarrow ||u|| = (\langle u, u \rangle)^{1/2}$

Properties :-

i) $||u|| \geq 0$ ii) $||u|| = 0$ if & only if $u = 0$ iii) $||Ku|| = K||u||$ iv) $||u+v|| \leq ||u|| + ||v||$ Triangle inequality

→ E) Distance in inner product space (u, v)

$\Rightarrow d = ||u-v|| = (\langle (u-v), (u-v) \rangle)^{1/2}$

→ F) Angle b/w vectors (u, v)

$\Rightarrow \cos\theta = \dfrac{\langle u, v \rangle}{||u|| \, ||v||}$

→ G) Orthogonality (u, v) $\Rightarrow \theta = \dfrac{\pi}{2} \Rightarrow \cos\theta = 0$

$\Rightarrow \langle u, v \rangle = 0$

→ H) Pythagorean Theorem $(||u+v||^2 = ||u||^2 + ||v||^2)$

→ I) Cauchy-Schwarz Inequalities $(|\langle u, v \rangle| \leq ||u|| \, ||v||)$ (Modulus)

→ J) Orthogonal & Orthonormal sets

let $S = \{u_1, u_2, \_\_\_, u_n\}$

a) orthogonal if

i) $\langle u_1, u_2 \rangle = 0$

ii) $\langle u_2, u_3 \rangle = 0$

$\vdots$

$\langle u_{n-1}, u_n \rangle = 0$

b) orthonormal if

i) orthogonal

ii) $||u_1|| = 1$

iii) $||u_2|| = 1$

$\vdots$

$||u_n|| = 1$

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→ K) Orthogonal & orthonormal Basis

i) Theorem 1 :- $S = \{v_1, v_2, ..., v_n\}$ is orthogonal
⇒ S is linearly independent.

ii) Theorem 2 :- Any orthogonal set of n non-zero vectors in R^n is a basis for R^n.

iii) Theorem 3 :- $S = \{v_1, v_2, ---, v_n\}$ is orthonormal then any vector u in V can be represented as a linear combination of S

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + ---- + \langle u, v_n \rangle v_n$$

here $\langle u, v_1 \rangle, \langle u, v_2 \rangle, ---, \langle u, v_n \rangle$ are coordinate vectors

$$\Rightarrow [u]_S = \{ \langle u, v_1 \rangle, \langle u, v_2 \rangle, ----, \langle u, v_n \rangle)$$

iv) corollary if $S = \{v_1, v_2, ---, v_n\}$ is orthonormal then any vector u in V can be represented as

$$u = \frac{\langle u, v_1 \rangle v_1}{\|v_1\|^2} + \frac{\langle u, v_2 \rangle v_2}{\|v_2\|^2} + ---- + \frac{\langle u, v_n \rangle v_n}{\|v_n\|^2}$$

v) Theorem 4 :- if S is an orthonormal basis and u & v are vectors in V having coordinate vectors of u & v are

$$[u]_S = (a_1, a_2, ---, a_n)$$
$$[v]_S = (b_1, b_2, ----, b_n)$$

i) $\|u\| = \sqrt{a_1^2 + a_2^2 + ---- + a_n^2}$

ii) $d(u, v) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + --- + (a_n - b_n)^2}$

iii) $\langle u, v \rangle = a_1 b_1 + a_2 b_2 + --- + a_n b_n = [u]_S \cdot [v]_S$

vi) Theorem 5 :- every non-zero finite dimensional inner product space has an orthonormal basis.

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* Gram-Schmidt orthogonalization process

→ let V be an inner product vector space, and $\{u_1, u_2, \ldots, u_n\}$ be an arbitary basis of V.

Then orthogonal basis of V is given by $\{v_1, v_2, \ldots, v_n\}$

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1,$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$v_n = u_n - \sum_{i=1}^{n-1} \frac{\langle u_n, v_i \rangle}{\|v_i\|^2} v_i$$

after normalizing we get orthonormal basis i.e $(w_1, w_2, \ldots, w_n) = \left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|} \right)$

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* Module-4 (Eigen Values & Eigen Vectors).

* Eigen Values & Eigen Vectors

$$\underbrace{(A-\lambda I)}_{\text{Charecteristic Matrix of A.}} x = 0$$

$\lambda \rightarrow$ eigenvalue of A.

$x \rightarrow$ eigenvector of A.

$$\underbrace{|A-\lambda I|}_{\text{Charecteristic Equation of A.}} = 0$$

1) Charecteristic equation of order 2.

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$S_2 \rightarrow$ Determinant A.

$S_1 \rightarrow$ Sum of principle diagonal elements

2) Charecteristic equation of order 3

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_3 \rightarrow$ determinant of A.

$S_2 \rightarrow$ Sum of minors of principle diagonal.

$S_1 \rightarrow$ Sum of principle diagonal elements.

$\rightarrow$ Sum of eigenvalues of a matrix is the sum of its principle diagonal elements.

$\rightarrow$ product of eigenvalues of a matrix is the determinant of the matrix.

$\rightarrow$ if λ is an eigen value of A

i) then $1/\lambda$ is an eigenvalue of A^{-1}

ii) then λ^k is an eigenvalue of A^k

iii) then $\lambda \pm k$ is an eigenvalue of $A \pm kI$

iv) then λ is also an eigenvalue of A^T

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* Algebraic & Geometric Multiplicity of Eigenvalue.

→ if $\lambda = 1, 2, 2$.

⇒ A.M for $\lambda = 1$ is 1

⇒ A.M for $\lambda = 2$ is 2

i.e AM is frequency of the $\{\lambda\}$ in set eigenvalues of A.

→ if no. of unknowns = n ⊛ $(A-\lambda I)$ (eigenvector). λ (eigenvalue)

& $\rho(A-\lambda I)$ = rank of matrix

↳ reduce $(A-\lambda I)$ to row echloen form

& then no. of nonzero rows is rank

⇒ $G.M = n - \rho(A-\lambda I)$

* Cayley-Hamilton Theorem

↳ Every square matrix satisfies its own cherrecteristic equation.

→ we can replace λ in (cherrecteristic equation of A) by matrix A.

* Similarity of matrices

A & B are 2 matrices & B is said to be similar to A. if there exists non-singular matrix P such that

⊛ $B = P^{-1} A P$

i) This is an Equivalance Relation $\begin{cases} \text{Reflexive : } (a,a) \in R \\ \text{Symmetric : } (a,b) \in R \Rightarrow (b,a) \in R \\ \text{Transitive : } (a,b) \,\&\, (b,c) \in R \Rightarrow (a,c) \in R. \end{cases}$

ii) They have same determinant.

iii) They have same cherrecteristic polynomial & hence same eigenvalues & eigenvectors.

$A \xrightarrow{EV} \lambda \xrightarrow{C.EVec} x$

$B \xrightarrow{EV} \lambda \xrightarrow{C.EVec} P^{-1}x$ where $B = P^{-1} A P$.

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★ Diagnolization

→ Matrix A is said to be diagonlizable if it is similar to a diagonal matrix.

i.e if there exists an invertible matrix P such that $P^{-1}AP = D$ → eigen values $(\lambda_1, \lambda_2, \lambda_3)$.

$\Rightarrow P = \begin{bmatrix} [x_1] & [x_2] & [x_3] \end{bmatrix}$, $\Rightarrow D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ D → diagonal ~~Modal~~ Matrix.

→ (P) is Modal Matrix. ↳ x_1, x_2, x_3 normalized eigen vectors

i) Eigenvalues of matrix are all distint then it is always similar to diagonal matrix

ii) A matrix $A_{n\times n}$ is only diagonalizable if & only if it posses (n) distinct eigenvalues.

iii) Nessary & sufficient condition to be a diagonal matrix ↳ that each eigenvalue has AM = GM

iv) Orthogonally similar Matrices (O.S.M)

$B = P^{-1}AP$ → similar matrices

$P^{-1} = P^T$ → orthogonal

$\Rightarrow B = P^{-1}AP = P^TAP$.

a) every real ~~so~~ symmetric matrix is orthogonally similar to diagonal matrix.

b) every real symmetric matrix of order n has n mutually orthogonal real eigenvectors.

⇒ i.e ⇒ $x_1 x_2^T = x_2 x_3^T = x_3 x_1^T = 0$ → Cond^n for O.S.M

→ To find orthogonal matrix P, each element of eigen vector is divided by its norm.

★ Power of a matrix

A is a n×n matrix & P is an invertible matrix

$\Rightarrow A^K = PD^KP^{-1}$

P → modal matrix

D → diagonal matrix

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★ Quadratic forms

$$\varphi = \underrightarrow{x^T A x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Symmetric.

→ coefficient of $x_i x_j = a_{ji} + a_{ij} = 2a_{ij} = 2a_{ji}$.

Coefficient of $x_i^2 = a_{ii}$

→ rank of quadratic form $= \rho(A)$ = number of non-zero eigen values of A.

→ if $\rho(A) < n$ (order of A) ⇒ singular.
otherwise, ⇒ non-singular.

→ Index ⇒ No. of positive eigen values (P)

→ Signature ⇒ ~~No. of~~ diff. b/w no. of positive & negative eigen values. $(2p - r)$

r → total no. of eigen values

→ Value Class / Nature of Quadratic form.

Value Class	Criteria	Canonical form	Eigen Values
i) +ve definite	$r = p = n$	only +ve terms	all +ve
ii) +ve Semidefinite	$r = p, p < n$	only +ve terms	+ve & atleast one zero
iii) -ve definite	$r = n, p = 0$	only -ve terms	all -ve
iv) -ve Semidefinite	$r < n, p = 0$	only -ve terms	-ve & atleast one zero.
v) Indefinite	otherwise	both +ve & -ve	+ve as well as -ve

★ Max & Min of Quadratic form $(x^T A x)$

$A \xrightarrow[\text{values}]{\text{eigen}} \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \xrightarrow[\text{Eigen Vectors}]{\text{Corresponding}} x_1, x_2, \ldots, x_n$

if $\lambda_1 > \lambda_2 > \lambda_3 \ldots > \lambda_n$

So at $\underbrace{\frac{x_1}{|x_1|}}_{\text{normalised } x_1}$ → Max & $\underbrace{\frac{x_n}{|x_n|}}_{\text{normalised } x_n}$ → Min.

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⊛ Methods to reduce Quadratic form to Canonical form.

1) Orthogonal Transformation

⊛ $x = PY$ be L.T which transforms quadratic to canonical

$\varphi = x^T A x$

↳ orthogonal Transformation ⇒ $\overbrace{x = Py}$

⊛ orthogonal matrix

⊛ $\varphi = y^T D y$ ⟶ gives canonical form

2) Congruent Transformation.

Perform (Row) Perform (Column)

⊛ $A = I_3 \, A \, I_3$

Convert to a Diagonal Matrix → (D)

DND → Upper Δ'arr. (P)

↳ make sure that Both Identity matrix are Transpose of each other

⊛ let $x = PY$ be L.T which transforms given quadratic into canonical form

& Canonical form is ⊛ $\varphi = y^T D y$

⊛ Conic Sections.

$ax^2 + 2bxy + by^2 + 2dx + 2ey + f = 0$

$$[x \;\; y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [d \;\; e]\begin{bmatrix} x \\ y \end{bmatrix} + f = 0$$

⟶ ⊛ $x = [x \;\; y]$, ⊛ $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, ⊛ $K = [d \;\; e]$

i) find A

ii) find eigen values (λ_1, λ_2)

iii) find eigen vectors (x_1, x_2)

iv) find length of eigen vectors

v) find norm of eigen vectors

vi) for orthogonal matrix $P = \left[\underset{\text{norm}}{[x_1]}\;\underset{\text{norm}}{[x_2]}\right]$, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

now apply Transformation $x = Px'$

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Module-5 (Linear Transformation)

★ Linear Transformation ($T: V \rightarrow W$)

a) $T(u+v) = T(u) + T(v)$

b) $T(ku) = kT(u)$

★ Linear Transformation from image of Basis vectors

let $V = (a, b, c)$ ~~in~~ any arbitrary vector which can be represented as (wrt)

$V = k_1 V_1 + k_2 V_2 + k_3 V_3$

find k_1, k_2, k_3 in terms of a, b, c

~~S~~ Then

$T(V) = k_1 T(V_1) + k_2 T(V_2) + k_3 T(V_3)$

and solve following equation.

★ composition of Linear Transformation

$(T_2 \circ T_1)(u) = T_2(T_1(u))$

$(T_3 \circ T_2 \circ T_1)(u) = T_3(T_2(T_1(u)))$