

# Real Analysis

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# Chapter 1

## Relations

### 1.1 Relations

**Definition 1.1.1** Let  $X$  and  $Y$  be sets. Then  $R \subseteq X \times Y$  is called a **Relation** over sets  $X$  and  $Y$ .

If  $(x, y) \in R$ , it is denoted as  $xRy$ , or "x is related to y", where  $x \in X$  and  $y \in Y$ . Sometimes, relations over two sets are called binary relations as in general, relations can be subsets of Cartesian Product of more than two sets.

### 1.2 Functions

**Definition 1.2.1** A Relation is called **functional** or right unique if:  
 $\forall x \in X, \forall y \in Y, \forall z \in Y, (x, y) \in R \wedge (x, z) \in R \implies y = z$

**Definition 1.2.2** A Relation is called **serial** or left total if:  
 $\forall x \in X, \exists y \in Y: (x, y) \in R$

**Definition 1.2.3** A Relation that is functional and serial is called a **function**.

A function is usually denoted by

$$f : X \rightarrow Y$$

and is usually thought of as a rule that assigns every element in  $X$  to exactly one element in  $Y$ .

**Definition 1.2.4** A function that is left unique is called **injective** or one-one.

**Definition 1.2.5** A function that is right total is called **surjective** or onto.

**Definition 1.2.6** A function that is both one-one and onto is called a **bijective function** or a **bijection**.

If a function  $f$  is bijective, we its **inverse** as

$$f^{-1}: Y \rightarrow X$$

## 1.3 Homogeneous Relations

**Definition 1.3.1** A **Homogeneous Relation** is a subset of a Cartesian Product of a set with itself.

**Definition 1.3.2** A Relation  $R$  is **Reflexive** if  
 $\forall x \in X, (x, x) \in R$

**Definition 1.3.3** A Relation is **Symmetric** if  
 $\forall x, y \in X, (x, y) \in R \implies (y, x) \in R$

**Definition 1.3.4** A Relation is **Transitive** if  
 $\forall x, y, z \in X, (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$

**Definition 1.3.5** A Relation is **Antisymmetric** if  
 $\forall x, y \in X, (x, y) \in R \implies (y, x) \notin R$

## 1.4 Equivalence Relations

**Definition 1.4.1** A Relation that is Reflexive, Symmetric and Transitive is called an **Equivalence Relation**.

Equivalence relations are usually denoted by  $\sim$ . Let  $\sim$  be an equivalence relation on a set  $X$ . If  $x, y \in X$  are related by  $\sim$  then it is denoted by  $x \sim y$ .

**Definition 1.4.2** Let  $\sim$  be an equivalence relation defined on a set  $X$ .

Let  $x \in X$ . Then the set

$$[x] := \{y \mid y \sim x\}$$

is called the **equivalence class** containing  $x$ .

Equivalence classes have the following properties -

1. Let  $x, y \in X$ , then either  $[x] = [y]$  or  $[x] \cap [y] = \emptyset$
2.  $\bigcup_{x \in X} [x] = X$ .

Thus equivalence classes form a partition of the set  $X$ .

## 1.5 Partial Orders

**Definition 1.5.1** A Relation that is Reflexive, Antisymmetric and Transitive is called a **Partial Order**.

**Definition 1.5.2** A set  $A \subseteq X$  is said to be **bounded above** if

$$\exists b \in X : \forall a \in A \quad b \geq a.$$

In this case  $b$  is said to be an **upper bound** of  $A$ .

**Definition 1.5.3** Let  $X$  be a set and  $A \subseteq X$ . The  $s \in X$  is called a **least upper bound** of  $A$  if it satisfies the following conditions

1.  $s$  is an upper bound of  $A$
2. If  $b$  is an upper bound of  $A$ , then  $s \leq b$ .

It is also known as the **supremum** of  $A$ . It is usually denoted by  $s = \text{lub } A$  or  $s = \sup A$ .

## Chapter 2

# Equinumerosity and Countability

## Chapter 3

# Real Numbers

**Axiom 3.0.1 (Axiom of Completeness)** *Every nonempty subset of  $\mathbb{R}$  that is bounded above has a least upper bound.*

The Axiom of Completeness is one of the defining characteristics of real numbers. Many important theorems about real numbers follow as a consequence of the Axiom of Completeness.

**Theorem 3.0.1 (Nested Interval Property)** *For each  $n \in \mathbb{N}$ , consider the closed intervals  $I_n = [a_n, b_n] = \{x \in \mathbb{R} | a_n \leq x \leq b_n\}$  such that  $I_1 \subseteq I_2 \subseteq \cdots$ . Then  $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$*

The Nested interval property is just another way of expressing the completeness of  $\mathbb{R}$  which will be useful to prove some theorems in real analysis. There are other ways of expressing the same which we will explore in the coming chapters.

## Chapter 4

# Sequences and Series of Real Numbers

### 4.1 Sequences

**Definition 4.1.1** A *sequence* is a function whose domain is  $\mathbb{N}$ .

**Definition 4.1.2** A sequence  $a_n$  is said to **converge** to a limit  $a$  if  $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - a| < \epsilon$

**Definition 4.1.3** A sequence  $x_n$  is said to be **bounded** if  $\exists M > 0 : \forall n \in \mathbb{N}, |x_n| < M$

**Theorem 4.1.1** All convergent sequences are bounded.

**Theorem 4.1.2 (Algebraic Limit Theorem)** Let  $a_n$  and  $b_n$  be sequences, such that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Then,

1.  $a_n + b_n \rightarrow a + b$
2.  $ca_n \rightarrow ca$
3.  $a_nb_n \rightarrow ab$
4.  $\frac{a_n}{b_n} \rightarrow \frac{a}{b}, b \neq 0$

**Theorem 4.1.3** Let  $a_n$  and  $b_n$  be sequences, such that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . Then,

1. If  $a_n \geq 0 \forall n$ , then  $a \geq 0$
2. If  $a_n \geq b_n \forall n$ , then  $a \geq b$
3. If  $\exists c \in \mathbb{R}$ , such that  $c \leq b_n \forall n \in \mathbb{N}$ , then  $c \leq b$ . Similarly, if  $c \geq a_n \forall n \in \mathbb{N}$ , then  $c \geq a$

**Definition 4.1.4** A sequence is increasing if  $a_{n+1} \geq a_n \forall n \in \mathbb{N}$  and decreasing if  $a_{n+1} \leq a_n \forall n \in \mathbb{N}$ .

A sequence is **monotone** if it is increasing or decreasing.

**Theorem 4.1.4 (Monotone Convergence Theorem)** If a sequence is bounded and monotone, it converges.

**Theorem 4.1.5 (Bolzano Weierstrass Theorem)** Every bounded sequence has a convergent subsequence.

**Definition 4.1.5** A sequence is **Cauchy** if  $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, |a_m - a_n| < \epsilon$ .

**Theorem 4.1.6** All convergent sequences are Cauchy sequences.

**Theorem 4.1.7** All Cauchy sequences are bounded.

**Theorem 4.1.8** A sequence is convergent if and only if it is Cauchy.

## 4.2 Series

**Definition 4.2.1** Let  $x_n$  be a sequence. A series is an expression of the form  $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \dots$  is called a **series**.

**Definition 4.2.2** The sequence  $s_n = x_1 + x_2 + \dots + x_n$  is called the **sequence of partial sums**.

The series  $\sum_{n=1}^{\infty} x_n$  is said to converge to a  $s$  if the corresponding sequence of partial sums  $s_n$  converges to  $s$ .  
 $\sum_{n=1}^{\infty} x_n = s \leftrightarrow s_n \rightarrow s$