# Real Analysis

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# Chapter 1

# Relations

## 1.1 Relations

**Definition 1.1.1** *Let $X$ and $Y$ be sets. Then $R \subseteq X \times Y$ is called a* ***Relation*** *over sets $X$ and $Y$.*

If $(x, y) \in$R, its is denoted as xRy, or "x is related to y", where $x \in$X and $y \in$Y. Sometimes, relations over two sets are called binary relations as in general, relations can be subsets of Cartesian Product of more than two sets.

## 1.2 Functions

**Definition 1.2.1** *A Relation is called* ***functional*** *or right unique if:*
$\forall x \in X$ , $\forall y \in Y$,$\forall z \in Y$, $(x, y) \in R \wedge (x, z) \in R \implies y = z$

**Definition 1.2.2** *A Relation is called* ***serial*** *or left total if:*
$\forall x \in X$, $\exists y \in Y$:$(x, y) \in R$

**Definition 1.2.3** *A Relation that is functional and serial is called a* ***function****.*

A function is usually denoted by

$f$ : X $\rightarrow$ Y

and is usually thought of as a rule that assigns every element in X to exactly one element in Y.

**Definition 1.2.4** *A function that is left unique is called* ***injective*** *or one-one.*

**Definition 1.2.5** *A function that is right total is called* ***surjective*** *or onto.*

**Definition 1.2.6** *A function that is both one-one and onto is called a* ***bijective function*** *or a* ***bijection****.*

If a function $f$ is bijective, we its **inverse** as

$f^{-1}$: Y $\rightarrow$ X

## 1.3 Homogeneous Relations

**Definition 1.3.1** *A **Homogeneous Relation** is a subset of a Cartesian Product of a set with itself.*

**Definition 1.3.2** *A Relation R is **Reflexive** if*
$\forall x \in X,\ (x,x) \in R$

**Definition 1.3.3** *A Relation is **Symmetric** if*
$\forall x,y \in X,\ (x,y) \in R \implies (y,x) \in R$

**Definition 1.3.4** *A Relation is **Transitive** if*
$\forall x,y,z \in X,\ (x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R$

**Definition 1.3.5** *A Relation is **Antisymmetric** if*
$\forall x,y \in X,\ (x,y) \in R \implies (y,x) \notin R$

## 1.4 Equivalence Relations

**Definition 1.4.1** *A Relation that is Reflexive, Symmetric and Transitive is called an **Equivalence Relation**.*

Equivalence relations are usually denoted by $\sim$. Let $\sim$ be an equivalence relation on a set X. If x,y $\in$ X are related by $\sim$ then it is denoted by x$\sim$y.

**Definition 1.4.2** *Let $\sim$ be an equivalence relation defined on a set X.*
*Let $x \in X$. Then the set*
$[x] := \{y \mid y \sim x\}$
*is called the **equivalence class** containing x.*

Equivalence classes have the following properties -

1. Let x,y $\in$ X, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$
2. $\bigcup_{x \in X}$ = X.

Thus equivalence classes form a partition of the set X.

## 1.5 Partial Orders

**Definition 1.5.1** *A Relation that is Reflexive, Antisymmetric and Transitive is called a* ***Partial Order****.*

**Definition 1.5.2** *A set $A \subseteq X$ is said to be* ***bounded above*** *if*
$\exists b \in X : \forall a \in A \ b \geq a$.
*In this case $b$ is said to be an* ***upper bound*** *of $A$.*

**Definition 1.5.3** *Let $X$ be a set and $A \subseteq X$. The $s \in X$ is called a* ***least upper bound*** *of $A$ if it satisfies the following conditions*

1. *$s$ an upper bound of $A$*
2. *If $b$ is an upper bound of $A$, then $s \geq b$.*

*It is also known as the* ***supremum*** *of $A$. It is usually denoted by $s = lub\ A$ or $s = sup\ A$*

# Chapter 2

# Equinumerosity and Countability

# Chapter 3

# Sequences and Series of Real Numbers

## 3.1 Sequences

**Definition 3.1.1** *A **sequence** is a function whose domain is* $\mathbb{N}$*.*

**Definition 3.1.2** *A sequence* $a_n$ *is said to **converge** to a limit* $a$ *if*
$\forall\ \epsilon > 0$ , $\exists N \in \mathbb{N}$:$\forall\ n \geq N$ , $|a_n - a| < \epsilon$

**Definition 3.1.3** *A sequence* $x_n$ *is said to be **bounded** if*
$\exists\ M > 0 : \forall\ n \in \mathbb{N}$ , $|x_n| < M$

**Theorem 3.1.1** *All convergent sequences are bounded.*

**Theorem 3.1.2 (Algebraic Limit Theorem)** *Let* $a_n$ *and* $b_n$ *be sequences, such that* $a_n \to a$ *and* $b_n \to b$*. Then,*

1. $a_n + b_n \to a + b$
2. $ca_n \to ca$
3. $a_n b_n \to ab$
4. $\frac{a_n}{b_n} \to \frac{a}{b}$, $b \neq 0$

**Theorem 3.1.3** *Let* $a_n$ *and* $b_n$ *be sequences, such that* $a_n \to a$ *and* $b_n \to b$*. Then,*

1. *If* $a_n \geq 0\ \forall\ n$*, then* $a \geq 0$
2. *If* $a_n \geq b_n\ \forall\ n$*, then* $a \geq b$
3. *If* $\exists\ c \in \mathbb{R}$*, such that* $c \leq b_n\ \forall\ n \in \mathbb{N}$*, then* $c \leq b$*. Similarly, if* $c \geq a_n$ $\forall n \in \mathbb{N}$*, then* $c \geq a$

**Definition 3.1.4** *A sequence is increasing if* $a_{n+1} \geq a_n \ \forall\ n \in \mathbb{N}$ *and decreasing if* $a_{n+1} \leq a_n \ \forall\ n \in \mathbb{N}$.
*A sequence is* ***monotone*** *if it is increasing or decreasing.*

**Theorem 3.1.4 (Monotone Convergence Theorem)** *If a sequence is bounded and monotone, it converges.*

**Theorem 3.1.5 (Bolzano Weierstrass Theorem)** *Every bounded sequence has a convergent subsequence.*

**Definition 3.1.5** *A sequence is* ***Cauchy*** *if*
$\forall\ \epsilon > 0\ ,\ \exists\ N \in \mathbb{N} : \forall\ m,n \geq N,\ |a_m - a_n| < \epsilon.$

**Theorem 3.1.6** *All convergent sequences are Cauchy sequences.*

**Theorem 3.1.7** *All Cauchy sequences are bounded.*

**Theorem 3.1.8** *A sequence in convergent if and only if it is Cauchy.*

## 3.2 Series

**Definition 3.2.1** *Let* $x_n$ *be a sequence. A series is an expression of the form* $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots$ *is called a* ***series***.

**Definition 3.2.2** *The sequence* $s_n = x_1 + x_2 + \cdots + x_n$ *is called the* ***sequence of partial sums***.

The series $\sum_{n=1}^{\infty}$ is said to converge to a s if the corresponding sequence of partial sums $s_n$ converges to s.
$\sum_{n=1}^{\infty} = \text{s} \leftrightarrow s_n \to \text{s}$