Real Analysis

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Chapter 1

Relations

1.1 Relations

Definition 1.1.1 *Let X and Y be sets. Then $R \subseteq X \times Y$ is called a* ***Relation*** *over sets X and Y.*

If $(x, y) \in$R, its is denoted as xRy, or "x is related to y", where $x \in$X and $y \in$Y. Sometimes, relations over two sets are called binary relations as in general, relations can be subsets of Cartesian Product of more than two sets.

1.2 Functions

Definition 1.2.1 *A Relation is called* ***functional*** *or right unique if:*
$\forall x \in X$, $\forall y \in Y$,$\forall z \in Y$, $(x, y) \in R \wedge (x, z) \in R \implies y = z$

Definition 1.2.2 *A Relation is called* ***serial*** *or left total if:*
$\forall x \in X$, $\exists y \in Y$:$(x, y) \in R$

Definition 1.2.3 *A Relation that is functional and serial is called a* ***function****.*

A function is usually denoted by

f : X $\rightarrow$ Y

and is usually thought of as a rule that assigns every element in X to exactly one element in Y.

Definition 1.2.4 *A function that is left unique is called* ***injective*** *or one-one.*

Definition 1.2.5 *A function that is right total is called* ***surjective*** *or onto.*

Definition 1.2.6 *A function that is both one-one and onto is called a* ***bijective function*** *or a* ***bijection****.*

If a function f is bijective, we its **inverse** as

f^{-1}: Y $\rightarrow$ X

1.3 Homogeneous Relations

Definition 1.3.1 *A **Homogeneous Relation** is a subset of a Cartesian Product of a set with itself.*

Definition 1.3.2 *A Relation R is **Reflexive** if*
$\forall x \in X$, $(x,x) \in R$

Definition 1.3.3 *A Relation is **Symmetric** if*
$\forall x,y \in X$, $(x,y) \in R \implies (y,x) \in R$

Definition 1.3.4 *A Relation is **Transitive** if*
$\forall x,y,z \in X$, $(x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R$

Definition 1.3.5 *A Relation is **Antisymmetric** if*
$\forall x,y \in X$, $(x,y) \in R \implies (y,x) \notin R$

1.4 Equivalence Relations

Definition 1.4.1 *A Relation that is Reflexive, Symmetric and Transitive is called an **Equivalence Relation**.*

Equivalence relations are usually denoted by $\sim$. Let $\sim$ be an equivalence relation on a set X. If x,y $\in$ X are related by $\sim$ then it is denoted by x$\sim$y.

Definition 1.4.2 *Let $\sim$ be an equivalence relation defined on a set X.*
Let $x \in X$. Then the set
$[x] := \{y \mid y \sim x\}$
*is called the **equivalence class** containing x.*

Equivalence classes have the following properties -

1. Let x,y $\in$ X, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$
2. $\bigcup_{x \in X}$ = X.

Thus equivalence classes form a partition of the set X.

1.5 Partial Orders

Definition 1.5.1 *A Relation that is Reflexive, Antisymmetric and Transitive is called a **Partial Order**.*

Chapter 2

Equinumerosity and Countability

Chapter 3

Sequences and Series of Real Numbers

3.1 Sequences

Definition 3.1.1 *A* ***sequence*** *is a function whose domain is* $\mathbb{N}$*.*

Definition 3.1.2 *A sequence* a_n *is said to* ***converge*** *to a limit* a *if*
$\forall\ \epsilon > 0$, $\exists N \in \mathbb{N}$:$\forall\ n \geq N$, $|a_n - a| < \epsilon$

Definition 3.1.3 *A sequence* x_n *is said to be* ***bounded*** *if*
$\exists\ M > 0 : \forall\ n \in \mathbb{N}$, $|x_n| < M$

Theorem 3.1.1 *All convergent sequences are bounded.*

Theorem 3.1.2 (Algebraic Limit Theorem) *Let* a_n *and* b_n *be sequences, such that* $a_n \to a$ *and* $b_n \to b$*. Then,*

1. $a_n + b_n \to a + b$
2. $ca_n \to ca$
3. $a_n b_n \to ab$
4. $\frac{a_n}{b_n} \to \frac{a}{b}$, $b \neq 0$

Theorem 3.1.3 *Let* a_n *and* b_n *be sequences, such that* $a_n \to a$ *and* $b_n \to b$*. Then,*

1. *If* $a_n \geq 0\ \forall\ n$*, then* $a \geq 0$
2. *If* $a_n \geq b_n\ \forall\ n$*, then* $a \geq b$
3. *If* $\exists\ c \in \mathbb{R}$*, such that* $c \leq b_n\ \forall\ n \in \mathbb{N}$*, then* $c \leq b$*. Similarly, if* $c \geq a_n$ $\forall n \in \mathbb{N}$*, then* $c \geq a$

Definition 3.1.4 *A sequence is increasing if* $a_{n+1} \geq a_n \; \forall \; n \in \mathbb{N}$ *and decreasing if* $a_{n+1} \leq a_n \; \forall \; n \in \mathbb{N}$.
A sequence is ***monotone*** *if it is increasing or decreasing.*

Theorem 3.1.4 (Monotone Convergence Theorem) *If a sequence is bounded and monotone, it converges.*

Theorem 3.1.5 (Bolzano Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

Definition 3.1.5 *A sequence is* ***Cauchy*** *if*
$\forall \; \epsilon > 0$, $\exists \; N \in \mathbb{N}$: $\forall$ $m,n \geq N$, $|a_m - a_n| < \epsilon$.

Theorem 3.1.6 *All convergent sequences are Cauchy sequences.*

Theorem 3.1.7 *All Cauchy sequences are bounded.*

Theorem 3.1.8 *A sequence in convergent if and only if it is Cauchy.*

3.2 Series

Definition 3.2.1 *Let* x_n *be a sequence. A series is an expression of the form* $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots$ *is called a* ***series***.

Definition 3.2.2 *The sequence* $s_n = x_1 + x_2 + \cdots + x_n$ *is called the* ***sequence of partial sums***.

The series $\sum_{n=1}^{\infty}$ is said to converge to a s if the corresponding sequence of partial sums s_n converges to s.
$\sum_{n=1}^{\infty}$ = s $\leftrightarrow$ $s_n \rightarrow$ s