Real Analysis

March 9, 2021

Chapter 1

Relations

1.1 Relations

Definition 1.1.1 *Let X and Y be sets. Then $R \subseteq X \times Y$ is called a* ***Relation*** *over sets X and Y.*

If $(x, y) \in$R, its is denoted as xRy, or "x is related to y", where $x \in$X and $y \in$Y. Sometimes, relations over two sets are called binary relations as in general, relations can be subsets of Cartesian Product of more than two sets.

1.2 Functions

Definition 1.2.1 *A Relation is called* ***functional*** *or right unique if:*
$\forall x \in X$, $\forall y \in Y$,$\forall z \in Y$, $(x, y) \in R \wedge (x, z) \in R \implies y = z$

Definition 1.2.2 *A Relation is called* ***serial*** *or left total if:*
$\forall x \in X$, $\exists y \in Y$:$(x, y) \in R$

Definition 1.2.3 *A Relation that is functional and serial is called a* ***function****.*

A function is usually denoted by

f : X $\rightarrow$ Y

and is usually thought of as a rule that assigns every element in X to exactly one element in Y.

Definition 1.2.4 *A function that is left unique is called* ***injective*** *or one-one.*

Definition 1.2.5 *A function that is right total is called* ***surjective*** *or onto.*

Definition 1.2.6 *A function that is both one-one and onto is called a* ***bijective function*** *or a* ***bijection****.*

If a function f is bijective, we its **inverse** as

f^{-1}: Y $\rightarrow$ X

1.3 Homogeneous Relations

Definition 1.3.1 *A **Homogeneous Relation** is a subset of a Cartesian Product of a set with itself.*

Definition 1.3.2 *A Relation R is **Reflexive** if*
$\forall x \in X,\ (x,x) \in R$

Definition 1.3.3 *A Relation is **Symmetric** if*
$\forall x,y \in X,\ (x,y) \in R \implies (y,x) \in R$

Definition 1.3.4 *A Relation is **Transitive** if*
$\forall x,y,z \in X,\ (x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R$

Definition 1.3.5 *A Relation is **Antisymmetric** if*
$\forall x,y \in X,\ (x,y) \in R \implies (y,x) \notin R$

1.4 Equivalence Relations

Definition 1.4.1 *A Relation that is Reflexive, Symmetric and Transitive is called an **Equivalence Relation**.*

Equivalence relations are usually denoted by $\sim$. Let $\sim$ be an equivalence relation on a set X. If x,y $\in$ X are related by $\sim$ then it is denoted by x$\sim$y.

Definition 1.4.2 *Let $\sim$ be an equivalence relation defined on a set X.*
Let $x \in X$. Then the set
$[x] := \{y \mid y \sim x\}$
*is called the **equivalence class** containing x.*

Equivalence classes have the following properties -

1. Let x,y $\in$ X, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$
2. $\bigcup_{x \in X}$ = X.

Thus equivalence classes form a partition of the set X.

1.5 Partial Orders

Definition 1.5.1 *A Relation that is Reflexive, Antisymmetric and Transitive is called a* ***Partial Order****.*

Definition 1.5.2 *A set $A \subseteq X$ is said to be* ***bounded above*** *if*
$\exists b \in X : \forall a \in A \; b \geq a$.
In this case b is said to be an ***upper bound*** *of A.*

Definition 1.5.3 *Let X be a set and $A \subseteq X$. The $s \in X$ is called a* ***least upper bound*** *of A if it satisfies the following conditions*

1. *s an upper bound of A*
2. *If b is an upper bound of A, then $s \geq b$.*

It is also known as the ***supremum*** *of A. It is usually denoted by $s = lub\ A$ or $s = sup\ A$.*

Chapter 2

Equinumerosity and Countability

Chapter 3

Real Numbers

3.1 Properties of Real Numbers

Axiom 3.1.1 (Axiom of Completeness) *Every nonempty subset of* $\mathbb{R}$ *that is bounded above has a least upper bound.*

The Axiom of Completeness is one of the defining characteristics of real numbers. Many important theorems about real numbers follow as a consequence of the Axiom of Completeness.

Theorem 3.1.1 (Nested Interval Property) *For each* $n \in \mathbb{N}$, *consider the closed intervals* $I_n = [a_n, b_n] = \{x \in \mathbb{R} | a_n \leq x \leq b_n\}$ *such that* $I_1 \subseteq I_2 \subseteq \cdots$. *Then* $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$

The Nested interval property is just another way of expressing the completeness of $\mathbb{R}$ which will be useful to prove some theorems in real analysis. There are other ways of expressing the same which we will explore in the coming chapters.

Theorem 3.1.2 (Archimedean Property)

1. *Let* $x \in \mathbb{R}$, *then* $\exists n \in \mathbb{N}$ *such that* $n > x$.
2. *Let* $y \in \mathbb{R}, y > 0$, *then* $\exists n \in \mathbb{N}$,*such that* $\frac{1}{n} < y$.

Theorem 3.1.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$) *Given real numbers a,b with* $a < b$, *there exists a rational number* r *such that* $a < r < b$.

Theorem 3.1.4 *There exists a real number* α *satisfying* $\alpha^2 = 2$.

Chapter 4

Sequences and Series of Real Numbers

4.1 Sequences

Definition 4.1.1 *A **sequence** is a function whose domain is* $\mathbb{N}$.

Definition 4.1.2 *A sequence* a_n *is said to **converge** to a limit* a *if*
$\forall\ \epsilon > 0$, $\exists N \in \mathbb{N}$:$\forall\ n \geq N$, $|a_n - a| < \epsilon$

Definition 4.1.3 *A sequence* x_n *is said to be **bounded** if*
$\exists\ M > 0 : \forall\ n \in \mathbb{N}$, $|x_n| < M$

Theorem 4.1.1 *All convergent sequences are bounded.*

Theorem 4.1.2 (Algebraic Limit Theorem) *Let* a_n *and* b_n *be sequences, such that* $a_n \rightarrow a$ *and* $b_n \rightarrow b$. *Then,*

1. $a_n + b_n \rightarrow a + b$
2. $ca_n \rightarrow ca$
3. $a_n b_n \rightarrow ab$
4. $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$, $b \neq 0$

Theorem 4.1.3 *Let* a_n *and* b_n *be sequences, such that* $a_n \rightarrow a$ *and* $b_n \rightarrow b$. *Then,*

1. *If* $a_n \geq 0\ \forall\ n$, *then* $a \geq 0$
2. *If* $a_n \geq b_n\ \forall\ n$, *then* $a \geq b$
3. *If* $\exists\ c \in \mathbb{R}$, *such that* $c \leq b_n\ \forall\ n \in \mathbb{N}$, *then* $c \leq b$. *Similarly, if* $c \geq a_n$ $\forall n \in \mathbb{N}$, *then* $c \geq a$

Definition 4.1.4 *A sequence is increasing if $a_{n+1} \geq a_n \ \forall \ n \in \mathbb{N}$ and decreasing if $a_{n+1} \leq a_n \ \forall \ n \in \mathbb{N}$.*
*A sequence is **monotone** if it is increasing or decreasing.*

Theorem 4.1.4 (Monotone Convergence Theorem) *If a sequence is bounded and monotone, it converges.*

Theorem 4.1.5 (Bolzano Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

Definition 4.1.5 *A sequence is **Cauchy** if*
$\forall \ \epsilon > 0 \ , \exists \ N \in \mathbb{N} : \forall \ m,n \geq N, \ |a_m - a_n| < \epsilon.$

Theorem 4.1.6 *All convergent sequences are Cauchy sequences.*

Theorem 4.1.7 *All Cauchy sequences are bounded.*

Theorem 4.1.8 (Cauchy Criterion) *A sequence in convergent if and only if it is Cauchy.*

The Axiom of Completeness, Nested Interval Property, Monotone Convergence Theorem, Bolzano Weierstrass Theorem and Cauchy's Criterion all assert the same fact - The completeness of $\mathbb{R}$ - in their own language. Any one of them can be taken as an axiom and used to prove the rest.

4.2 Series

Definition 4.2.1 *Let x_n be a sequence. A series is an expression of the form $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots$ is called a **series**.*

Definition 4.2.2 *The sequence $s_n = x_1 + x_2 + \cdots + x_n$ is called the **sequence of partial sums**.*

The series $\sum_{n=1}^{\infty} s_n$ is said to converge to a s if the corresponding sequence of partial sums s_n converges to s.
$\sum_{n=1}^{\infty} =$ s $\leftrightarrow s_n \rightarrow$ s.

Theorem 4.2.1 (Algebraic Limit Theorem for Series) *Let $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$. Then*

1. $\sum_{n=1}^{\infty} ca_n = cA \ \forall c \in \mathbb{R}$.
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$.

Theorem 4.2.2 (Cauchy Criterion for Series) *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if*
$\forall \epsilon > 0, \exists N \in \mathbb{N}$ *such that when* $n \geq m \geq N$, $|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$.

Theorem 4.2.3 *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \rightarrow 0$.*

Theorem 4.2.4 (Comparison Test) *Let a_k, b_k be sequences satisfying $0 \leq a_k \leq b_k$ $\forall k \in \mathbb{N}$, then*

1. *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
2. *$\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

Theorem 4.2.5 (Absolute Convergence Test) *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.*

Definition 4.2.3 *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ is said to converge **absolutely**. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is said to converge **conditionally**.*