

Real Analysis

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Chapter 1

Relations

1.1 Relations

Definition 1.1.1 Let X and Y be sets. Then $R \subseteq X \times Y$ is called a **Relation** over sets X and Y .

If $(x, y) \in R$, it is denoted as xRy , or "x is related to y", where $x \in X$ and $y \in Y$. Sometimes, relations over two sets are called binary relations as in general, relations can be subsets of Cartesian Product of more than two sets.

1.2 Functions

Definition 1.2.1 A Relation is called **functional** or right unique if:
 $\forall x \in X, \forall y \in Y, \forall z \in Y, (x, y) \in R \wedge (x, z) \in R \implies y = z$

Definition 1.2.2 A Relation is called **serial** or left total if:
 $\forall x \in X, \exists y \in Y: (x, y) \in R$

Definition 1.2.3 A Relation that is functional and serial is called a **function**.

A function is usually denoted by

$$f : X \rightarrow Y$$

and is usually thought of as a rule that assigns every element in X to exactly one element in Y .

Definition 1.2.4 A function that is left unique is called **injective** or one-one.

Definition 1.2.5 A function that is right total is called **surjective** or onto.

Definition 1.2.6 A function that is both one-one and onto is called a **bijective function** or a **bijection**.

If a function f is bijective, we its **inverse** as

$$f^{-1}: Y \rightarrow X$$

1.3 Homogeneous Relations

Definition 1.3.1 A **Homogeneous Relation** is a subset of a Cartesian Product of a set with itself.

Definition 1.3.2 A Relation R is **Reflexive** if
 $\forall x \in X, (x, x) \in R$

Definition 1.3.3 A Relation is **Symmetric** if
 $\forall x, y \in X, (x, y) \in R \implies (y, x) \in R$

Definition 1.3.4 A Relation is **Transitive** if
 $\forall x, y, z \in X, (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$

Definition 1.3.5 A Relation is **Antisymmetric** if
 $\forall x, y \in X, (x, y) \in R \implies (y, x) \notin R$

1.4 Equivalence Relations

Definition 1.4.1 A Relation that is Reflexive, Symmetric and Transitive is called an **Equivalence Relation**.

Equivalence relations are usually denoted by \sim . Let \sim be an equivalence relation on a set X . If $x, y \in X$ are related by \sim then it is denoted by $x \sim y$.

Definition 1.4.2 Let \sim be an equivalence relation defined on a set X .

Let $x \in X$. Then the set

$$[x] := \{y \mid y \sim x\}$$

is called the **equivalence class** containing x .

Equivalence classes have the following properties -

1. Let $x, y \in X$, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$
2. $\bigcup_{x \in X} [x] = X$.

Thus equivalence classes form a partition of the set X .

1.5 Partial Orders

Definition 1.5.1 A Relation that is Reflexive, Antisymmetric and Transitive is called a **Partial Order**.

Chapter 2

Equinumerosity and Countability

Chapter 3

Sequences and Series of Real Numbers

3.1 Sequences

Definition 3.1.1 A *sequence* is a function whose domain is \mathbb{N} .

Definition 3.1.2 A sequence a_n is said to **converge** to a limit a if $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - a| < \epsilon$

Definition 3.1.3 A sequence x_n is said to be **bounded** if $\exists M > 0 : \forall n \in \mathbb{N}, |x_n| < M$

Theorem 3.1.1 All convergent sequences are bounded.

Theorem 3.1.2 (Algebraic Limit Theorem) Let a_n and b_n be sequences, such that $a_n \rightarrow a$ and $b_n \rightarrow b$. Then,

1. $a_n + b_n \rightarrow a + b$
2. $ca_n \rightarrow ca$
3. $a_nb_n \rightarrow ab$
4. $\frac{a_n}{b_n} \rightarrow \frac{a}{b}, b \neq 0$

Theorem 3.1.3 Let a_n and b_n be sequences, such that $a_n \rightarrow a$ and $b_n \rightarrow b$. Then,

1. If $a_n \geq 0 \forall n$, then $a \geq 0$
2. If $a_n \geq b_n \forall n$, then $a \geq b$
3. If $\exists c \in \mathbb{R}$, such that $c \leq b_n \forall n \in \mathbb{N}$, then $c \leq b$. Similarly, if $c \geq a_n \forall n \in \mathbb{N}$, then $c \geq a$

Definition 3.1.4 A sequence is increasing if $a_{n+1} \geq a_n \forall n \in \mathbb{N}$ and decreasing if $a_{n+1} \leq a_n \forall n \in \mathbb{N}$.
A sequence is **monotone** if it is increasing or decreasing.

Theorem 3.1.4 (Monotone Convergence Theorem) If a sequence is bounded and monotone, it converges.

Theorem 3.1.5 (Bolzano Weierstrass Theorem) Every bounded sequence has a convergent subsequence.

Definition 3.1.5 A sequence is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, |a_m - a_n| < \epsilon$.

Theorem 3.1.6 All convergent sequences are Cauchy sequences.

Theorem 3.1.7 All Cauchy sequences are bounded.

Theorem 3.1.8 A sequence is convergent if and only if it is Cauchy.

3.2 Series

Definition 3.2.1 Let x_n be a sequence. A series is an expression of the form $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \dots$ is called a **series**.