CHAPTER 2

Relations and Functions

The word "relation" suggests some familiar examples such as the relation of father to sons, brother to sister etc. In mathematics and computer science, familiar relations are "greater than", "less than", "is a subset of", "is equal to" etc. These are the examples of relationship between two objects. There exists relationship between three or more objects.

In this chapter, we will study relation between a pair of objects, called binary relation. We will discuss different properties that a binary relation may possess and discuss different types of relations. We introduce a special type of relation, a function that plays an important role in mathematics, computer science and many applications.

2.1 RELATIONS

We have already defined cartesian product of sets A and B in chapter 1. We have seen that any element in $A \times B$ is just an ordered pair.

☞ **DEF.** Let A and B be sets. A **binary relation** or simply a **relation** R from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a, b) \in R$ for $a \in A$, $b \in B$, then we say that *a is related to b by R*, or we write aRb. If a is not related to b by R, we write $a \not R\ b$.

If $A = B$, then we say R is a binary relation on A.

☞ **DEF.** Let R be a relation from a set A to a set B. The **domain of R**, denoted by Dom (R), is defined as

$$\text{Dom}(R) = \{x \mid x \in A \text{ and } (x, y) \in R \text{ for some } y \in B\}$$

Similarly, we can define range of a relation.

☞ **DEF.** The **range of R**, denoted by ran (R), is defined as

$$\text{Ran}(R) = \{y \mid y \in B \text{ and } (x, y) \in R \text{ for some } x \in A\}.$$

Note that Dom (R) $\subseteq A$ and Ran (R) $\subseteq B$.

We can give a number of examples.

Example 1. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$

Then $R = \{(1, x), (2, y), (3, z)\}$ is a relation from A to B.

Here Dom (R) = $\{1, 2, 3\} = A$ and Ran (R) = $\{x, y, z\} = B$.

Example 2. Let $A = \{1, 2, 3, 4\}$. Define a relation R on A as aRb if and only if $a < b$.

Then $R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

Here Dom $(R) = \{1, 2, 3\}$ and Ran $(R) = \{2, 3, 4\}$

Example 3. Let $A = \{1, 2, 3, 4, 5\}$.

Define a relation R on A as aRb iff $a = b$.

Then $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.

Here Dom $(R) = A =$ Ran (R).

Example 4. Consider the set Z of integers. Define a relation R on Z as $x\,R\,y$ iff y is a square of x, where $x, y \in Z$.

Then $R = \{(1, 1), (2, 4), (3, 9), (-2, 4), \ldots\ldots\}$.

Here dom $(R) = Z$ and Ran $(R) = \{0, 1, 4, 9, 16, 25, \ldots\ldots\}$.

Example 5. Consider any collection of sets C. Then set inclusion $\subseteq$ is a relation on C since for any given pair of sets A and B, either $A \subseteq B$ or $A \not\subseteq B$.

Example 6. Let $A = Z^+$, the set of positive integers. Define a relation R on A as : aRb if and only if a divides b. Then $2R8$ but $2 \not R\ 7$.

☞ **DEF.** Let R be any relation from a set A to a set B. The **inverse relation** of R, denoted by R^{-1}, is the relation from B to A, defined as

$$R^{-1} = \{(b, a) \mid (a, b) \in R\} \text{ where } a \in A \text{ and } b \in B.$$

i.e., $$R^{-1} \subseteq B \times A.$$

Example 7. Inverse of the relation $R = \{(1, a), (2, b), (3, c), (3, a)\}$ is given by $R^{-1} = \{(a, 1), (b, 2), (c, 3), (a, 3)\}$.

Note that $(R^{-1})^{-1} = R$, Dom $(R) =$ Ran (R^{-1}) and Ran $(R) =$ Dom (R^{-1}).

Example 8. *How many distinct binary relations are there on a finite ~~site~~ set A ?*

(UPTU, B.Tech., 2002-03)

Solution. Let the number of elements in A be n, then the number of elements in $A \times A = n^2$ and the number of subsets of $A \times A = 2^{n^2}$. Since every subset of $A \times A$ is a relation on a finite set A, there are 2^{n^2} binary relations on a binary set A.

2.2 TYPES OF RELATIONS

In this section we will discuss some special types of relations which are defined on a set A, rather than the relations from A to B.

☞ **DEF.** A relation R on a set A is **reflexive** if $(a, a) \in R$ for every $a \in A$, *i.e.*, if aRa for all $a \in A$. A relation R on a set A is **irreflexive** if $a \not R\ a$ for every $a \in A$.

Example 9. Let $A = \{1, 2, 3\}$. Then the relation

$$R = \{(1, 1), (1, 2), (3, 1), (3, 3)\}$$

is not reflexive since $(2, 2) \notin R$, whereas the relation

$$S = \{(1, 1), (2, 2), (3, 3)\}$$

is a reflexive relation.

Example 10. Set inclusion $\subseteq$ on a collection C of sets is a reflexive relation since every set is a subset of itself.

Example 11. Let L be set of lines in the plane. Then the relation "is perpendicular to" is not reflexive since no line is perpendicular to itself.

DEF. A relation R on a set A is said to be a **symmetric relation** if whenever aRb, then bRa, *i.e.*, if whenever $(a, b) \in R$ then $(b, a) \in R$, for every $a, b \in A$.

Example 12. The relation '$\leq$' defined on the set of real numbers is not symmetric, since $2 \leq 3$ but $3 \nleq 2$

Example 13. Consider the set of triangles T, in a plane then the relation "is similar to" is both reflexive and symmetric.

DEF. A relation R on a set A is called **antisymmetric** if for all $a, b \in A$, aRb and $bRa \Rightarrow a = b$. *i.e.*, if whenever $(a, b), (b, a) \in R$ then $a = b$.

Note that the properties of being symmetric and being antisymmetric are not negatives of each other.

Example 14. Let $A = \{1, 2, 3\}$ then the relation $R = \{(1, 2), (2, 1), (2, 3)\}$ is not symmetric because $(2, 3) \in R$ but $(3, 2) \notin R$. It is not antisymmetric because $(1, 2), (2, 1) \in R$ but $2 \neq 1$, whereas the relation $R_1 = \{(1, 1)\ (2, 2)\}$ is symmetric as well as antisymmetric.

DEF. For a set A, a relation R on A is called **transitive** if for all $a, b, c \in A$ $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$.

Example 15. The relation R defined in Examples 9, 12 are transitive whereas relation R of Example 14 is not transitive because $(1, 2), (2, 1) \in R$ but $(1, 1), (2, 2) \notin R$.

2.3 EQUIVALENCE RELATIONS

DEF. A relation R in a set A is called an **equivalence relation** if it is reflexive, symmetric and transitive.

Example 16. *Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x, y) \mid x - y$ is divisible by $3\}$ Check whether R is an equivalence relation or not. Also draw the graph of R.*

Solution. (*i*) Reflexive : For any $a \in A$,

$$a - a \text{ is divisible by 3.}$$

Hence aRa, *i.e.*, R is reflexive.

(*ii*) Symmetric : For $a, b \in A$, let aRb

$\Rightarrow$ $a - b$ is divisible by 3.

$\Rightarrow$ $-(a - b)$ is divisible by 3.

$\Rightarrow$ $b - a$ is divisible by 3.

$\Rightarrow$ bRa

i.e., R is symmetric.

(*iii*) Transitive : For any $a, b, c \in A$, let aRb and bRc

$\Rightarrow$ $a - b$ and $b - c$ are divisible by 3

$\Rightarrow$ $(a - b) + (b - c)$ is divisible by 3

$\Rightarrow \quad a - c$ is divisible by 3

$\Rightarrow \quad aRc$

Thus R is transitive.

Hence, the relation R is an equivalence relation.

Graph of the relation R is given in Fig. 2.1.

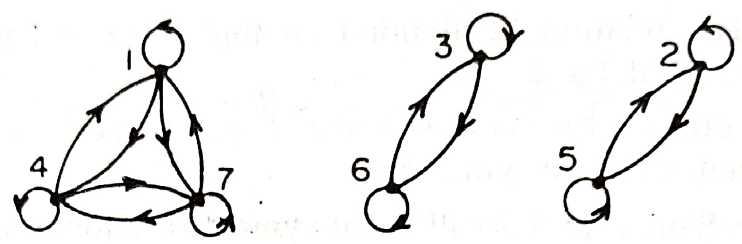

Fig. 2.1.

In general, for a fixed positive integer m, "$x - y$ is divisible by m" can also be written as "$x \equiv y \pmod{m}$" read as "x is congruent to y mod m", *e.g.*, $11 \equiv 3 \pmod{4}$ since $11 - 3 = 8$ and 4 divides 8.

This relation of congruence modulo m is an equivalence relation over the set Z of integers. Example 16 can be generalised as follows :

Example 17. *Let $A = Z$ and let $m \in Z^+$.*
Then $R = \{(a, b) \in A \times A \mid a \equiv b \pmod{m}\}$ is an equivalence relation.

Solution. Solution of this is analogous to Example 16.

Example 18. *Show that the set inclusion $\subseteq$ relation on a collection C of sets is not an equivalence relation.*

Solution. It is reflexive since every set is a subset of itself. It is transitive since for $A, B, C \in$ C if

$$A \subseteq B \text{ and } B \subseteq C$$

then
$$A \subseteq C$$

But it is not symmetric since for $A, B \in$ C, $A \subseteq B$ does not imply $B \subseteq A$.

Example 19. *Let $A = Z^+$. The relation R on $A \times A$ is defined as : $(a, b)\, R\, (c, d)$ iff $ad = bc$. Prove that R is an equivalence relation.*

Solution. We must show that R is reflexive, symmetric and transitive.

(*i*) **Reflexivity :** Since for each $(a, b) \in A \times A$, we have

$$ab = ba$$

i.e., $\quad (a, b)\, R\, (a, b)$.

Hence R is reflexive.

(*ii*) **Symmetry :** Let us suppose $(a, b)\, R\, (c, d)$

$\Rightarrow \quad ad = bc$

$\Rightarrow \quad cb = da$

$\Rightarrow \quad (c, d)\, R\, (a, b)$

Thus, R is symmetric.

(*iii*) **Transitivity :** Suppose $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$.

i.e., $\quad ad = bc$ and $cf = de$

$\Rightarrow \quad (ad)(cf) = (bc)(de)$ [Multiplying corr. sides of the eqns.]

$\Rightarrow \quad a(dc)f = b(cd)e$

$\Rightarrow \quad af = be$ (dividing by cd, $\because cd \neq 0$)

$\Rightarrow \quad (a, b)\ R\ (e, f)$

Thus, R is transitive.

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example 20. *Let $A = R \times R$ (R is the set of real numbers) and define the following relation on $A : (a, b)\ R\ (c, d)$ iff $a^2 + b^2 = c^2 + d^2$.*

(*i*) *Verify that (A, R) is an equivalence relation.*

(*ii*) *Describe geometrically what the equivalence classes are for this relation (Justify).*

Solution. (*i*) **Reflexive.** For $(a, a) \in R \times R$

$$a^2 + a^2 = a^2 + a^2$$

$\Rightarrow \quad (a, a)\ R\ (a, a)$

$\Rightarrow$ R is reflexive.

Symmetric. For $(a, b), (c, d) \in R \times R$

Let $\quad (a, b)\ R\ (c, d)$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$

$\Rightarrow \quad c^2 + d^2 = a^2 + b^2$

$\Rightarrow \quad (c, d)\ R\ (a, b)$

$\therefore \quad$ R is symmetric.

Transitive. For $(a, b), (c, d), (e, f) \in R \times R$

Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$

$\Rightarrow \quad a^2 + b^2 = e^2 + f^2$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

$\Rightarrow \quad$ R is transitive.

Hence, the relation R is an equivalence relation.

(*ii*) For any point (a, b), the sum $a^2 + b^2$ is the square of the distance from the origin. The equivalence classes are, therefore, the set of points in the plane which have the same distance from the origin. Hence, the equivalence classes are concentric circles centered on the origin.

Example 21. *Let R be a relation in the set of all lines in a plane defined by "aRb if line a is parallel to line b". Then, show that R is an equivalence relation.*

Solution. For any three lines $l_1 m_1 n$ of the given set

(*i*) **Reflexive.** For each l, $l \parallel l$

$\Rightarrow$ $l \, R \, l$

$\Rightarrow$ R is reflexive.

(*ii*) **Symmetry.** Let lRm

$\Rightarrow$ $l \parallel m$

$\Rightarrow$ $m \parallel l$

$\Rightarrow$ $m \, R \, l$

i.e., R is symmetric.

(*iii*) **Transitive.** Let lRm and mRn

i.e., $l \parallel m$ and $m \parallel n$

$\Rightarrow$ $l \parallel n$

$\Rightarrow$ $l \, R \, n$

i.e., R is transitive.

Hence, from (*i*), (*ii*) and (*iii*) we conclude that R is an equivalence relation.

Example 22. *Let R be the binary relation defined as*

$$R = \{(a, b) \in R^2 : a - b \leq 3\}$$

Determine whether R is reflexive, symmetric, antisymmetric and transitive.

Solution. Reflexive. For $(a, a) \in R^2$

$$a - a = 0 \leq 3$$

$\Rightarrow$ aRa

$\Rightarrow$ R is reflexive.

Symmetric. R is not symmetric since $(4, 8) \in R$ as $4 - 8 = -4 \nless 3$

and $(8, 4) \notin R$ as $8 - 4 = 4 \nless 3$.

Transitive. R is not transitive since

$(8, 6), (6, 4) \in R$ as $8 - 6 = 2 < 3$ and $6 - 4 = 2 < 3$

but $(8, 4) \notin R$ as $8 - 4 = 4 > 3$.

Antisymmetric. R is not antisymmetric as $(8, 6) \in R$ and $(6, 8) \in R$.

But $8 \neq 6$.

Example 23. *Let R be a binary relation on the set of all positive integers such that*

$$R = \{(a, b) : a - b \text{ is an odd positive integer}\}$$

Is R reflexive ? Symmetric ? Antisymmetric ? Transitive ? An equivalence ? A partial ordering relation ?

Solution. Let $a, b, c \in Z^+$, the set of all the integers, be any elements.

Reflexive. $a - a = 0$ which is not odd.

$\therefore$ R is not reflexive.

Symmetric. Let aRb

i.e., $a - b$ is an odd positive integer.

But $b - a = -(a - b)$

$= -$ (odd positive integer)

$=$ odd negative integer

$\therefore \quad b \not R\, a.$

i.e., R is not symmetric.

Transitive. Let aRb and bRc

$\Rightarrow \quad a - b,\ b - c$ are odd positive integers.

Now $a - c = (a - b) + (b - c)$

$=$ odd positive integer + odd positive integer

$=$ even positive integer

$\Rightarrow \quad a \not R\, c \quad \Rightarrow R$ not transitive.

Antisymmetric. Let aRb and bRa.

i.e., $a - b$ and $b - a$ are odd positive integers.

This is possible only if $a = b$.

i.e., R is antisymmetric.

Hence, R is not an equivalence relation. Also, R is not a partial ordering relation.

Example 24. *Let R be a binary relation on the set of all strings of 0's and 1's such that $R = \{(a, b) : a$ and b are strings that have the same number of 0's$\}$. If R reflexive ? Symmetric ? Antisymmetric ? Transitive ? An equivalence relation ? A partial ordering relation ?*

Solution. Let S denote the set of all strings of 0's and 1's and let $a, b, c \in S$ be any elements.

Reflexive. Since aRa for all $a \in S$

$\therefore \quad R$ is reflexive.

Symmetric. Let aRb

i.e., a and b have the same number of 0s.

i.e., b and a have same number of 0s.

$\Rightarrow \quad bRa \quad \Rightarrow \quad R$ is symmetric.

Antisymmetric. Let aRb and bRa

i.e., a and b have equal number of 0s, this does not imply that $a = b$.

$\because \quad$ for $a = 1001$ and $b = 010$

aRb but $a \neq b$

$\therefore \quad R$ is into antisymmetric.

Transitive. Let aRb and bRc

i.e., a and b have same number of 0s and b and c have same number of 0s.

$\Rightarrow \quad a$ and c have same number of 0s.

$\Rightarrow \quad aRc \quad \Rightarrow \quad R$ is transitive.

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation. But R is not a partial ordering relation.

Example 25. *Let S be the set of all points in a plane. Let R be a relation such that for any two points, a and b : $(a, b) \in R$ if b is within two centimetre from a, show that R is an equivalence relation.*

Solution. Reflexive. Let $a \in S$, then $(a, a) \in R$ as $d(a, a) = 0 \leq 2$

$\therefore$ R is reflexive.

Symmetric. Let $a, b \in S$ be such that $(a, b) \in R$

$\Rightarrow$ $d(a, b) \leq 2$

$\Rightarrow$ $d(b, a) \leq 2$

$\Rightarrow$ $(b, a) \in R$

$\therefore$ R is symmetric.

Transitive. Let $a, b, c \in S$ be such that

$$d(a, b) \leq 2, d(b, c) \leq 2.$$

but $d(a, c) \nleq 2$

For example take points $a = (2, 0), b = (3, 0), C = (4.5, 0)$, then

$$d(a, b) = 1 \leq 2$$

$$d(b, c) = 1.5 \leq 2$$

but $$d(a, c) = 2.5 > 2$$

$\therefore$ R is not transitive.

Hence, R is not an equivalence relation.

Example 26. *If R and S are equivalence relations on a set A, show that the following are equivalence relations :* **(UPTU, B.Tech. 2005-04)**

(*i*) $R \cap S$ (*ii*) $R \cup S$

Solution. Let $a, b, c \in A$ be any elements.

(*i*) (*a*) **Reflexive.**

$$(a, a) \in R \text{ and } (a, a) \in S$$

as R and S are equivalence relations.

Hence $(a, a) \in R \cap S$, for all $a \in A$

$\Rightarrow$ $R \cap S$ is a reflexive relation.

(*b*) **Symmetric.** Let $(a, b) \in R \cap S$

$\Rightarrow$ $(a, b) \in R$ and $(a, b) \in S$

$\Rightarrow$ $(b, a) \in R$ and $(b, a) \in S$

$\Rightarrow$ $(b, a) \in R \cap S$. [$\because$ R and S are symmetric]

$\therefore$ $R \cap S$ is a symmetric relation.

(*c*) **Transitive.** Let $(a, b), (b, c) \in R \cap S$

$\Rightarrow$ $(a, b) \in R, (a, b) \in S$ and $(b, c) \in R, (b, c) \in S$

$\Rightarrow$ $(a, b) \in R, (b, c) \in R$ and $(a, b) \in S, (b, c) \in S$

$\Rightarrow$ $(a, c) \in R$ and $(a, c) \in S$

$\Rightarrow$ $(a, c) \in R \cap S$. [$\because$ R and S are transitive relation]

$\therefore$ $R \cap S$ is a transitive relation.

Hence, from (*a*), (*b*) and (*c*), $R \cap S$ is a equivalence relation.

(ii) The relation $R \cup S$ need not be an equivalence relation.

For example :

Let $R = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$

$S = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$

The R and S are equivalence relations on $A = \{1, 2, 3\}$

Now $R \cup S = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2), (1, 3), (3, 1)\}$

$(1, 3), (3, 2) \in (R \cup S)$ but $(1, 2) \notin R \cup S$

$\therefore$ $R \cup S$ is not transitive and hence not an equivalence relation.

Equivalence Relations and Partitions

☞ **DEF.** **A partition P of a set S** is a collection $\{P_i\}_{i=1}^{n}$ of non-empty subsets of S with the following properties :

1. Each $a \in S$ belongs to some P_i or we can say $P_1 \cup P_2 \cup \ldots\ldots, \cup P_n = S$
2. If $P_i \neq P_j$ then $P_i \cap P_j = \phi$

The sets in P are called the **blocks or cells** of the partition.

Example 27. Let $A = \{1, 2, 3, 4, 5\}$.

Consider the subsets A_1, A_2, A_3, A_4, A_5 of A where $A_1 = \{1, 2\}$, $A_2 = \{1, 2, 5\}$, $A_3 = \{3, 4, 5\}$, $A_4 = \{3, 4\}$ and $A_5 = \{1, 3, 4, 5\}$.

Then $\{A_1, A_5\}$ is not a partition since $A_1 \cap A_5 \neq \phi$, $\{A_1, A_4\}$ so not a partition because 5 does not belong to either A_1 or A_4.

$\{A_1, A_3\}$ is a partition of A since $A_1 \cap A_3 = \phi$ and $A_1 \cup A_3 = A$.

☞ **DEF.** Let R be an equivalence relation on a set S. For any $a \in S$, the set $[a]_R \subseteq S$ defined by

$$[a]_R = \{x \mid x \in S \text{ and } (a, x) \in R\}.$$

is called an **R–equivalence class of a in S.**

☞ **DEF.** The collection of all equivalence classes of elements of S under an equivalence relation R, denoted by S/R *i.e.,*

$$S/R = \{[a] \mid a \in S\}$$

is called the **quotient set** of S by R.

Example 28. *Let R be an equivalence relation on a non-empty set S. Let $a, b \in S$ be any elements. Then prove that*

(i) *For each $a \in S$, $a \in [a]$*

(ii) *$[a] = [b]$ if and only if aRb*

(iii) *If $[a] \neq [b]$ then $[a] \cap [b] = \phi$ i.e., Equivalence classes of a and b are either disjoint or identical.*

Solution. (i) Since R is reflexive therefore for each $a \in S$, $aRa \Rightarrow a \in [a]$

(ii) Suppose aRb. We want to show that $[a] = [b]$.

Let $x \in [b]$ be any element

i.e., $(b, x) \in R$ and $(a, b) \in R$ (by supposition)

$\therefore$ by transitivity $(a, x) \in R$

$\Rightarrow$ $x \in [a]$

$\Rightarrow$ $[b] \subseteq [a]$

Similarly, we can also show that $[a] \subseteq [b]$.

Thus $aRb \Rightarrow [a] = [b]$.

Conversely, if $[a] = [b]$ then show that aRb

By (*i*) $b \in [b] = [a]$

$\Rightarrow$ $b \in [a]$

$\Rightarrow$ $aRb \Rightarrow (a, b) \in R$

(*iii*) To prove this, we prove that

If $[a] \cap [b] \neq \phi$ then $[a] = [b]$

If $[a] \cap [b] \neq \phi$, then there exists an element $x \in S$ such that $x \in [a] \cap [b]$.

$\Rightarrow$ $x \in [a]$ and $x \in [b]$

$\Rightarrow$ $(a, x) \in R$ and $(b, x) \in R$

$\Rightarrow$ $(a, x) \in R$ and $(x, b) \in R$ (by symmetry)

$\Rightarrow$ $(a, b) \in R$ (by transitivity)

$\Rightarrow$ $[a] = [b]$ [by (*ii*)]

These properties of equivalence classes give us an important result which can be stated as :

Theorem 1. *An equivalence relation R in a non-empty set S generates a unique partition of S and conversely, a partition of S defines an equivalence relation in S.*

Example 29. Let Z be the set of integers and let R be the relation called "congruence modulo 5" which is defined as

$$R = \{(x, y) \in z \times z \mid x - y \text{ is divisible by } 5\}.$$

Then R is an equivalence relation on Z and the equivalence classes are

$$[0]_R = \{......, -15, -10, -5, 0, 5, 10, 15, ...\}$$
$$[1]_R = \{......, -9, -4, 1, 6, 11, 16, ...\}$$
$$[2]_R = \{......, -8, -3, 2, 7, 12, ...\}$$
$$[3]_R = \{......, -7, -2, 3, 8, 13, ...\}$$
$$[4]_R = \{......, -6, -1, 4, 9, 14, ...\}$$
$$Z/R = \{[0]_R, [1]_R, [2]_R, [3]_R, [4]_R\}$$

Example 30. *Let* $X = \{1, 2, 3, 4, 5\}$ *and let* $P = \{\{1, 2\}, \{3\}, \{4, 5\}\}$. *Find the equivalence relation defined by the partition P.*

Solution. The relation defined by partition P is

$$R = \{(1, 1), (2, 2), (1, 2), (2, 1), (3, 3), (4, 4), (5, 5), (4, 5), (5, 4)\}.$$

2.4 THE MATRIX OF A RELATION

☞ **DEF.** A relation R from a finite set A to a finite set B can be represented by a matrix called the **matrix of the relation** or **relation matrix.**

Let $A = \{a_1, a_2, \ldots\ldots, a_n\}$ and $B = \{b_1, b_2, \ldots\ldots, b_n\}$

are finite sets and R be a relation from A to B. Then R can be represented by the $m \times n$ matrix $M_R = [m_{ij}]$ where m_{ij} is defined by

$$m_{ij} = \begin{cases} 1 & \text{if } a_i R b_j \quad \text{or } (a_i, b_j) \in R \\ 0 & \text{if } a_i \not R\, b_j\,; \text{or } (a_i, b_j) \notin R \end{cases}$$

Conversely, if sets A and B are given, then from the relation matrix whose entries are zeroes and ones, we can determine a relation.

Example 31. Let R be a relation defined by $R = \{(1, a), (2, b), (3, a)\}$ where $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then the matrix of R is given by

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 32. Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Since M is 3×4 matrix, we can write $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$

Then $\quad R = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_4), (a_3, b_1), (a_3, b_4)\}$,

since $(a_i, b_j) \in R$ if and only if $m_{ij} = 1$.

The matrix of the relation can be used to determine whether the relation has various properties. Let $M = [m_{ij}]$ represents the matrix of relation R.

(*i*) **Reflexive.** A relation is reflexive if all the elements in the main diagonal of the relation matrix are 1.

i.e., if $m_{ii} = 1$, then the relation is reflexive.

Thus, $$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the relation matrix of a reflexive relation R. If all $m_{ii} = 0$, then the relation is irreflexive.

(*ii*) **Symmetric.** If matrix of the relation R is a symmetric matrix *i.e.*, if $m_{ij} = m_{ji}$ for all values of i and j, then the relation is symmetric. A relation is antisymmetric if and only if $m_{ij} = 1$ necessitates that $m_{ji} = 0$. For example,

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) **Transitive.** There is no simple way to test whether a relation R is transitive by examining its relation matrix. A relation R is transitive if and only if its matrix $M_R = [m_{ij}]$ has the property, if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$. Thus, the transitivity of R means that if $M_R^2 = M_R \cdot M_R$ has a 1 in any position than M_R must have 1 in the same position. Thus, in particular, if $M_R^2 = M_R$, then R is transitive. But the converse is not true.

Example 33. *Let $A = \{1, 2, 3\}$ and let, R be a relation on A whose matrix is*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that R is transitive.

Solution. $M_R^2 = M_R \cdot M_R$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The relation is transitive as M_R^2 has 1 in the position then M_R must have 1 in the same position.

2.5 PICTORIAL REPRESENTATION OR GRAPHS OF RELATIONS

If R is a relation on a finite set $A = \{a_1, a_2, \ldots\ldots, a_n\}$ then we can also represent a relation pictorially by drawing graph as follows. The elements of A are represented by points or circles called nodes or vertices. If $a_i R a_j$ then draw an arrow, called an edge, from vertex a_i to vertex a_j. This type of graph of relation R is called a **directed graph or digraph.**

Example 34. Let $A = \{1, 2, 3, 4\}$ and

$$R = \{(1, 1), (1, 3), (2, 3), (2, 4), (1, 4), (3, 4), (1, 2)\}$$

Then the digraph is as shown in Fig. 2.2.

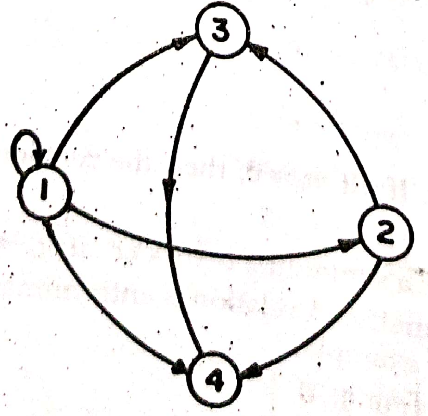

Fig. 2.2.

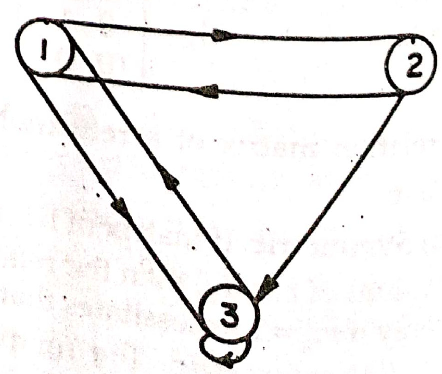

Fig. 2.3.

Conversely, from the graph we can determine the relation.

Example 35. Find the relation determined by the graph shown in Fig. 2.3.

Since a_iRa_j if there is an edge from a_i to a_j, we have

$$R = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 3)\}$$

The number of edges in the graph of R is same as the number of ordered pairs in R.

The graphical representation of a graph can be used to determine whether the relation has various properties.

(*i*) A relation is *reflexive* if and only if there is a loop at every vertex of the directed graph. If no vertex has a loop, then the relation is irreflexive.

(*ii*) A relation is *symmetric* if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction too *i.e.*, if (u, b) is in the relation than (b, a) is also there in the relation. A relation is *antisymmetric* if no two distinct points in the digraph have an edge going between them in both directions.

(*iii*) A relation is *transitive* if and only if whenever there is a directed edge from a vertex a to a vertex b and from a vertex b to vertex c, then there is also a directed edge from a to c.

Example 36. *Determine whether the relation for the directed graph shown in Fig.* 2.4 *are reflexive, symmetric, antisymmetric, transitive.*

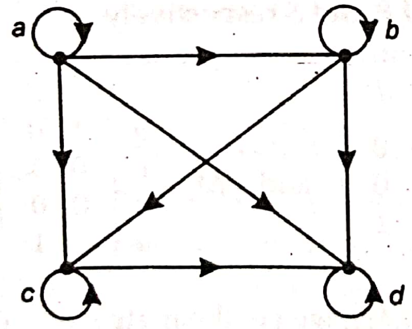

Fig. 2.4

Solution. Reflexive. The given relation is *reflexive* since there is a loop at every vertex of the digraph of the relation.

Symmetric. The relation is not symmetric since there is a directed edge from a to b but there is no directed edge from b to a, in the digraph of the relation *i.e.*, aRb but $b\not R\, a$.

Antisymmetric. The relation is antisymmetric, since there is at most one directed edge between each pair of vertices in the digraph of the relation.

Transitive. The relation is transitive. The digraph of the relation has the property whenever there are directed edges from a and b and from b and c, there is also a directed edge from a and c.

2.6 COMPOSITION OF RELATIONS

Let A, B and C be sets and let R be a relation from A to B and S be a relation from B to C. Then the composition of relations R and S, denoted by RoS, is a relation from A to C, defined by

$$RoS = \{(a, c) \in A \times C \mid (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B\}.$$

The composition of R and S is sometimes denoted by RS.

Example 37. Let $R = \{(1, 2), (3, 4), (2, 2)\}$ and $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$ be relations on a set $A = \{1, 2, 3, 4, 5\}$.

Then $\quad RoS = \{(1, 5), (3, 2), (2, 5)\}$

$\quad SoR = \{(4, 2), (3, 2), (1, 4)\}$

Note that $\quad RoS \neq SoR$

$\quad RoR = \{(1, 2), (2, 2)\}$

$\quad SoS = \{(4, 5), (3, 3), (1, 1)\}$

$\quad Ro(SoR) = \{(3, 2)\}$

$\quad (RoS)oR = \{(3, 2)\} = Ro(SoR)$

We have seen that $RoS \neq SoR$ which shows that composition of relations is not commutative. But $(RoS)oR = Ro(SoR)$ *i.e.*, composition of relations is associative.

We can also find the composition of relations R and S using matrices. Let M_R and M_S denote the matrices of R and S respectively.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \end{array}$$

Then multiplying M_R and M_S, we get the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \end{array}$$

The non-zero entries in this matrix M tell us which elements are related by RoS. Thus we can find RoS by multiplying M_R and M_S. *i.e.*, $M_R M_S$ and M_{RoS} have the same non-zero entries.

2.7 CLOSURES OF RELATIONS

Let R be a relation on a non-empty set A. R may or may not be reflexive, symmetric and transitive . A relation R can be made reflexive by adding a few ordered pairs (x, x), then a relation R_1, so obtained is the smallest relation containing R which is reflexive. Such R_1 will be called the reflexive closure of R.

It may be noted here that the closure of a relation with respect to a property may not exist.

Reflexive closure. The reflexive closure of a relation R defined on a non-empty set is the smallest reflexive relation that contains R as a subset.

Given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R. Thus, reflexive closure of $R = R \cup \Delta$

where $\Delta = \{(a, a) \mid a \in A\}$ is the diagonal relation on A.

Example 38. Let $A = \{1, 2, 3\}$ and R be a relation on A defined $R = \{(1, 1), (2, 1), (1, 2), (3, 3)\}$. Then, R is not a reflexive relation as $(2, 2) \notin R$. This is done by adding $(2, 2)$ to R since this is the only pair of the form (a, a) that is not in R. Clearly, this new relation contains R. Also, any reflexive relation that contains R must also contain $(2, 2)$.

Example 39. *Find the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ where $a, b \in Z$, the set of integers.*

Solution. The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in Z\}$$
$$= \{(a, b) \mid a \leq b\} \quad \text{where } a, b \in Z.$$

Symmetric closure. Let R be a relation defined on a non-empty set. The symmetric closure of R is the smallest symmetric relation that contains R as a subset.

A symmetric relation contains (a, b) if it contains (b, a). The inverse relation of R *i.e.,* R^{-1} contains (a, b) if (b, a) is in R. Therefore, the symmetric closure of R can be constructed by taking the union of a relation with its inverse, *i.e.,* $R \cup R^{-1}$ is the symmetric closure of R. Geometrically, the graph of a symmetric closure of R is simply the digraph of R with all edges made bidirectional.

Example 40. *Let $R = \{(1, 2), (1, 1), (1, 4), (3, 4), (2, 2)\}$ be a relation on a set $A = \{1, 2, 3, 4\}$. Find symmetric closure of R.*

Solution. The symmetric closure of R is the relation

$$S = R \cup R^{-1} = \{(1, 2), (1, 1), (1, 4), (3, 4), (2, 2)\}$$
$$\cup \{(2, 1), (1, 1), (4, 1), (4, 3), (2, 2)\}$$
$$= \{(1, 2), (2, 1), (1, 1), (1, 4), (4, 1), (3, 4), (4, 3), (2, 2)\}$$

Clearly, S is symmetric and it contains R and is in any symmetric relation containing R.

Example 41. *Find the symmetric closure of the relation*

$$R = \{(a, b) \mid a > b\} \text{ where } a, b \in Z^+, \text{ the set of positive integers.}$$

Solution. The symmetric closure of R is the relation

$$S = R \cup R^{-1} \{(a,b) \mid a > b\} \cup \{(b,a)\ (a > b)\}$$
$$= \{(a,b) \mid (a \neq b)\}$$

Transitive closure

Let R be a relation defined on a non-empty set A. The transitive closure of R is the smallest transitive relation containing R. The transitive closure of R is denoted by R^* or R^∞.

The method to find transitive closure of a given relation is not as straight forward as we have seen in case of reflexive and symmetric closure. In this case, we shall have to use boolean matrix multiplication. We shall explain with an example.

Example 42. *Let* $A = \{1, 2, 3\}$ *and* R *be a relation on* A *given as* $R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$. *Find the transitive closure of* R.

Solution. Step 1. First write the matrix of the relation R and denote it by M_R.

So

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 2. Compute different powers of M_R, using boolean matrix multiplication. Stop the computation when M_R^n is equal to any of $M_R, M_R^2, \ldots\ldots, M_R^{n-1}$. Here M_R^n is the n^{th} power of M_R.

$$M_R^2 = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_R^3 = M_R^2 \cdot M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = M_R^2$$

[In Boolean multiplication, $0 \cdot 0 = 0$, $1 \cdot 0 = 0 \cdot 1 = 0$; $1 \cdot 1 = 1$ $1 + 1 = 1$; $0 + 1 = 1 = 1 + 0$; and $0 + 0 = 0$]

Step 3. Select distinct M_R^n from the computed list of matrices.

Step 4. Join the matrices so selected in step 3 using boolean OR on the elements of matrices while adding these matrices.

$\therefore$ Matrix for transitive closure of R denoted by M_R^∞ is given as

$$M_R^\infty = M_R \vee M_R^2$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 5. Convert the M_R^∞ in R^∞ *i.e.*, transitive closure

$$\therefore \quad R^\infty = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,2),(3,3)\}$$

This completes the answer.

We shall learn a more efficient method for computing the transitive closure of a relation, which is called **Warshall's** algorithm.

Warshall's Algorithm

Given a relation R on a non-empty set A.

Step 1. Convert the given relation into matrix form M_R and denote it by W_0, the initial Warshall matrix.

Step 2. For $k = 0$ to $|A|$, compute W_k called Warshall matrix at k^{th} state. To compute W_k from W_{k-1}, we shall proceed as follows :

(*i*) Transfer all 1's from W_{k-1} to W_k ;

(*ii*) List the rows in column k of W_{k-1} where the entry in W_{k-1} is 1, say $p_1, p_2, p_3, \ldots\ldots$

Similarly, list columns in row k of W_{k-1} where entry in W_{k-1} is 1, say it $q_1, q_2, q_3, \ldots\ldots$

(*iii*) Place 1 at all locations (p_i, q_i) in W_k, if 1 is not already there.

Let us now use this algorithm to find transitive closure of the given relation R.

Example 43. *Let* $A = \{a, b, c, d\}$ *and* R *be a relation on* A *given as* $R = \{(a,d),(b,c),(b,a),(c,a),(c,d),(d,c)\}$. *Find the transitive closure of* R.

Solution. Step 1. Represent the given matrix into matrix form M_R and denote it by W_0.

$$\therefore \quad W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 2. Here we have to find W_1, W_2, W_3 and W_4 since $|A| = 4$.

To compute W_1

(*i*) Transfer all 1's from W_0 to W_1.

(*ii*) $k = 1$, in column 1, W_0 has 1 in row 2 and 3 and in row 1, W_0 has 1 in column 4.

(*iii*) In W_1, we have additional 1 at (2, 4).

$$\therefore \quad W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To compute W_2

(*i*) Transfer all 1's from W_1 to W_2.

(*ii*) $k = 2$, in column 2, W_1 has no 1's and in row 2, W_1 has 1 in columns 1, 3 and 4.

(*iii*) W_2 is same as W_1, *i.e.*, $W_2 = W_1$.

To compute W_3

(*i*) Transfer all 1's from W_2 to W_3.

(*ii*) $k = 3$, in column 3, W_2 has 1 in row 2 and 4 and in row 3, W_2 has 1 in columns 1 and 4.

(*iii*) In W_3, we have additional 1's at (4, 1) and (4, 4)

$$\therefore \quad W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

To compute W_4

(*i*) Transfer all 1's from W_3 to W_4.

(*ii*) $k = 4$, in column 4, W_3 has 1 in all rows and in row 4, W_3 has 1 in columns 1, 3 and 4.

(*iii*) In W_4, we have additional 1's at (1, 1), (1, 3), (3, 3).

$$\therefore \quad W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and W_4 is the matrix of the transitive closure of the given relation.

Hence, $R^{\infty} = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$

The number of bit operations used by Algorithm 1 are more than the number of bit operations used by Warshall's algorithm. Also in Warshall's algorithm, it is not required to store past results as in previous method.

In Warshall's algorithm, the result is available once $W_{|A|}$ is computed. Boolean addition is not required as in previous method.

Note. *Sometimes in Warshall's algorithm, say on a relation R on a set A having $|A| = n$, we may have $W_k = W_{k+1}$ for $k, k + 1 < n$ it does not mean that we should stop procedure there. In Warshall's algorithm the computation must proceed upto $|A|$ as shown in above example.*

Example 44. *Let $A = \{1, 2, 3, 4, 5\}$ and R is a relation defined on A as*

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}.$$

Find transitive closure of R using Warshall's algorithm.

Solution. Step 1. Write the matrix representation of R and call it W_0.

$$\therefore \quad W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Step 2. To compute W_1

(*i*) Transfer all the 1's from W_0 to W_1.

(*ii*) $k=1$, in column 1, W_0 has 1 in rows 1 and 2 and in row 1, W_0 has 1 in columns 1 and 2.

(*iii*) 1 is already there at (1, 1), (1, 2), (2, 1), (2, 2) and therefore $W_1 = W_0$.

To compute W_2 :

(*i*) Transfer all the 1's from W_1 to W_2.

(*ii*) $k=2$, in column 2, W_1 has 1 in rows 1 and 2 and in row 2, W_1 has 1 in columns 1 and 2.

(*iii*) 1 is already there at (1, 1), (1, 2), (2, 1), (2, 2) and therefore $W_2 = W_1$.

To compute W_3 :

(*i*) Transfer all the 1's from W_2 to W_3.

(*ii*) $k=3$, in column 3, W_2 has 1 in row 3 and 4 and in row 3, W_2 has 1 in columns 3 and 4.

(*iii*) 1 is already there at (3, 3), (3, 4), (4, 3), (4, 4). Therefore, $W_2 = W_3$.

To compute W_4 :

(*i*) Transfer all the 1's from W_3 to W_4.

(*ii*) $k=4$, in column 4, W_3 has 1's in column 3, 4 and 5 and in row 4, W_3 has 1's in column 3, 4 and 5.

(*iii*) In W_4, we have additional 1's at (3, 5), (5, 3).

$$\therefore \quad W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

To compute W_5 :

(*i*) Transfer all the 1's from W_4 to W_5.

(*ii*) $k=5$. In column 5, W_4 has 1's in rows 3, 4, 5 and in row 5 W_4 has 1's in columns 3, 4, 5.

(*iii*) No additional 1 is added to W_5.

$$\therefore \quad W_5 = W_4.$$

W_5 is the matrix for transitive closure of the given relation R.

$$\therefore \quad R^{\infty} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$$

Exercise 2.1

1. Let A be set of real numbers. Consider the relation R on A defined as ; aRb if and only if $a^2 + b^2 = 25$. Find Dom (R) and Ran (R).
2. Let $A = \{1, 2, 3, 4, 6\}$ and a relation R on A is defined as : aRb if and only if a is a multiple of B. Find Dom (R) and Ran (R) and matrix of the relation R.

3. Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$ and a relation R from A to B is defined as: aRb if and only if $b < a$. Find Dom (R), Ran (R), and matrix of R.

4. Let R be a relation in the set $Z^+ = \{1, 2, 3, ...\}$ defined by "$x + 2y = 10$". Find Dom (R), Ran (R) and R^{-1}.

5. Let $A = \{1, 2, 3, 4, 8\}$ and let R be a relation on A defined by "aRb if and only if $a + b \leq 9$"

(a) Write R as a set of ordered pairs.

(b) Draw its digraph

(c) Find R^{-1}.

6. Find the number of relations from $A = \{1, 2, 3\}$ to $B = \{x, y\}$.

7. Let R and S be two relations on Z^+, the set of positive integers, where

$$R = \{(x, 2x) \mid x \in Z^+\} \quad \text{and} \quad S = \{(x, 7x) \mid \in Z^+\}$$

Find RoS, RoR, $RoRoR$, $RoSoR$, SoR.

8. The relation matrices M_R and M_S are given as

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Find M_{RoS}.

9. Find the relation determined by the digraph and give its matrix.

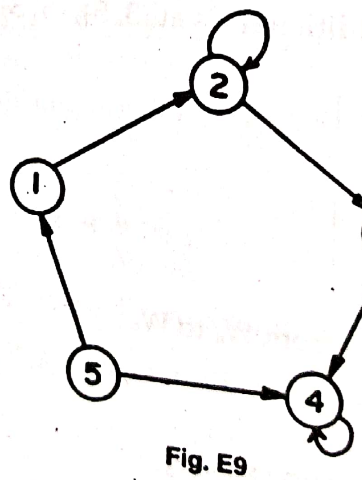

Fig. E9

10. Let $A = \{1, 2, 3, 4\}$ and let R be a relation on A defined as

$$R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$$

Is R reflexive, irreflexive, symmetric, antisymmetric or transitive?

11. Determine whether the relation R defined in a set Z^+, the set of positive integers, as "for $a, b \in Z^+$, aRb iff $a = b^K$ for some $K \in Z^+$" is reflexive, symmetric, antisymmetric or transitive.

12. Show that relation of "similarity" on the set of all triangles in a plane is an equivalence relation.

13. Let $A = \{1, 2, 3, 4, 5\}$. Determine whether relation R whose digraph given is reflexive, symmetric, antisymmetric or transitive.

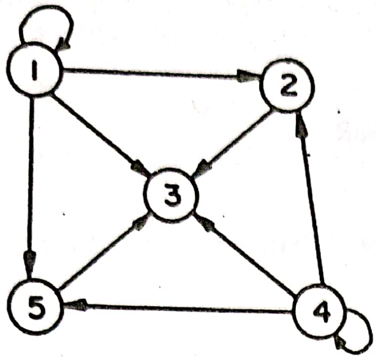

Fig. E13

14. Let $A = \{1, 2, 3, 4\}$ and let $S = A \times A$. Define the relation R on S as :
$(a, b) R (c, d)$ iff $a + b = c + d$.
Show that R is an equivalence relation and find S/R.
15. If R and S are equivalence relations on a set A, then show that $R \cap S$ is an equivalence relation in A.
16. If $\{\{1, 3, 5\}, \{2, 4\}\}$ is a partition of a set $A = \{1, 2, 3, 4, 5\}$, find the corresponding equivalence relation R.
17. Let P and Q be relation on $\{1, 2, 3, 4\}$ defined by $P = \{(a, b) \mid |a - b| = 1\}$ and $Q = \{(a, b) \mid a - b$ is even$\}$
 (a) Represent P and Q as both graphs and matrices.
 (b) Determine PoQ, PoP, QoQ.
 (c) Determine whether P and Q are reflexive, symmetric, antisymmetric or transitive.
18. Let $A = \{0, 1, 2, 3\}$ and $R = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (3, 2), (2, 3), (3, 1), (1, 3)\}$.
Show that R is an equivalence relation on A. Find equivalence classes for each $a \in A$. Determine A/R.
19. Let R be a relation on set $A = \{a, b, c, d\}$ defined by
$$R = \{(a, b), (b, c), (d, c), (d, a), (a, d), (d, d)\}$$
Determine :
 (i) Reflexive closure of R. (ii) Symmetric closure of R.
 (iii) Transitive closure of R. (UPTU, B.Tech. 2003-04)

Answers 2.1

1. Dom $(R) = \{-5, 5\}$ Ran $(R) = \{-5, 5\}$.
2. Dom $(R) = A =$ Ran (R)

and $$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Dom $= \{3, 5, 7, 9\}$ Ran $= \{2, 4, 6, 8\}$

$$M_R = \begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

6. $|A| = 3$

$|B| = 2$

$(A \times B) = 3.2 = 6$

7. $RoS = \{(x, 14x) \mid x \in Z^+\} = SoR$

$RoR = \{(x, 4x) \mid x \in Z^+\}$

$RoRoR = \{(x, 8x) \mid x \in Z^+\}$

$RoSoR = \{(x, 8x) \mid x \in Z^+\}$

8. $M_{RoS} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

9. $R = \{(1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 1), (5, 4)\}$

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

10. None.

11. Reflexive, antisymmetric, transitive.

13. Antisymmetric, transitive.

2.8 FUNCTIONS

A function is a special type of relation that plays an important role in mathematics, computer science and many applications. The words mapping, transformation, correspondence and operator are also used as synonyms for functions.

☞ **DEF.** Let A and B be non-empty sets. A **function** from A to B denoted by $f: A \rightarrow B$ is a relation from A into B such that each element of A is related to exactly one element of B. The set A is called the **domain** of the function and set B is called the **co-domain**.

We often write $f(a) = b$ when (a, b) is an ordered pair in the function f.

For $(a, b) \in f$, b is called the **image** of a under f, whereas a is a **preimage** of b.

Note that f is a relation from A to B with two restrictions or we can say function is different from relation in two ways :

1. Each element in A, the domain of f, must be used by the rule defining f, and
2. If $(a, b) \in f$ and $(a, c) \in f$ then $b = c$. This is what we mean by "is related to exactly one element of B".

Example 1. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z, w\}$ and $f = \{(1, w), (2, x), (3, x)\}$ is a function and consequently a relation from A to B. But $R_1 = \{(1, w), (3, x)\}$ and $R_2 = \{(1, w), (2, w), (2, x), (3, z)\}$ are relations but not functions, from A to B because in R_1, $2 \in A$ is not related to any element and in R_2, $2 \in A$ has two different images w and x in B.

☞ **DEF.** Let A and B be two non-empty sets. The **range of a function** $f: A \to B$ is the set of images of its domain, denoted by

$$\begin{aligned} f(A) &= \{f(a) \mid a \in A\} \\ &= \{b \in B \mid b = f(a) \text{ for some } a \in A\} \end{aligned}$$

Note that $f(A) \subseteq B$.

Pictorially, a function is generally shown as in Fig. 2.5.

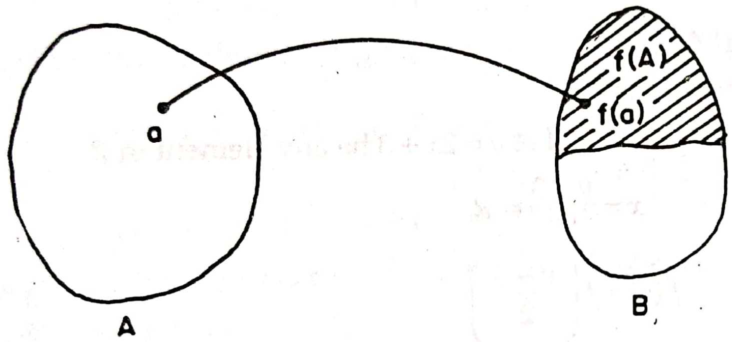

Fig. 2.5

Example 2. Let $f: Z \to Z$ be defined by $f(a) = 2a$ for $a \in Z$, then f is a function.

Example 3. Let $f: R \to R$, where R is the set of real numbers, be defined by, $f(x) = x^2$ for all $x \in R$.

Then f is a function from R to R because square of a real number is unique.

Dom $(f) = R$ and Range of $f = f(R) = \{1, 4, 9, 16, \ldots\ldots\}$

2.9 TYPES OF FUNCTIONS

☞ **DEF.** A function $f: A \to B$ is called **one-to-one or in jective,** if distinct elements of A are mapped into distinct elements of B.

Equivalently, f is one-to-one if

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$$

or

$$f(x_1) = f(x_2) \Rightarrow \quad x_1 = x_2.$$

☞ **DEF.** A function $f: A \to B$ is called an **onto or surjective function** if range of $f = f(A) = B$, *i.e.*, if for all $b \in B$, there is at least one $a \in A$ with $f(a) = b$. Otherwise, it is called into.

☞ **DEF.** A mapping $f: A \to B$ is called **one-to-one correspondence or bijective** if it is both one-to-one and onto.

Example 4. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$.

Given $f = \{(1, z), (2, y), (3, x), (4, y)\}$ and $g = \{(1, x), (2, y), (3, y), (4, x)\}$

then f is an onto function but not one-to-one. ($\because f(A) = \{x, y, z\} = B$) and g is neither onto nor one-to-one because $g(A) = \{x, y\} \subset B$.

Example 5. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$. The func $f = \{(1, a), (2, b), (3, c)\}$ is a one-to-one function from A to B but not onto.

Example 6. *The function $f : R \to R$ defined as $f(x) = 2x + 3$ for all $x \in R$ is b one-to-one and onto functions.*

Solution. To show that f is *one-to-one*. Let $x_1, x_2 \in R$ be two different elemen

i.e., $\quad x_1 \neq x_2$

$\Rightarrow \quad 2x_1 \neq 2x_2$

$\Rightarrow \quad 2x_1 + 3 \neq 2x_2 + 3$

$\Rightarrow \quad f(x_1) \neq f(x_2)$

$\Rightarrow \quad f$ is one-to-one.

To show that f is *onto* : Let $y = 2x + 3$ be any element in R

then $$x = \frac{y-3}{2} \in R$$

and $$f(x) = f\left(\frac{y-3}{2}\right)$$

$$= 2.\left(\frac{y-3}{2}\right) + 3 = y - 3 + 3 = y$$

This shows that for every element $y \in R$, there exists an element $x \in R$ suc that $f(x) = y$. Thus f is onto.

Hence f is a one-to-one correspondence from R into R.

☞ **DEF.** Let A be a non-empty set. The function $f : A \to A$ defined by $f(a) = a$ f all $a \in A$, is called an **identity function** for A. It is generally denoted b I_A.

☞ **DEF.** A function $f : A \to B$ is called a **constant function,** if some element $b \in$ is assigned to every element of A *i.e.*, $f(a) = b$ for all $a \in A$. Equivalently, $f : A \to B$ is a constant function if the range of f consists only one element.

☞ **DEF.** Let A, B be non-empty sets. Then $f, g : A \to B$ are equal and we wri $f = g$, if $f(a) = g(a)$ for all $a \in A$.

Example 7. Let $A = \{1, 2, 3\}$ then the function $f : A \to A$ defined by $f(1) =$ $f(2) = 2, f(3) = 3$ is an identity function.

Identity function is a bijective function.

Example 8. A function $f : R \to R$ such that $f(x) = x$ for all $x \in R$ is an identi function.

2.10 COMPOSITION OF FUNCTIONS

We will define an operation for combining two appropriate functions.

☞ **DEF.** Let $f : A \to B$ and $g : B \to C$ be two functions. Then the **composition** of f and g, written gof, is a function from A into C defined by

$$(gof)(a) = g(f(a)) \text{ for all } a \in A.$$

Note that in the definition of composition of function f and g, it is required that codomain of f = domain of g.

Example 9. Let $f: \{1, 2, 3\} \rightarrow \{a, b\}$ be given by $f = \{(1, a), (2, a), (3, b)\}$ and let $g: \{a, b\} \rightarrow \{5, 6, 7\}$ be defined by $g = \{(a, 5), (b, 7)\}$.

Then $gof: \{1, 2, 3\} \rightarrow \{5, 6, 7\}$ is defined by

$$(gof)(1) = g(f(1)) = g(a) = 5$$
$$(gof)(2) = g(f(2)) = g(a) = 5$$
$$(gof)(3) = g(f(3)) = g(b) = 7$$

i.e., $$gof = \{(1, 5), (2, 5), (3, 7)\}$$

fog is not defined because codomain of $g \neq \text{Dom}(f)$.

Example 10. Let $f: R \rightarrow R$ and $g: R \rightarrow R$ be defined by $f(x) = x^2$ and $g(x) = 2x + 1$. Then

$$(gof)(x) = g(f(x)) = g(x^2) = 2x^2 + 1$$

and $$(fog)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 = 4x^2 + 4x + 1.$$

Here $gof: R \rightarrow R$ and $fog: R \rightarrow R$

but $(gof)(1) = 3$ and $(fog)(1) = 9$

$\Rightarrow$ $gof \neq fog$, so the composition of functions is not commutative in general.

However, the composition of functions is associative which can be stated in form of theorem.

Theorem 2. *If $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$, then $ho(gof) = (hog)of$.*

☞ **DEF.** A function $f: A \rightarrow B$ is called an **invertible function** if and only if f is both one-to-one and onto and inverse of function f is defined as $f^{-1}: B \rightarrow A$.

Example 11. Consider the function f defined in example 6. We have shown that f is a one-to-one correspondence. Thus f is invertible and its inverse f^{-1} is given by

$$f^{-1}(x) = \frac{x - 3}{2}.$$

Example 12. Let $A = \{1, 2, 3\}$ and let f be a function defined on A, such that

$$f = \{(1, 2), (2, 3), (3, 1)\}.$$

Then $f^{-1}: A \rightarrow A$ is defined by

$$f^{-1} = \{(2, 1), (3, 2), (1, 3)\}$$

and $$fof^{-1} = \{(1, 1), (2, 2), (3, 3)\}$$

and $$f^{-1}of = \{(1, 1), (2, 2), (3, 3)\}$$

i.e., $$fof^{-1} = f^{-1}of = I_A.$$

Thus, inverse of a function can also be defined as

☞ **DEF.** Let $f: A \rightarrow B$. If there exists a function $g: B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$, then g is called the **inverse of f** and is denoted by f^{-1}.

Example 13. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$ and let $f : A \to B$ be defined $f(1) = x, f(2) = y, f(3) = z.$

Then $g : B \to A$ defined by $g(x) = 1, g(y) = 2, g(z) = 3$ is the inverse of f sin

$$(gof)(1) = g(f(1)) = g(x) = 1$$
$$(gof)(2) = g(f(2)) = g(y) = 2$$
$$(gof)(3) = g(f(3)) = g(z) = 3$$
$$\Rightarrow \quad gof = I_A$$

and $(fog)(x) = x,\ (fog)(y) = 4,\ (fog)(z) = z \ \Rightarrow\ fog = I_B.$

Example 14. *Let $X = \{a, b, c\}$. Define $f : X \to X$ such that $f = \{(a, b), (b, a), (c, c)\}$* **(UPTU, B.Tech. 2003)**

Find (i) f^{-1} (ii) f^2 (iii) f^3 (iv) f^4.

Solution. Here $f = \{(a, b), (b, a), (c, c)\}$

i.e., $f(a) = b, f(b) = a, f(c) = c.$

(i) $f^{-1} = \{(b, a), (a, b), (c, c)\} = f$

(ii) $f^2 = fof.$

Now $(fof)(a) = f(f(a)) = f(b) = a$

$(fof)(b) = f(f(b)) = f(a) = b$

$(fof)(c) = f(f(c)) = f(c) = c$

$\therefore \quad f^2 = \{(a, a), (b, b), (c, c)\}$

(iii) $f^3 = fof^2$

$(fof^2)(a) = f(f^2(a)) = f(a) = b$

$(fof^2)(b) = f(f^2(b)) = f(b) = a$

$(fof^2)(c) = f(f^2(c)) = f(c) = c$

$\therefore \quad f^3 = \{(a, b), (b, a), (c, c)\} = f$

(iv) $f^4 = fof^3$

$= fof = f^2$

$\therefore \quad f^4 = \{(a, a), (b, b), (c, c)\}$

Some properties of invertible functions are stated as follows :

If $f : A \to B$ is one-one, onto function from a non-empty set A to a non-em set B, then

1. $f^{-1} : B \to A$ **is also one-one and onto.** **2.** $f^{-1} : B \to A$ **is uniqu**

2.11 RECURSIVELY DEFINED FUNCTIONS

☞ **DEF.** A function is said to be **recursively defined** if the function defini refers to itself.

A recursively defined function must have the following two properties :

1. There must be certain arguments, called base values, for which function does not refer to itself.
2. Each time the function does refer to itself, the argument of the fu tion must be closer to the base value.