Ring:

An Algebric structure $(R, +, \cdot)$ is called a Ring

If (i) $(R, +)$ is an abelain Group

(ii) $(R, \cdot)$ is semigroup

ie $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in R$

(iii) Left hand and Right hand distributive Laws holds.

ie $\forall\, a, b, c \in R$

$a \cdot (b + c) = a \cdot b + a \cdot c$ and

$(b + c) \cdot a = b \cdot a + c \cdot a$

eg. $(Z, +, \cdot)$, $(Q, +, \cdot)$, $(R, +, \cdot)$, $(C, +, \cdot)$

$(Z_n, +_n, \times_n)$

Types of Ring:

① Ring with Unity: If there exist an element in R denoted by 1 such that

$a \times 1 = 1 \times a = a \quad \forall\, a \in R$

then the Ring is called Ring with unity

② Commutative Ring: If the multiplication in the ring R is also commutative, the the ring is called commutative Ring

Ring without Zero divisor:

A Ring $(R, +, \cdot)$ is called a Ring without Zero divisor If it is not possible to find $a, b \in R$ s.t $a \neq 0$, $b \neq 0$ and $a \cdot b = 0$

e.g. ie If $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

e.g. ① $(Z, +, \cdot)$ $(Q, +, \times)$, $(R, +, \cdot)$

② $(Z_5, +_5, \times_5)$

$Z_5 = \{0, 1, 2, 3, 4\}$

for all $a, b \in Z_5$, If $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

$\therefore (Z_5, +_5, \times_5)$ is Ring without Zero divisor

In general $(Z_p, +_p, \times_p)$ where p is prime is Ring without Zero divisor

Ring with Zero divisor:

A Ring $(R, +, \cdot)$ is called a Ring with Zero divisor If it is possible to find $a, b \in R$ s.t $a \neq 0$, $b \neq 0$ and $a \cdot b = 0$

e.g. $(Z_6, +_6, \times_6)$

$Z_6 = \{0, 1, 2, 3, 4, 5\}$

$a = 2 \neq 0$, $b = 3 \neq 0 \in Z_6$ But $a \cdot b = 2 \cdot 3 = 6 = 0 \in Z_6$

$\therefore (Z_6, +_6, \times_6)$ is Ring with Zero divisor

here 2 is called Zero divisor of 3
and 3 is called Zero divisor of 2

In general $(Z_n, +_n, \times_n)$, is Ring with Zero divisor
where $n \neq P$

Integral Domain:

A Commutative Ring with unity and without Zero divisor is called an Integral Domain

eg $(Z, +, \cdot)$, $(Q, +, \cdot)$ $(R, +, \cdot)$ $(Z_p, +_p, \times_p)$

Field:

A Commutative Ring with unity is called Field if every nonzero element has multiplicative inverse

e.g. $(Q, +, \cdot)$, $(R, +, \cdot)$

$(Z_5, +_5, \times_5)$

$Z_5^* = \{1, 2, 3, 4\}$

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\therefore (Z_5, +_5, \times_5)$ is Field

In general
$(Z_p, +_p, \times_p)$ is Field

Every Field is Integral domain But Converse need not be true
eg $(Z, +, \cdot)$ is an Integral domain But not Field because for element $a = 2 \in Z$ But $a^{-1} \notin Z$ (multiplicative Inverse)

Lattice

Consider a Relation R on a set A satisfying the following Properties

(1) R is Reflexive
(2) R is antisymmetric
(3) R is transitive

Then R is called a partial order relation and the set A together with partial order is called a partially ordered set or Poset and is denoted by $(A, \leq)$

Elements of Poset:

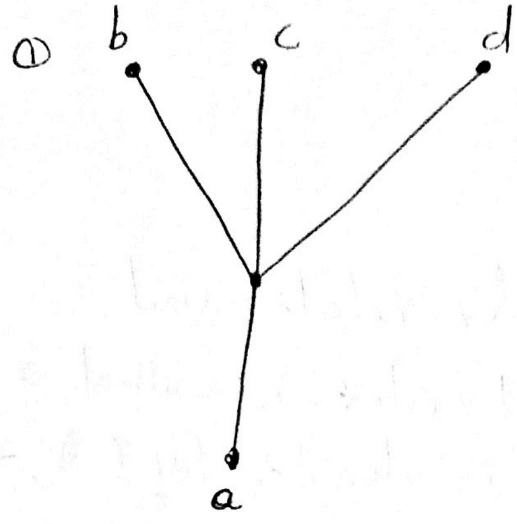
(1) **Maximal Element:** An element $a \in A$ is called a maximal element of A if there is no element c in A such that $a \leq c$

(2) **Minimal Element:** An element $b \in A$ is called a minimal element of A if there is no element c in A such that $c \leq b$

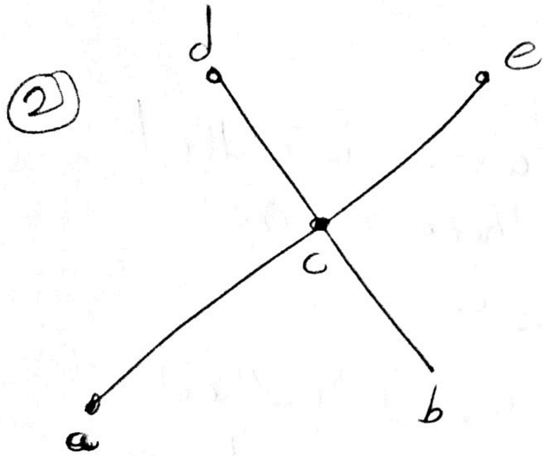
(3) **Greatest element:** An element $a \in A$ is called a greatest element of A if $x \leq a$ for all $x \in A$

(4) **Least element:** An element $a \in A$ is called a least element of A if $a \leq x$ for all $x \in A$

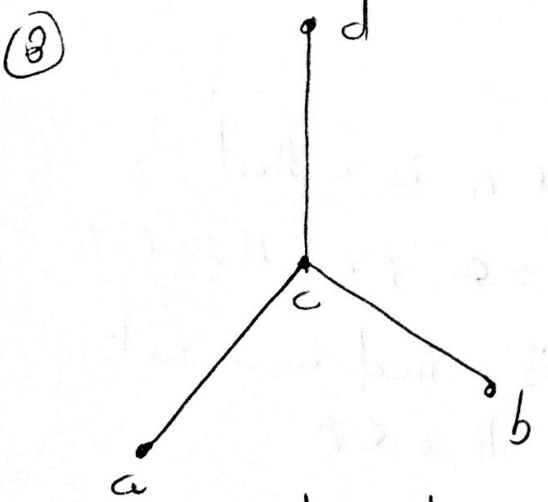
Find the maximal element, minimal element, greatest element and least element of the following Posets



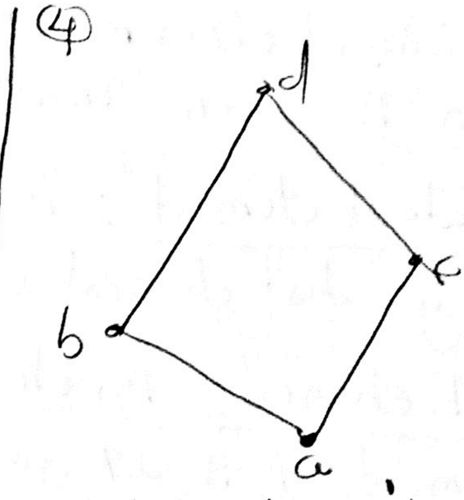
Maximal elements: b, c, d
Minimal element - a
(Least element)



Maximal elements - d, e
Minimal elements - a, b



Minimal element - a, b
greatest element - d



greatest element - d
least element - a

The greatest element of a Poset, If it exists is denoted by I and and the least element of a Poset If it exists is denoted by O

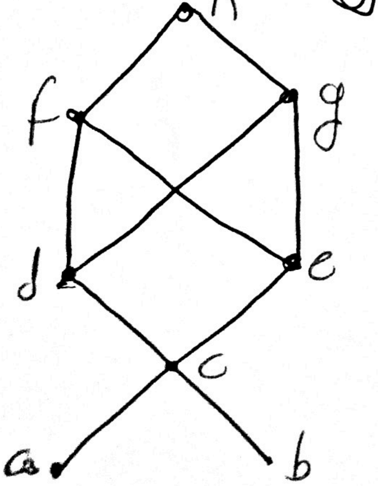
Upper Bound: Consider a Poset A and Subset B of A, an element $a \in A$ is called an upper Bound of B If $b \leq a$ for all $b \in B$

Lower Bound: Consider a Poset A and Subset B of A, an element $a \in A$ is called an Lower Bound of B If $a \leq b$, for all ~~$a \in A$~~ $b \in B$

Least Upper Bound of the set (L.U.B)
an upper Bound of the ~~set~~ that is less than all other upper bounds

greatest Lower Bound of the set (G.L.B)
a lower bound of the set that is greater than all other lower bounds

eg. Consider the poset $A = \{a, b, c, d, e, f, g, h\}$ whose Hasse diagram is shown in figure
Find all Upper Bounds, L.U.B, all Lower Bounds, G.L.B of the following Subsets of A (a) $B = \{a, b\}$
(b) $B = \{c, d, e\}$



(a) $B = \{a, b\}$

Upper Bounds of B ~~is {c, d, e, f, g, h}~~ are c, d, e, f, g, h

L.U.B of B = h

B has no Lower Bounds

(b) $B = \{c, d, e\}$

Upper Bounds of B are f, g, h

B has no L.U.B (f and g are not comparable)

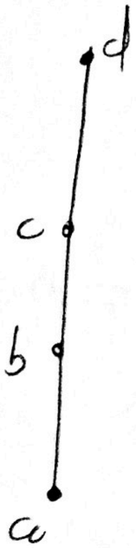
Lower Bounds of B are a, b, c

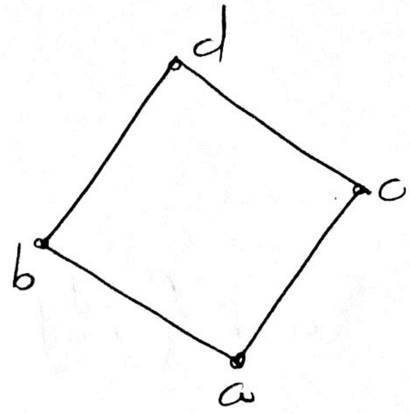
G.L.B of B = c

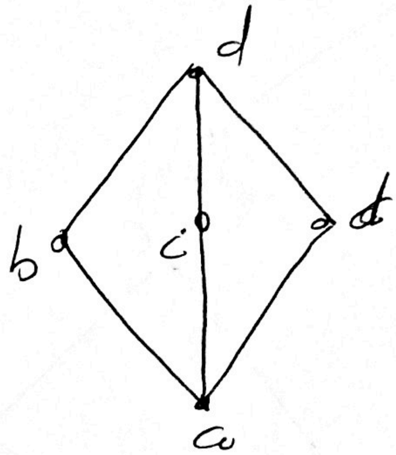
Lattice:

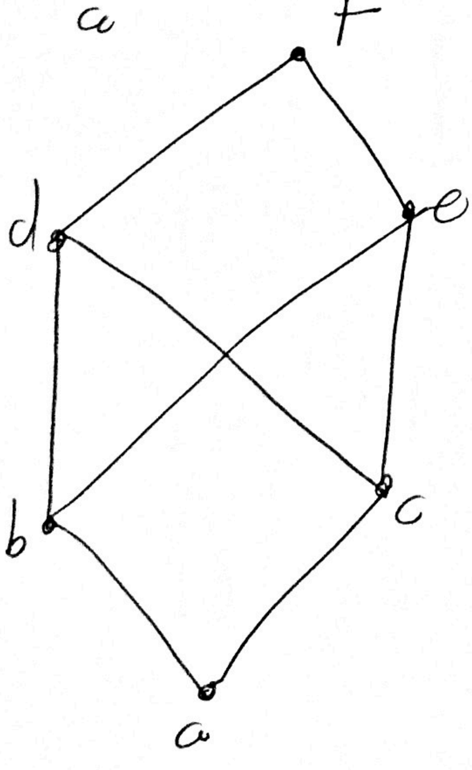
A Lattice is a Poset in which every subset $\{a, b\}$ consisting of two elements has a least upper bound and greatest lower Bound. We denote L.U.B $(\{a, b\})$ by $a \vee b$ and call it the join of a and b. Similarly we denote G.L.B $(\{a, b\})$ by $a \wedge b$ and call it the meet of a and b

e.g



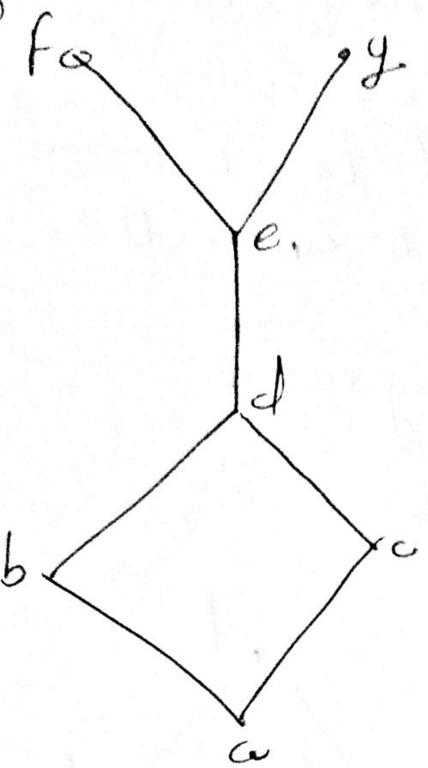






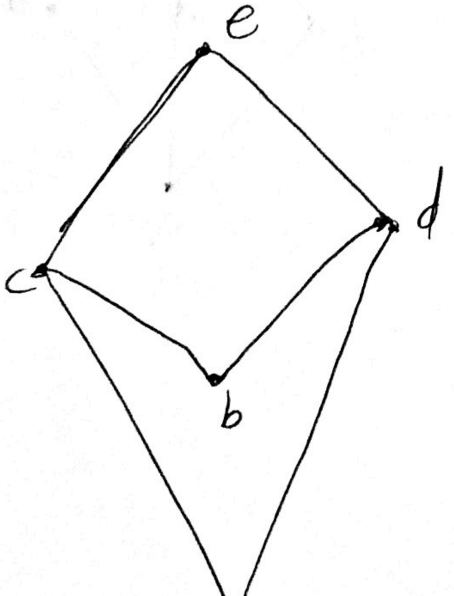
Poset which is not Lattice.

①



$L.U.B\{f, g\} = f \vee g =$ not exist

②



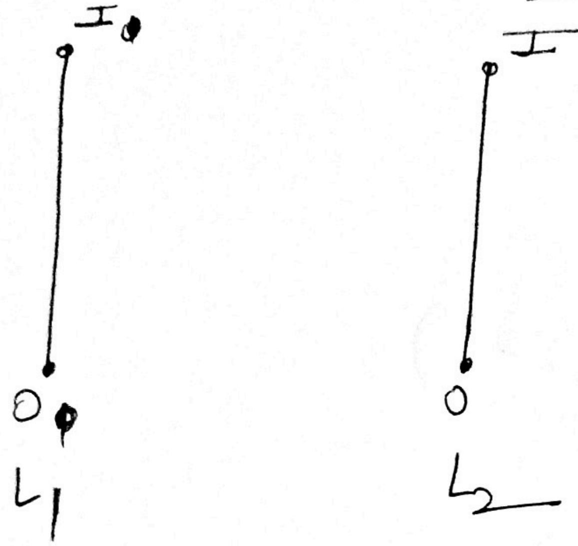
$G.L.B\{c, d\} = c \wedge d =$ not exist

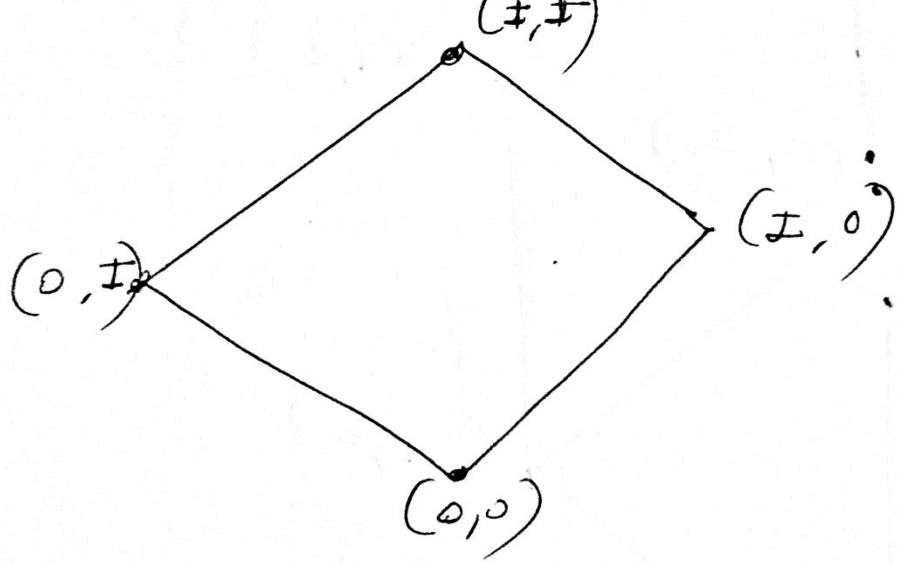
If $(L_1, \leq)$ and $(L_2, \leq)$ are lattices

then $(L, \leq)$ is a Lattice

where $L = L_1 \times L_2$

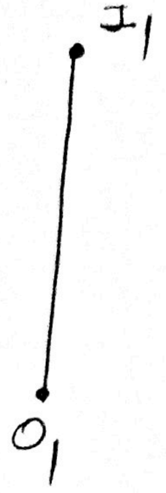
(1)

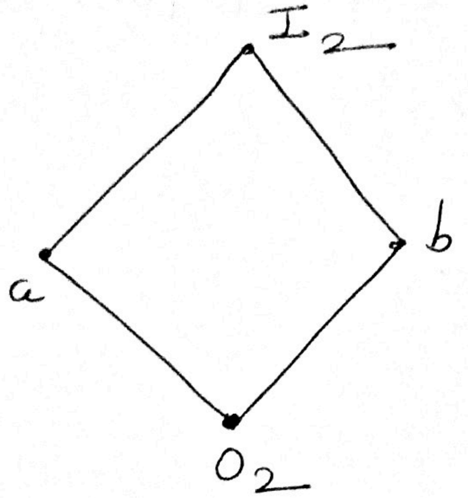




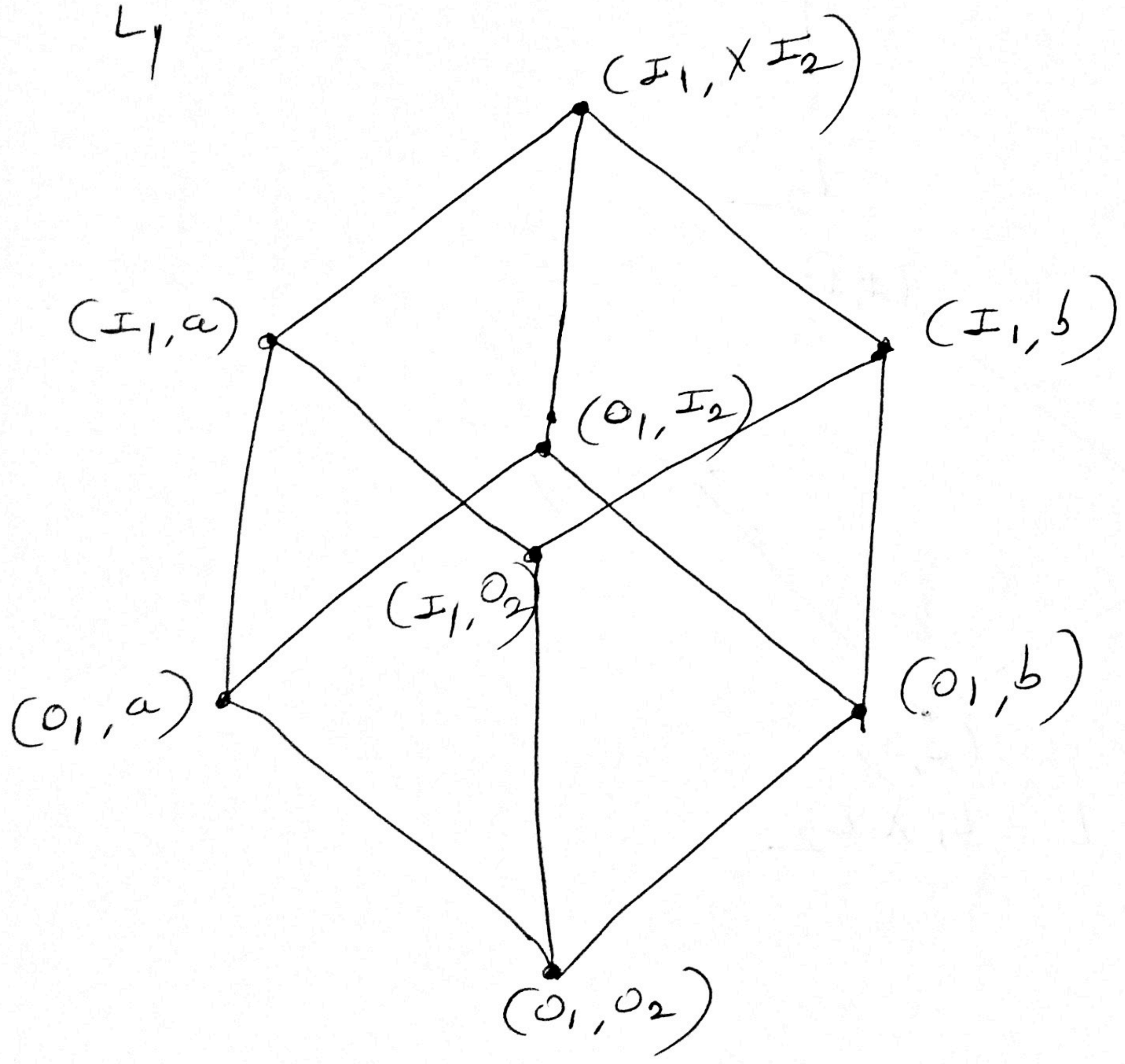
$L = L_1 \times L_2$

②





L_1

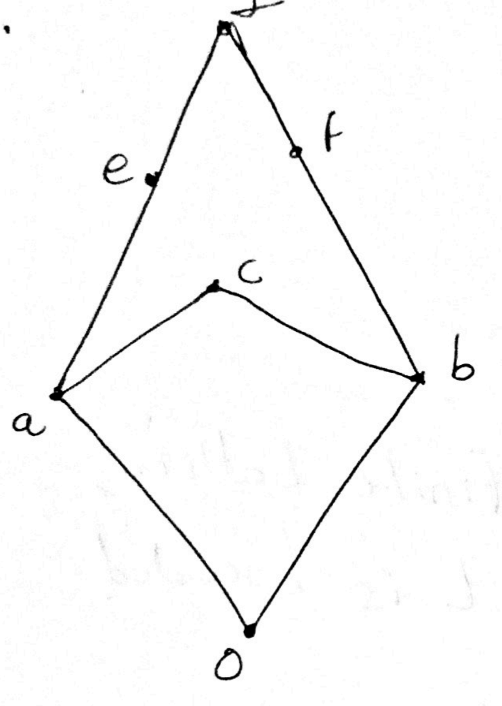


$L = L_1 \times L_2$

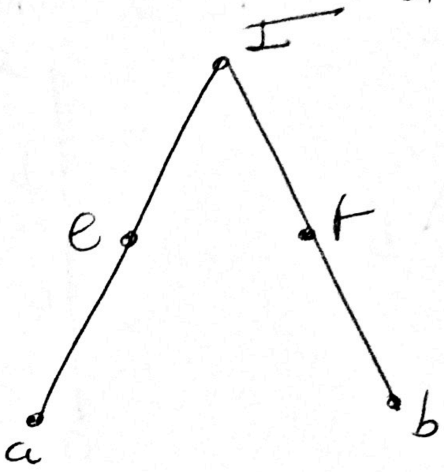
Sublattice of Lattice:

Let $(L, \leq)$ be a Lattice. A nonempty subset S of L is called a sublattice of L If $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$

e.g.

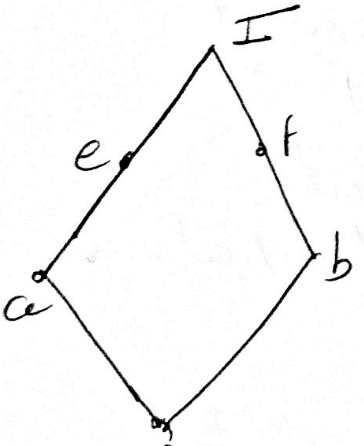


L



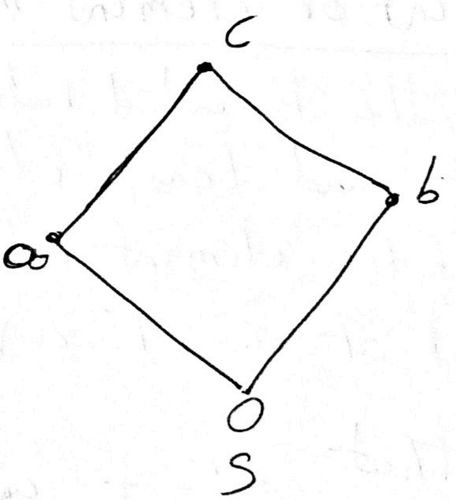
$a \wedge b$ not exist

$\Rightarrow$ not sublattice



$a \vee b$ not exist

$\Rightarrow$ not sublattice



S

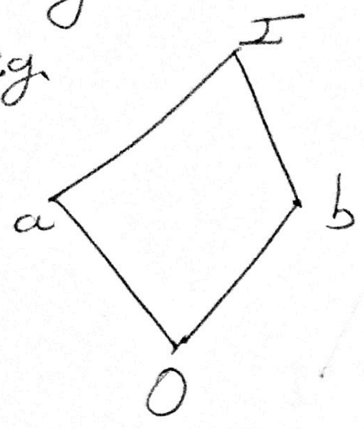
S is sublattice of given Lattice L

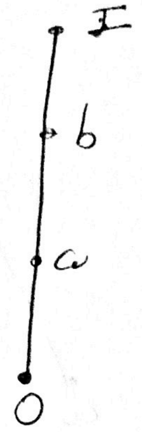
Types of Lattices:

(1) Bounded Lattice:

A Lattice L is said to be bounded If it has greatest element I and Least element 0

eg.





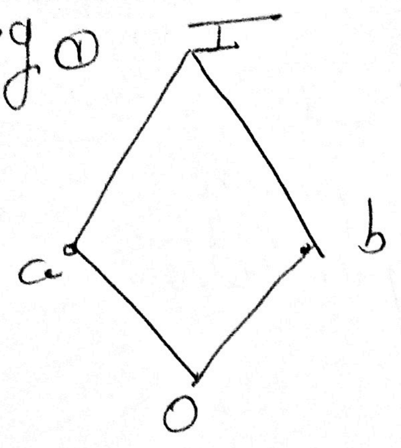
Let $L = \{a_1, a_2, \ldots a_n\}$ be a finite Lattice, then L is bounded

Complement of element in Lattice

Let L be the bounded lattice with greatest element I and Least element 0 and Let $a \in L$. An element $a' \in L$ is called complement of a If $a \vee a' = I$ and $a \wedge a' = 0$.

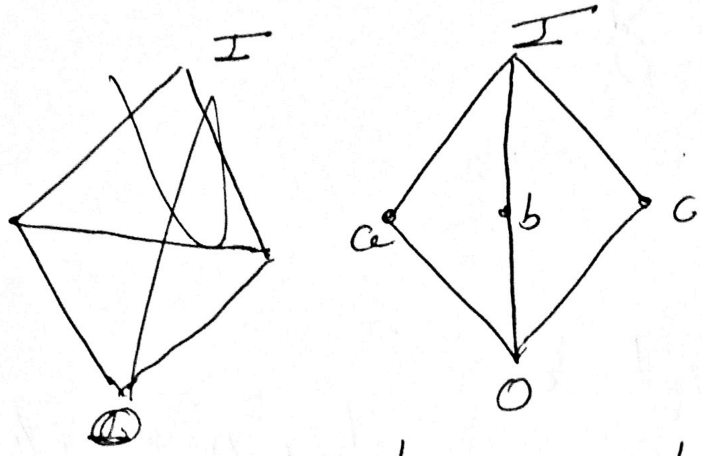
Observe that $0' = I$ and $I' = 0$

eg ①



$O' = I, \quad I' = 0, \quad a' = b, \quad b' = a$

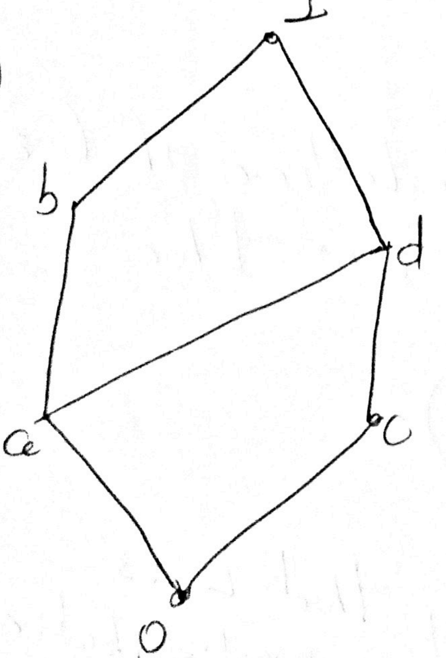
②



$O' = I, \quad I' = 0, \quad a' = b, c$

$b' = a, c, \quad c' = a, b$

③



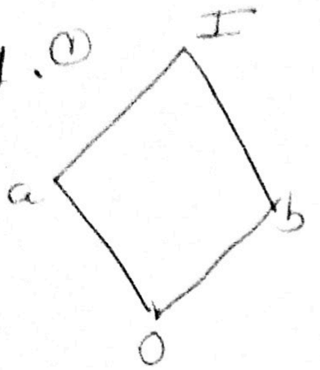
$a \vee d = d$

$\Rightarrow$ the element a has no complement

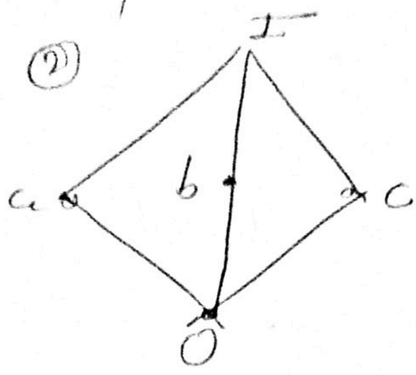
Complemented Lattice:

A Lattice L is called Complemented if it is bounded and if every element in L has a complement

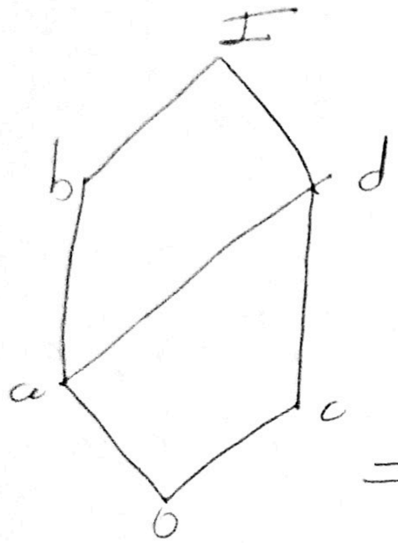
eg. ①



②



③



$a \vee d = d$

for element a has no Complement

$\Longrightarrow$ Lattice is not complemented

distributed lattice:

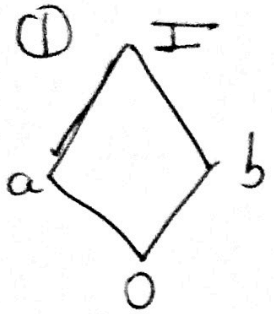
A Lattice L is called distributive Lattice if for any elements a, b, c of L it satisfies the following distributive properties:

① $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

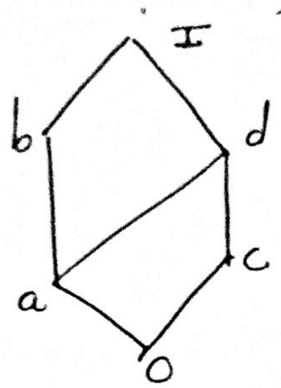
② $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If L is not distributive, we say that L is nondistributive

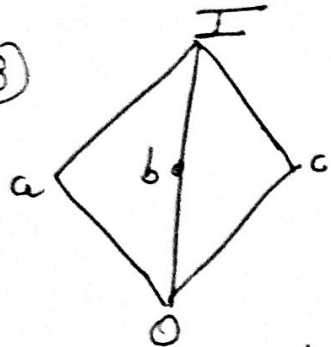
A Lattice L is said to be distributive if for every element of L has at most one complement eg ①



②



③



$a \wedge (b \vee c) = a \wedge I = a$

$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

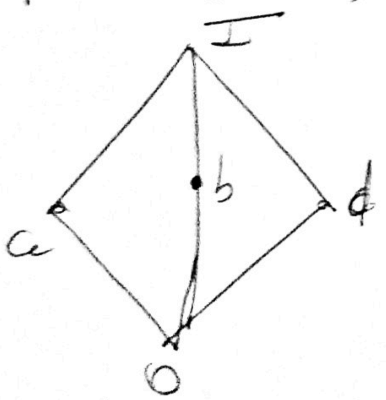
$\Rightarrow a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

L is nondistributive Lattice

Modular Lattice:

A Lattice $(L, \wedge, \vee)$ is called Modular lattice if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$

eg



L

$c = I$

$a \leq I = c$

$a \vee (b \wedge c) = a \vee (b \wedge I)$

$= a \vee b = I$

$(a \vee b) \wedge c = (a \vee b) \wedge I = I \wedge I = I$

$\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

$\therefore L$ is Modular Lattice