Relations:

Cartesian Product of Sets:

Let A and B sets. The Cartesian product of A and B, denoted by $A \times B$ is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

Hence

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

* Let A and B be sets. A binary relation from A to B is a subset of $A \times B$

* A relation on set A is relation from A to A

In other words, a relation on set A is a subset of $A \times A$

for eg $A = \{1, 2, 3\}$

$$A \times A = \{(1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\ (2,3)\ (3,1)\ (3,2)\ (3,3)\}$$

① $R_1 = \{\ \} = \phi$

(2) $R_2 = \{(1,1)\ (2,2)\ (3,3)\}$

(3) $R_3 = \{(1,2)\ (2,1)\}$

(4) $R_4 = \{(1,1)\ (1,2)\ (2,1)\}$

(5) $R_5 = A \times A$

The relations R_1, R_2, R_3, R_4, R_5 are relations on set A

~~x~~ If $|A| = n$ then number of Relations on set $A = 2^{n^2}$

Types of Relations:

① Reflexive Relation:

A relation on set A is called reflexive if $(a,a) \in R$ for every element $a \in A$

e.g. $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

$R_2 = A \times A$

$R_3 = \{(1,1)\ (2,2)\ (3,1)\ (3,3)\ (3,2)\ (4,4)\}$

The Relations R_1, R_2, R_3 are reflexive because they contains all pairs of the form (a,a)

$R_4 = \{(1,1)\ (2,2)\ (4,4)\}$

$R_5 = \{(1,1)\ (2,2)\ (1,2)\ (1,3)\ (3,3)\}$

R_4 and R_5 not reflexive

(2) Symmetric relation:

A relation R on set A is called symmetric If $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$

eg $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1)\ (1,2)\ (2,1)\ (3,3)\}$

$R_2 = A \times A$

$R_3 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

$R_1 . R_2, R_3$ Symmetric relation

$R_4 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\}$

$R_5 = \{(1,1)\ (3,2)\ (2,3)\ (1,4)\}$

But R_4 and R_5 not symmetric relation

(8) Antisymmetric Relation

A relation R on set A such that for all $a, b \in A$ If $(a,b) \in R$ and $(b,a) \in R$ then $a = b$ is called antisymmetric

In other words If $(a,b) \in R$ then $(b,a) \notin R$ for all $a, b \in A$ and $a \neq b$

e.g $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1)\ (1,2)\ (1,3)\ (2,2)\}$

$R_2 = \{(1,1)\ (2,2)\ (3,3)\}$

$R_3 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

$R_4 = \{(1,2)\ (1,3)\ (2,1)\}$

$R_5 = A \times A$

R_1, R_2, R_3 Antisymmetric But

R_4 and R_5 not Antisymmetric

(4) Transitive Relation:

A relation R on set A is called transitive If whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a, b, c \in A$

eg $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1)\ (1,2)\ (2,1)\}$

$R_2 = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

$R_3 = A \times A$

$R_4 = \{(1,1)\ (2,2)\ (2,3)\ (3,1)\}$

$R_5 = \{(1,1)\ (2,2)\ (1,3)\ (3,4)\}$

relation R_1, R_2, R_3 transitive But R_4 and R_5 not transitive.

Representing Relations:

Representing Relations Using Matrices:

Suppose that R is a relation from $A = \{a_1, a_2, \ldots a_m\}$ to $B = \{b_1, b_2, \ldots b_n\}$

The relation R can be represented by the matrix $M_R = [m_{ij}]$,

where $m_{ij} = \begin{cases} 1 & \text{If } (a_i, b_j) \in R \\ 0 & \text{If } (a_i, b_j) \notin R \end{cases}$

e.g. ① $A = \{1, 2, 3\}$ and $B = \{1, 2\}$

$R = \{(2,1)\ (3,1)\ (3,2)\}$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(2) $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\ (2,1)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3) $A = \{1, 2, 3, 4\}$

$R = \{(1,2)\ (2,1)\ (1,3)\ (3,1)\ (2,3)\ (3,2)\}$

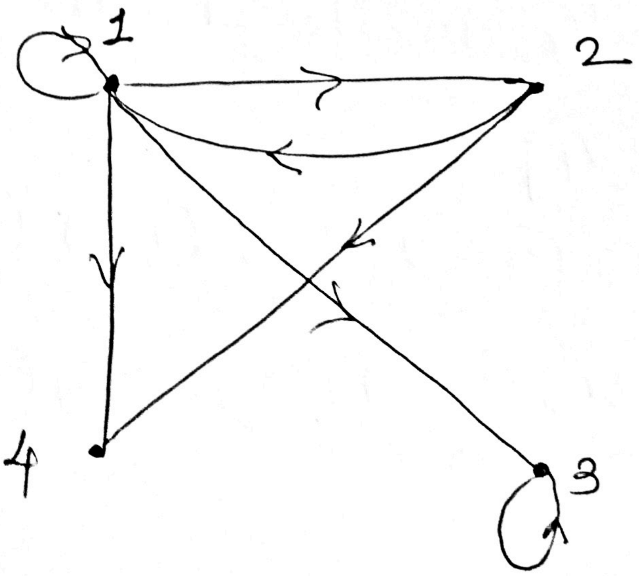
$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs:

A directed graph or digraph, consists of set V of vertices together with a set E of ordered pairs of elements of V called edges. The vertex a is called initial vertex of the edge (a, b) and the vertex b is called the terminal vertex of the edge. An edge of the form (a, a) is represented using an arc from a to itself, such edge is called loop

e.g. $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (1,2)\ (2,1)\ (1,3)\ (2,4)\ (3,3)\ (1,4)\}$



Closure of Relation:

Let R be the relation on a set A

Reflexive closure of R is the smallest reflexive relation R_r which contains R

$$\overline{R_r} = R \cup \Delta$$

where Δ is the diagonal relation on A

e.g $A = \{1, 2, 3, 4\}$

$$R = \{(1,1)\,(1,3)\,(3,2)\,(2,2)\,(4,2)\,(4,4)\}$$

then $\overline{R_r} = \{(1,1)\,(1,3)\,(3,2)\,(2,2)\,(4,2)\,(4,4)\,(3,3)\}$

Symmetric closure of R is the smallest symmetric relation containing R that is

$$\overline{R_s} = R \cup R^{-1}$$

where R^{-1} is the inverse of the relation R

e.g. $A = \{1, 2, 3, 4\}$

$$R = \{(1,1)\,(2,2)\,(1,3)\,(1,2)\,(3,1)\}$$

then $\overline{R_s} = \{(1,1)\,(2,2)\,(1,3)\,(1,2)\,(3,1)\,(2,1)\}$

Transitive closure of relation R is the Smallest transitive relation containing R

e.g $A = \{1, 2, 3, 4\}$

$R = \{(1,2)\ (2,1)\ (2,3)\}$

then transitive closure of R is

$\overline{R}_t = \{(1,2)\ (2,1)\ (2,3)\ (1,1)\ (1,3)\}$

Warshall's Algorithm:

Use Warshall's Algorithm to find the transitive closure of R

where $A = \{1, 2, 3, 4\}$ and $R = \{(1,2)\ (2,4)\ (1,3)\ (3,2)\}$

Soln. $\Rightarrow M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow W_0 = M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow W_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_0$$

$$\Rightarrow W_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow W_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_2$$

$$W_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = W_3$$

Then transitive closure of R is

$$\overline{R}_t = \{(1,2)\,(1,3)\,(1,4)\,(2,4)\,(3,2)\,(3,4)\}$$

Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1)\ (1,4)\ (2,1)\ (2,2)\ (3,3)\ (4,4)\}$

Find the transitive closure of R

Solution:

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow W_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then transitive closure of R is

$$\bar{R}_t = \{(1,1)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$$

~~If $A = \{1,2,3,4\}$ and $R = \{(1,1)\ (1,4)\ (2,2)\ (2,4)\}$~~

If $A = \{1,2,3,4\}$ and $R = \{(1,2)\ (2,3)\ (3,4)\ (2,1)\}$

Find its transitive closure

* Equivalence Relations:

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive

e.g ① $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2), (2,1)\}$

② Let R be the relation on the set of real numbers such that aRb if and only if a-b is an integer
Is R is an equivalence relation

Soln:
Given
aRb if and only if a-b is an integer

ie $(a,b) \in R \iff a-b$ is an integer

① $0 = a - a$ is an integer

$\iff (a,a) \in R$

$\therefore$ R is Reflexive

(ii) If $aRb \iff a-b$ is an integer.

If aRb ie $(a,b) \in R$

$\iff a-b$ is an integer

$\iff b-a$ is an integer

$\iff (b,a) \in R$

$\iff bRa$

$\therefore R$ is Symmetric relation

(iii) If aRb and bRc

ie $(a,b) \in R$ and $(b,c) \in R$

If $(a,b) \in R$

$\iff a-b$ is an integer

If $(b,c) \in R$

$\iff b-c$ is an integer

Consider $a-c = (a-b)+(b-c)$

$\therefore a-c$ is an integer

$\implies (a,c) \in R$

$\therefore R$ is transitive

$\therefore R$ is an equivalence relation

Q) let m be an integer with $m \geq 1$.
Show that the relation
$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$
is an equivalence relation on the set of integers

* Equivalence classes

let R be an equivalence relation on set A. The set of all elements that are related to an element a of A is called the equivalence class of A. The equivalence class of a with respect to R is denoted by $[a]_R$

and defined by
$$[a]_R = \{s \mid (a,s) \in R\}$$

eg ① $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\ (2,1)\}$

be the equivalence Relation on A

$[a]_R = \{s \mid (a,s) \in R\}$

$[1]_R = \{s \mid (1,s) \in R\}$

$\Rightarrow [1]_R = \{1, 2\}$

$[2]_R = \{2, 1\}$

$[3]_R = \{3\}$

$[4]_R = \{4\}$

② What are the equivalence classes of 0, 1, 2 and 3 for congruence modulo 4?

Soln: m be the +ve integer with $m > 1$

$R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers

$R = \{(a, b) \mid a \equiv b \pmod{4}\}$

$[0]_R = \{\ldots -8, -4, 0, 4, 8 \ldots\}$

$[1]_R = \{\ldots -7, -3, 1, 5, 9 \ldots\}$

$[2]_R = \{\ldots -6, -2, 2, 6, 10, \ldots\}$

$[3]_R = \{\ldots -5, -1, 3, 7, 11 \ldots\}$

* The equivalence classes of the relation congruence modulo m are called congruence classes modulo m. The congruence class of an integer a modulo m denoted by $[a]_m$

$[a]_m = \{\ldots a-2m, a-m, a, a+m, a+2m \ldots\}$

* Partial Orderings or Partial order

A relation R on set S is called a <u>partial order</u> If it is reflexive, antisymmetric, and transitive

e.g. $A = \{1, 2, 3, 4\}$

$$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\ (1,3)\}$$

be the partial order relation on set A

A set S together with a partial ordering R is called a partially ordered set or poset and is denoted by (S, R)

The ordered pair (S, R) is called partially ordered set or Poset

* Hasse diagram:

A partial ordering $\leq$ on S Poset is represented by diagram called Hasse diagram

Hasse diagram is a simpler version of a diagraph

* To draw the Hasse diagram of partial order, apply the following points:

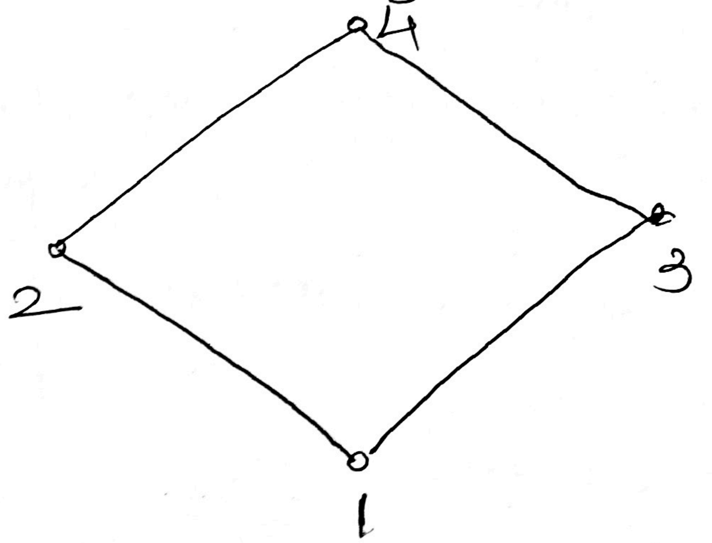
1. Delete all edges implied by reflexive property
2. Delete all edges implied by transitive property
3. Omit the arrows

① Let $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (1,2)\ (2,2)\ (2,4)\ (1,3)\ (3,3)\ (3,4)\ (1,4)\ (4,4)\}$

then show that R is partial order and draw its Hasse diagram

Soln: Hasse diagram

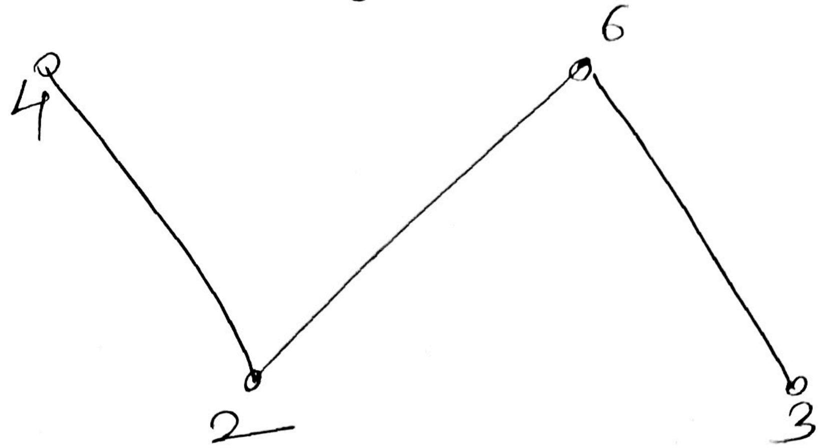


② Let $A = \{2, 3, 4, 6\}$ and let aRb if a divides b

Show that R is partial order and draw its Hasse diagram

Soln: $R = \{(2,2)\ (2,4)\ (2,6)\ (3,3)\ (3,6)\ (4,4)\ (6,6)\}$

. Hasse diagram



(3) Draw the Hasse diagram representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on $\{1,2,3,4,6,8,12\}$

(4) Draw the Hasse diagram for the less than or equal to relation on $\{0,1,2,3,4,5\}$