ME5204 Finite Element Analysis

Assignment 3

Pratham Sunkad

ME21B145

Contents

1 Problem 1

1.1 Problem Statement

Let V be the space of all continuous piecewise linear polynomials on the interval $I = [0, 3]$. Given the function $f(x) = \exp(\sin(0.25\pi x^2))$, the objective is to determine the L^2-projection of $f(x)$ onto V.

The tasks are as follows:

1. **Interpolating Function (L2 Projection)**:

   Determine the interpolating function, which is the L^2-projection of $f(x)$ onto V, using a piecewise linear approximation.

2. **L2 Norm Error**:

   Compute the error in the L^2-norm between the actual function $f(x)$ and its interpolation for different mesh sizes.

3. **Optimal Mesh Size**:

   Find the minimum number of points (mesh size) required such that the error in the L^2-norm is less than 1×10^{-5}.

4. **Rate of Convergence**:

   Plot the L^2-norm error as a function of mesh size and compute the rate of convergence.

5. **Overlay Plot**:

   For a specific mesh size, overlay the plot of the actual function $f(x)$ (evaluated at a finer set of points) and the interpolated function.

1.2 Solution

1.2.1 1D L2 projection

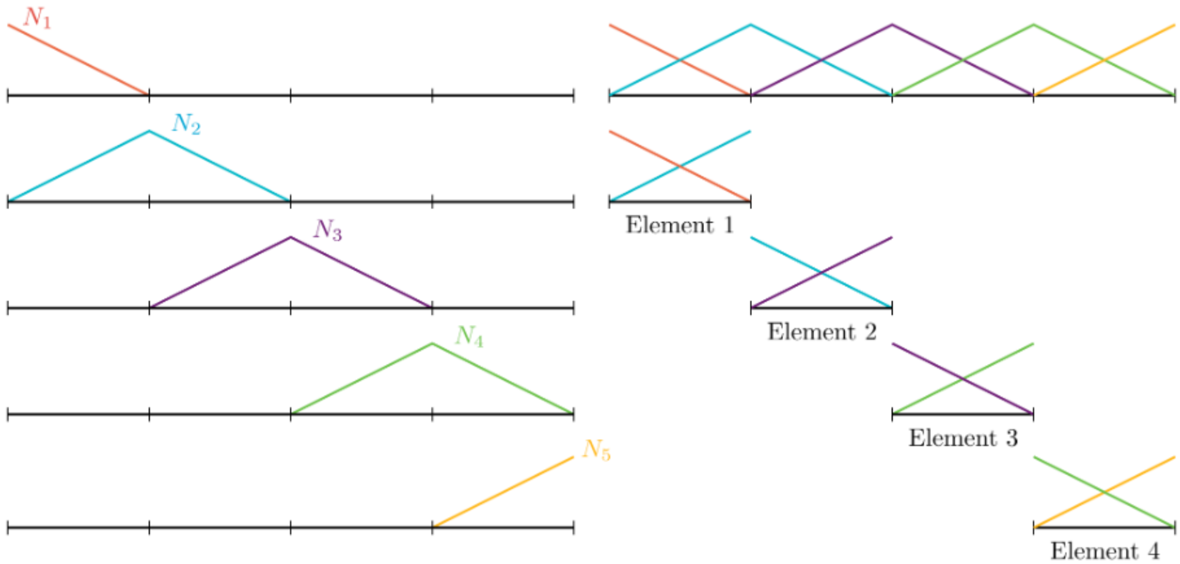


Figure 1: Linear Interpolating Functions in 1D element

Let $f(x)$ be a given function, and $I_h f(x)$ be the interpolated (piecewise linear) function. The goal is to find $I_h f(x) \in V$, the space of continuous piecewise linear polynomials, such that the following condition holds:

$$\int_0^3 \left(f(x) - I_h f(x)\right) \varphi_j(x)\, dx = 0, \quad \forall \varphi_j \in V$$

where $\varphi_j(x)$ are the piecewise linear basis functions. This ensures that the error between $f(x)$ and $I_h f(x)$ is orthogonal to all basis functions in V with respect to the L^2-norm.

1. **Interpolating Function**

The interpolated function $I_h f(x)$ can be expressed as a linear combination of the basis functions:

$$I_h f(x) = \sum_{i=1}^{n} c_i \varphi_i(x)$$

where c_i are the coefficients to be determined.

2. **L2 Projection Matrix Equation**

Substituting $I_h f(x)$ into the projection equation:

$$\int_0^3 \left(f(x) - \sum_{i=1}^{n} c_i \varphi_i(x) \right) \varphi_j(x)\, dx = 0, \quad \forall j$$

After taking derivatives w.r.t to c_i and equating to 0, the equations simplify to the matrix form:

$$Ac = f$$

Here, A is the stiffness matrix (or mass matrix), c is the vector of coefficients, and f is the right-hand side vector representing the projection of $f(x)$ onto the basis functions.

3. **Matrix Form of A in Terms of Basis Functions**

Each entry in the matrix A corresponds to an inner product between two basis functions. If we have two basis functions φ_1 and φ_2 on a specific element e_n, the entries in A are computed as:

$$A_{ij}^n = \int_{x_n}^{x_{n+1}} \varphi_1^n(x) \varphi_2^n(x)\, dx$$

where $\varphi_1^n(x)$ and $\varphi_2^n(x)$ are the local linear basis functions on the element e_n, and the integral is taken over the corresponding element.

On element $e_n = [x_n, x_{n+1}]$, the local basis functions can be written as:

$$\varphi_1^n(x) = \frac{x_{n+1} - x}{x_{n+1} - x_n}, \quad \varphi_2^n(x) = \frac{x - x_n}{x_{n+1} - x_n}$$

Transformed Coordinates:

We can transform the coordinates from the reference element $[-1, 1]$ to the actual element $[x_n, x_{n+1}]$ using the transformation:

$$x = \frac{1-\xi}{2}x_n + \frac{1+\xi}{2}x_{n+1}, \quad \text{where } \xi \in [-1, 1]$$

The local basis functions in the reference element become:

$$\varphi_1^n(\xi) = \frac{1-\xi}{2}, \quad \varphi_2^n(\xi) = \frac{1+\xi}{2}$$

Thus, the entries in the stiffness matrix A can be expressed in terms of integrals of these linear functions.

Products of Basis Functions:

- $\varphi_1^n(\xi)\varphi_1^n(\xi) = \frac{(1-\xi)^2}{4}$ - $\varphi_1^n(\xi)\varphi_2^n(\xi) = \frac{(1-\xi)(1+\xi)}{4} = \frac{1-\xi^2}{4}$ - $\varphi_2^n(\xi)\varphi_2^n(\xi) = \frac{(1+\xi)^2}{4}$

Numerical Integration:

We approximate the integral using Gaussian quadrature with weights w_i and corresponding quadrature points ξ_i. The Jacobian for the transformation from ξ to x is:

$$J = \frac{x_{n+1} - x_n}{2}$$

Thus, the entries of the matrix A are computed as:

$$A_{ij}^n = \sum_{i=1}^{N} w_i \, \varphi_i^n(\xi_i)\varphi_j^n(\xi_i) \cdot J$$

where N is the number of quadrature points.

Modified Matrix A:

Using this, the matrix A for a single element e_n is:

$$A^n = J \cdot \begin{bmatrix} \sum_{i=1}^{N} w_i \frac{(1-\xi_i)^2}{4} & \sum_{i=1}^{N} w_i \frac{1-\xi_i^2}{4} \\ \sum_{i=1}^{N} w_i \frac{1-\xi_i^2}{4} & \sum_{i=1}^{N} w_i \frac{(1+\xi_i)^2}{4} \end{bmatrix}$$

where ξ_i are the quadrature points and w_i are the corresponding weights.

4. **Final System of Equations**

In summary, the L2 projection problem leads to the system of equations:

$$Ac = f$$

The matrix A is sparse, with non-zero entries only on the principal diagonal and the neighboring diagonals (sub-diagonal and super-diagonal). It can be written as:

$$A = \begin{bmatrix} \int_{x_1}^{x_2} \varphi_1^1 \varphi_1^1 \, dx & \int_{x_1}^{x_2} \varphi_1^1 \varphi_2^1 \, dx & 0 & \dots & 0 \\ \int_{x_1}^{x_2} \varphi_2^1 \varphi_1^1 \, dx & \int_{x_1}^{x_2} \varphi_2^1 \varphi_2^1 \, dx + \int_{x_2}^{x_3} \varphi_1^2 \varphi_1^2 \, dx & \int_{x_2}^{x_3} \varphi_1^2 \varphi_2^2 \, dx & \dots & 0 \\ 0 & \int_{x_2}^{x_3} \varphi_2^2 \varphi_1^2 \, dx & \int_{x_2}^{x_3} \varphi_2^2 \varphi_2^2 \, dx + \int_{x_3}^{x_4} \varphi_1^3 \varphi_1^3 \, dx & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \int_{x_n}^{x_{n+1}} \varphi_2^n \varphi_2^n \, dx \end{bmatrix}$$

where φ_1^n and φ_2^n are the linear basis functions on the element $[x_n, x_{n+1}]$.

Diagonal Elements:

$$A_{ii} = \int_{x_{i+1}}^{x_{i+2}} \varphi_1^{i+1} \varphi_1^{i+1} \, dx + \int_{x_i}^{x_{i+1}} \varphi_2^i \varphi_2^i \, dx$$

Off-Diagonal Elements:

$$A_{i(i+1)} = \int_{x_i}^{x_{i+1}} \varphi_1^i \varphi_2^i \, dx, \quad A_{(i+1)i} = \int_{x_i}^{x_{i+1}} \varphi_2^i \varphi_1^i \, dx$$

Thus, the matrix A is banded, with non-zero elements restricted to the principal diagonal and the adjacent diagonals.

Vector f Representation

The vector f corresponds to the inner product of the function $f(x)$ with the local basis functions. For each element $e_n = [x_n, x_{n+1}]$, the entries of f are computed as:

$$f_i^n = \int_{x_n}^{x_{n+1}} f(x) \varphi_i^n(x) \, dx$$

Local Basis Functions:

On element $e_n = [x_n, x_{n+1}]$, the local basis functions are:

$$\varphi_1^n(x) = \frac{x_{n+1} - x}{x_{n+1} - x_n}, \quad \varphi_2^n(x) = \frac{x - x_n}{x_{n+1} - x_n}$$

Transformed Coordinates:

We transform the coordinates from the reference element $[-1, 1]$ to the actual element $[x_n, x_{n+1}]$ using:

$$x = \frac{1 - \xi}{2} x_n + \frac{1 + \xi}{2} x_{n+1}, \quad \text{where } \xi \in [-1, 1]$$

The basis functions in the reference element are:

$$\varphi_1^n(\xi) = \frac{1-\xi}{2}, \quad \varphi_2^n(\xi) = \frac{1+\xi}{2}$$

Function Value Transformation:

To compute the integrals involving $f(x)$, we use the transformed basis functions in the reference element:

$$f_i^n = \int_{-1}^{1} f\left(\frac{1-\xi}{2}x_n + \frac{1+\xi}{2}x_{n+1}\right)\varphi_i^n(\xi) \cdot \frac{x_{n+1}-x_n}{2}\, d\xi$$

Numerical Integration:

We approximate the integral using Gaussian quadrature with weights w_i and quadrature points ξ_i. Thus:

$$f_i^n = \sum_{i=1}^{N} w_i\, f\left(\frac{1-\xi_i}{2}x_n + \frac{1+\xi_i}{2}x_{n+1}\right)\varphi_i^n(\xi_i) \cdot J$$

where $J = \frac{x_{n+1}-x_n}{2}$ is the Jacobian, and N is the number of quadrature points.

Modified Matrix f:

Using this, the vector f for a single element e_n is:

$$f^n = \begin{bmatrix} \sum_{i=1}^{N} w_i\, f\left(\frac{1-\xi_i}{2}x_n + \frac{1+\xi_i}{2}x_{n+1}\right)\frac{1-\xi_i}{2} \cdot J \\ \sum_{i=1}^{N} w_i\, f\left(\frac{1-\xi_i}{2}x_n + \frac{1+\xi_i}{2}x_{n+1}\right)\frac{1+\xi_i}{2} \cdot J \end{bmatrix}$$

where ξ_i are the quadrature points and w_i are the corresponding weights.

The vector f can be written as:

$$f = \begin{bmatrix} \int_{x_1}^{x_2} f(x)\varphi_1^1(x)\,dx \\ \int_{x_1}^{x_2} f(x)\varphi_2^1(x)\,dx + \int_{x_2}^{x_3} f(x)\varphi_1^2(x)\,dx \\ \vdots \\ \int_{x_{n-1}}^{x_n} f(x)\varphi_2^{n-1}(x)\,dx + \int_{x_n}^{x_{n+1}} f(x)\varphi_1^n(x)\,dx \\ \int_{x_n}^{x_{n+1}} f(x)\varphi_2^n(x)\,dx \end{bmatrix}$$

This vector represents the projection of $f(x)$ onto the basis functions $\varphi_j^n(x)$.

The coefficient vector c is given by:

$$c = A^{-1} f$$

where A is the stiffness matrix and f is the load vector.

Interpolated Function

The interpolated function can be expressed as:

$$I_h f(x) = c_1\varphi_1^1(x) + c_2\left(\varphi_2^1(x) + \varphi_1^2(x)\right) + c_3\left(\varphi_2^2(x) + \varphi_1^3(x)\right) + \cdots + c_n\varphi_2^n(x)$$

Here: - $c_1, c_2, c_3, \ldots, c_n$ are the coefficients. - $\varphi_1^n(x)$ and $\varphi_2^n(x)$ are the basis functions defined over each element.

1.2.2 Error in L2 Norm

The L2 norm of the error between $f(x)$ and its projection $\Pi_h f(x)$ is given by:

$$\|f(x) - \Pi_h f(x)\| = \sqrt{\int_\Omega (u - u_h)\cdot(u - u_h)\,d\Omega}$$

where: - u is the exact solution, - u_h is the interpolated solution, - Ω is the domain over which the norm is computed.

The number of elements required to achieve an L2 error less than 1×10^{-5} is **700**.

1.2.3 Plots and Results

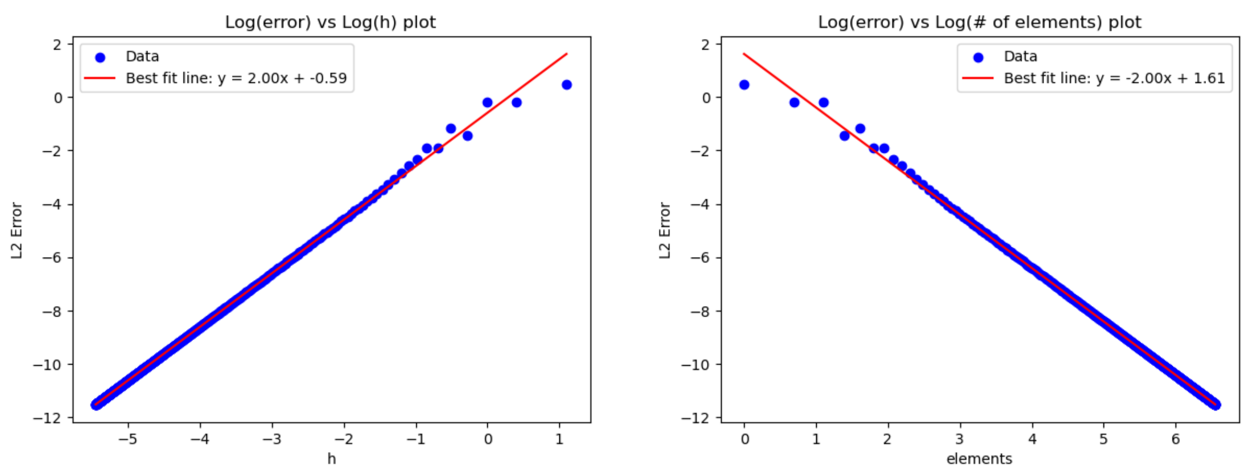


Figure 2: L2 Error Plot

Let L denote the total length of the domain, and let N be the number of elements. The length of each element h is given by:

$$h = \frac{L}{N}$$

Slope of Log(L2 error) vs Log(h) graph = 2. This is consistent with our hypothesis

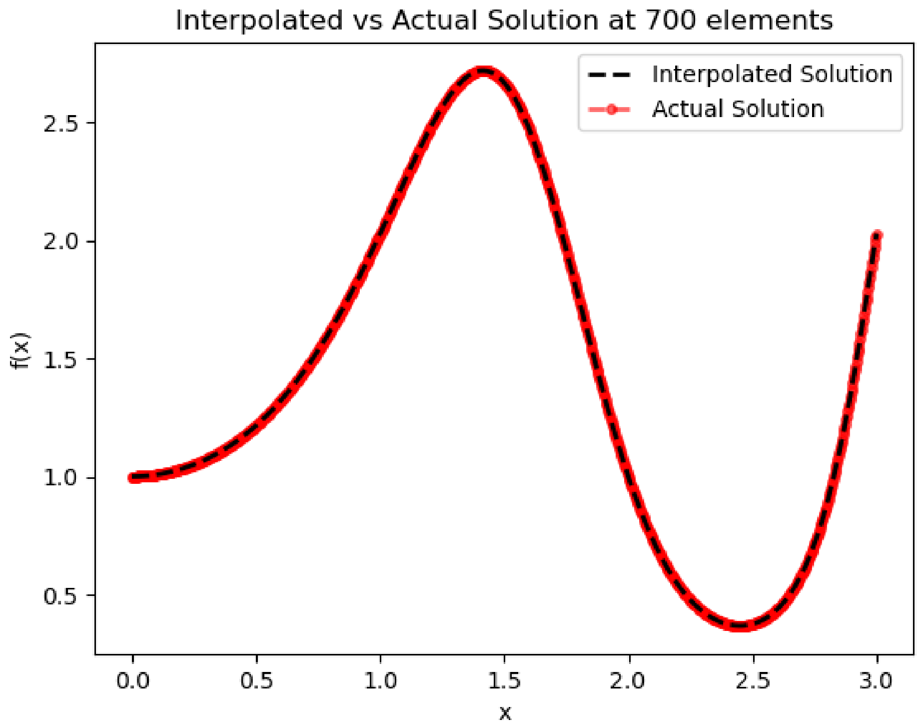


Figure 3

2 Problem 2

2.1 Problem Statement

The center of a heat source will be focused at the center of the Gajendra circle with the base radius equal to the diameter of the inner circle. The heat source follows a Gaussian distribution centered at the black circle's center.

1. Obtain an L2 projection of the heat source using an appropriate finite element (FE) mesh.

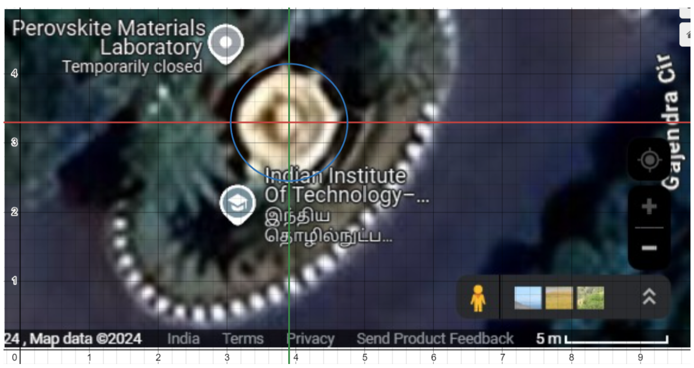
2. Plot the error of the L2 projection as a function of mesh size and compute the rate of convergence.

3. For a particular mesh size, generate contour plots of both the actual function and the interpolated function, showing them side by side. Additionally, include an error plot that depicts the difference between the actual and interpolated functions.

2.2 Solution

2.2.1 Determining Radius of GC and σ of the Gaussian Distribution

The inner Radius of the Gajendra Circle is unknown. Using this screenshot from Google Maps as the reference and modifying it in desmos, we get radius of Gajendra Circle = 2.4 m.



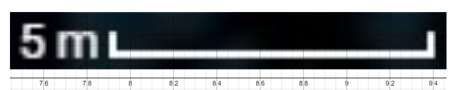


Figure 4: Gajendra Circle Coordinates

Steps:

- Open the image of Gajendra Circle taken from Google Maps on desmos
- Identify the coordinates of the centre of GC. (x,y) = (3.9, 3.3)
- Draw a circle that is similar size as on showed in the Assignment. Radius of this circle = 0.72 desmos units
- Using the scale given in Google Maps, we find that 5 m = 1.5 desmos units.
- Therefore Radius of GC = 2.4 m and Diameter of GC = 4.8 m

Form of our Gaussian Distribution: We will be using a symmetric bivariate gaussian of the form:

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

- σ = Diameter of GC (in the mesh coordinate system) = $\frac{4.8\,\text{m}}{\text{Scaling Factor}}$
- Scaling Factor = 2.7 (as calculated in the previous assignment)
- Since the heat source is concentrated inside the black circle, we assume that there will be Temperatures $\gg 0$ within this circle.
- Therefore, σ = Diameter of GC.

2.2.2 Refined Mesh:

We now propose using a mesh which is slightly more refined in the region near the Gajendra Circle. This is to help cases where the size of the element is $\gg$ diameter of GC. In this case the calculated error has a lesser magnitude than it should have, as gauss points chosen for numerical integration fails to capture the region where f(x,y) is non zero.

Since f(x,y) has a steep gradient near GC, we need more refined elements in that region to effectively capture the error.

Element Size Factor:

- This is a variable in GMSH, when changed makes the mesh more coarse or fine
- Lesser the element size factor, finer the mesh
- Experiments were done by varying the element size factor

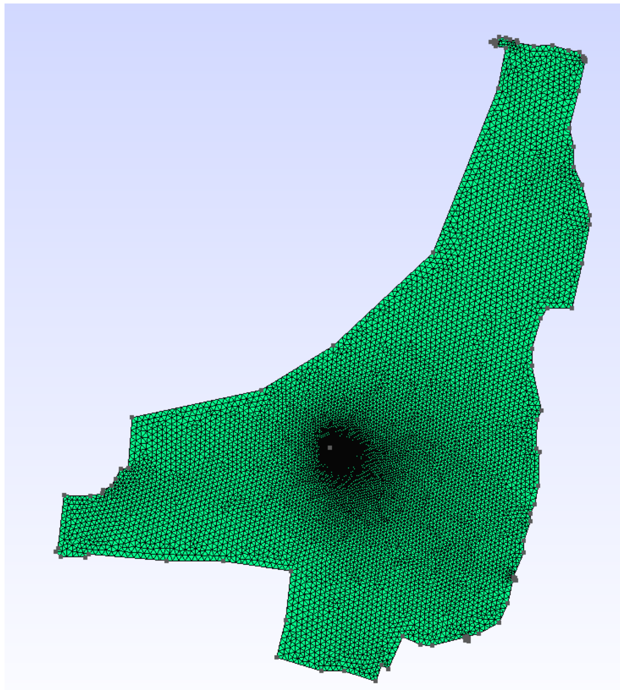


Figure 5: Mesh for Element Size Factor = 10

2.2.3 2D L2 Projection :

Transformed Coordinates:

We transform the coordinates from the reference element 6 to the actual element $[x_1, y_1, x_2, y_2, x_3, y_3]$ using:

$$x = (1 - \xi - \eta)x_1 + \xi x_2 + \eta x_3,$$

$$y = (1 - \xi - \eta)y_1 + \xi y_2 + \eta y_3$$

The basis functions in the reference element are:

$$\varphi_1^n(\xi, \eta) = 1 - \xi - \eta, \quad \varphi_2^n(\xi, \eta) = \xi, \quad \varphi_3^n(\xi, \eta) = \eta$$

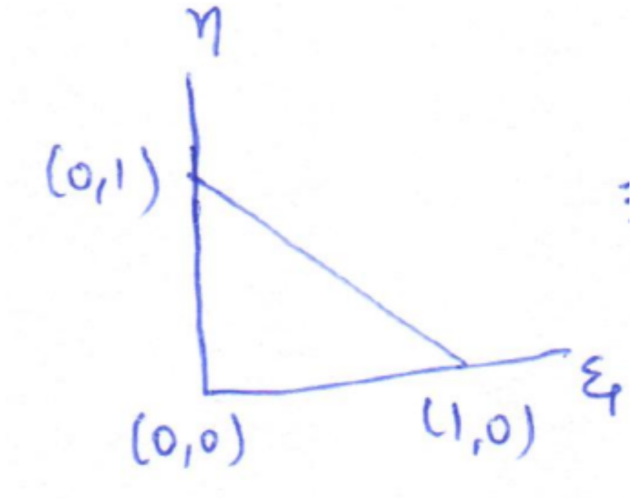


Figure 6: Reference Triangle

Products of Basis Functions:

$$\varphi_1^n(\xi,\eta)\cdot\varphi_1^n(\xi,\eta) = (1-\xi-\eta)^2,$$

$$\varphi_1^n(\xi,\eta)\cdot\varphi_2^n(\xi,\eta) = (1-\xi-\eta)\cdot\xi,$$

$$\varphi_1^n(\xi,\eta)\cdot\varphi_3^n(\xi,\eta) = (1-\xi-\eta)\cdot\eta,$$

$$\varphi_2^n(\xi,\eta)\cdot\varphi_2^n(\xi,\eta) = \xi^2,$$

$$\varphi_2^n(\xi,\eta)\cdot\varphi_3^n(\xi,\eta) = \xi\cdot\eta,$$

$$\varphi_3^n(\xi,\eta)\cdot\varphi_3^n(\xi,\eta) = \eta^2.$$

Numerical Integration:

We approximate the integral using Gaussian quadrature with weights w_i and corresponding quadrature points (ξ_i,η_i). The Jacobian for the transformation from (ξ,η) to (x,y) is:

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} \end{vmatrix}$$

Thus, the entries of the matrix A are computed as:

$$A_{ij}^n = \sum_{i=1}^{N} w_i \, \varphi_i^n(\xi_i, \eta_i)\varphi_j^n(\xi_i, \eta_i) \cdot J$$

where N is the number of quadrature points.

Modified Matrix A:

Using this, the matrix A for a single element e_n is:

$$A^n = J \cdot \begin{bmatrix} \sum_{i=1}^{N} w_i(1-\xi_i-\eta_i)^2 & \sum_{i=1}^{N} w_i(1-\xi_i-\eta_i)\xi_i & \sum_{i=1}^{N} w_i(1-\xi_i-\eta_i)\eta_i \\ \sum_{i=1}^{N} w_i(1-\xi_i-\eta_i)\xi_i & \sum_{i=1}^{N} w_i\xi_i^2 & \sum_{i=1}^{N} w_i\xi_i\eta_i \\ \sum_{i=1}^{N} w_i(1-\xi_i-\eta_i)\eta_i & \sum_{i=1}^{N} w_i\xi_i\eta_i & \sum_{i=1}^{N} w_i\eta_i^2 \end{bmatrix}$$

where (ξ_i, η_i) are the quadrature points and w_i are the corresponding weights.

Vector f Representation

The vector f corresponds to the inner product of the function $f(x)$ with the local basis functions. For each element $e_n = [x_n, x_{n+1}]$, the entries of f are computed as:

Function Value Transformation:

To compute the integrals involving $f(x, y)$, we use the transformed basis functions in the reference element:

$$f_i^n = \int_{-1}^{1}\int_{-1}^{1} f\left((1-\xi-\eta)x_1 + \xi x_2 + \eta x_3,\, (1-\xi-\eta)y_1 + \xi y_2 + \eta y_3\right) \varphi_i^n(\xi, \eta) \cdot J \, d\xi \, d\eta$$

where $J = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|$ is the Jacobian determinant.

Numerical Integration:

We approximate the integral using Gaussian quadrature with weights w_i and quadrature points (ξ_i, η_i). Thus:

$$f_i^n = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j\, f\left((1-\xi_i-\eta_j)x_1 + \xi_i x_2 + \eta_j x_3,\ (1-\xi_i-\eta_j)y_1 + \xi_i y_2 + \eta_j y_3\right)\varphi_i^n(\xi_i, \eta_j)\cdot J$$

where $J = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|$ is the Jacobian determinant, and N is the number of quadrature points in each dimension.

Modified Vector f:

Using this, the vector f for a single element e_n is:

$$f^n = \begin{bmatrix} \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j\, f\left((1-\xi_i-\eta_j)x_1 + \xi_i x_2 + \eta_j x_3,\ (1-\xi_i-\eta_j)y_1 + \xi_i y_2 + \eta_j y_3\right)\varphi_1^n(\xi_i, \eta_j)\cdot J \\ \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j\, f\left((1-\xi_i-\eta_j)x_1 + \xi_i x_2 + \eta_j x_3,\ (1-\xi_i-\eta_j)y_1 + \xi_i y_2 + \eta_j y_3\right)\varphi_2^n(\xi_i, \eta_j)\cdot J \\ \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j\, f\left((1-\xi_i-\eta_j)x_1 + \xi_i x_2 + \eta_j x_3,\ (1-\xi_i-\eta_j)y_1 + \xi_i y_2 + \eta_j y_3\right)\varphi_3^n(\xi_i, \eta_j)\cdot J \end{bmatrix}$$

where (ξ_i, η_j) are the quadrature points, and w_i and w_j are the corresponding weights.

2.2.4 Constructing A and f

To construct the global matrix A and the global vector f for the finite element system, we aggregate the contributions from each element e_n. For each element e_n, the local matrix A^n and the local vector f^n are added to their corresponding locations in the global matrix and vector.

The global matrix A is constructed by adding the elements of A^n to the corresponding locations in A based on the connectivity of the nodes:

$$A_{\text{global}} = \sum_{n=1}^{E} A^n$$

where E is the number of elements.

For example, if element e_n connects nodes i and j, the entries of A^n are added to the entries in A at positions corresponding to these nodes.

Similarly, the global vector f is constructed by adding the elements of f^n to the corresponding locations in f:

$$f_{\text{global}} = \sum_{n=1}^{E} f^n$$

For instance, if element e_n has contributions to nodes i and j, the entries of f^n are added to the entries in f at positions corresponding to these nodes.

2.2.5 System of Equations

The finite element system is given by the equation:

$$Ac = f$$

where A is the global matrix, c is the vector of unknowns, and f is the global vector.

To solve for c, we rearrange the equation as follows:

$$c = A^{-1} f$$

Here, A^{-1} denotes the inverse of the matrix A.

2.2.6 Plots and Results

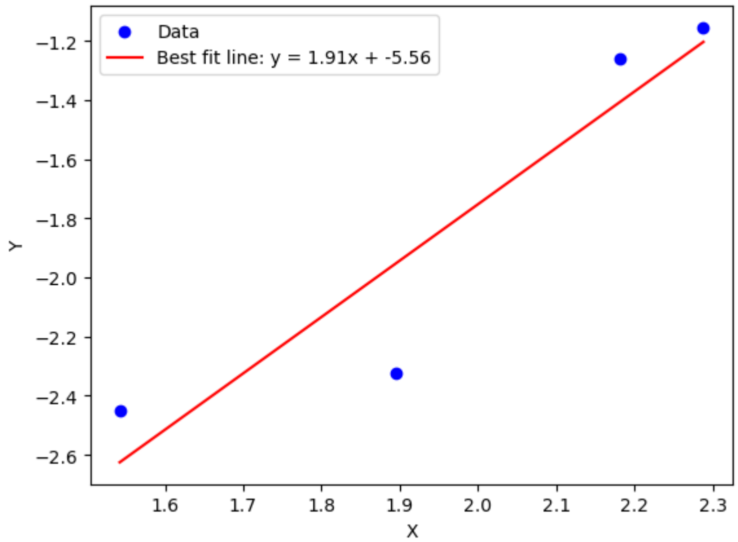


Figure 7: Log(error) vs Log(h)

The characteristic length h is calculated as follows:

$$h = \sqrt{\frac{\text{Total Area of IITM Map}}{\text{Total Number of Elements}}}$$

The slope of the graph is approximately 2, which is consistent with our initial hypothesis

Table 1: Summary of Mesh Details

Element Size Factor	No. of Nodes	No. of Elements	Error
10	15466	30928	0.0862
15	7644	15284	0.0979
20	4305	8606	0.284
25	3480	6956	0.315

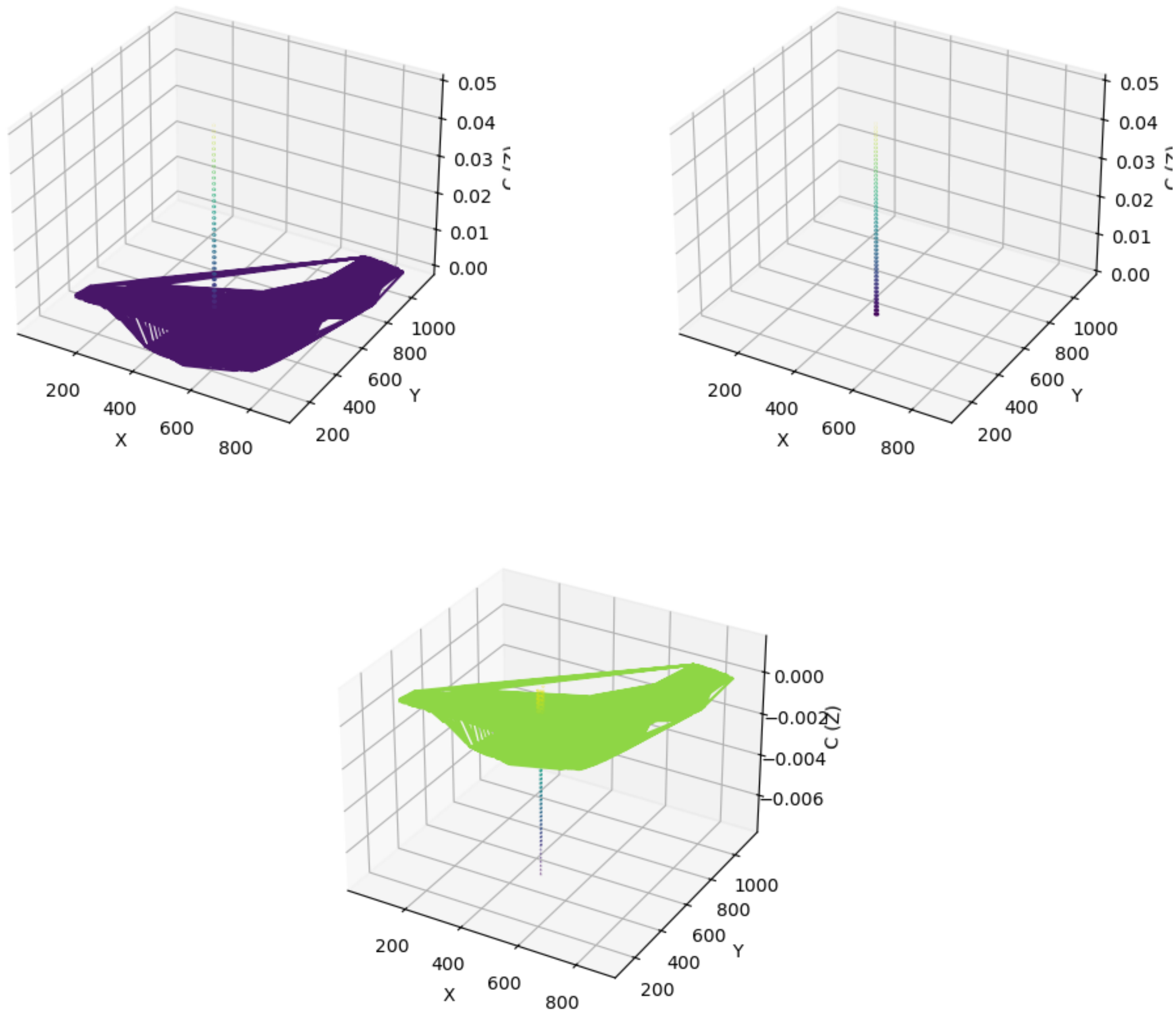


Figure 8: Interpolated function, True function and Pointwise error for element size factor = 10

****