ME5204 Finite Element Analysis

Assignment 5

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ME21B145

Contents

1 Problem 1

1.1 Problem Statement

- Government authorities are planning a visit to IITM and intend to arrive via helicopter. They are considering using one of the campus water tanks as a helipad. After careful evaluation, the intelligence unit has shortlisted two potential tanks: the MSB tank and the one near Gymkhana.
- As finite element analysis experts, it is our responsibility to conduct a comprehensive stress and free vibration analysis to determine which tank would be the safest for landing purposes.
- For all further analysis, we shall assume the tanks to be axisymmetric. Hence, performing the stress and strain analysis on any cross-section of the tank will suffice.

1.2 Strong Form and Boundary Conditions

Let:

- u and v be the displacements in the x and y directions, respectively.
- σ_x, σ_y, σ_{xy} be the stress components.
- ϵ_x, ϵ_y, ν_{xy} be the strain components.
- ρ be the density of the material.
- b_x and b_y be the forces per unit volume in the x and y directions, respectively.
- t be time.

1. Momentum Balance Equations (Cauchy Momentum Equations)

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x \tag{1}$$

$$\rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + b_y \tag{2}$$

2. Strain-Displacement Relations

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{3}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \tag{4}$$

$$\nu_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{5}$$

3. Constitutive Relationship Stress and strain can be related via the constitutive relationship, which is given as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = C \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \nu_{xy} \end{bmatrix}$$

Under plane stress conditions, the constitutive matrix C for an isotropic material is:

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

- E is the Young's modulus
- ν is the Poisson's ratio

The boundary conditions imposed are:

- u and $v = 0$ for the bottom points that are in contact with the ground as they are fixed.
- $\sigma \cdot \mathbf{n} = 0$ (or traction = 0) everywhere else.

Note: We will discuss boundary conditions in further detail in the weak form sub-part.

1.3 Weak Form

Invoking the Galerkin Orthogonality property, it can be stated that:

$$\int_\Omega R(x,y,t)v(x,y,t)\,d\Omega = 0 \quad \forall g(x,y,t) \tag{6}$$

where the residual $R(x,y,t)$ is derived from the governing equation, and $g(x,y,t)$ is an arbitrary function. As there are two governing equations in the present formulation, the weak form is expressed as follows:

$$\int_\Omega R_1(x,y,t)g_1(x,y,t)\,d\Omega + \int_\Omega R_2(x,y,t)g_2(x,y,t)\,d\Omega = 0 \quad \forall g_1(x,y,t), g_2(x,y,t) \tag{7}$$

where

$$R_1(x,y,t) = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x - \rho\frac{\partial^2 u}{\partial t^2} \tag{8}$$

$$R_2(x,y,t) = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y - \rho\frac{\partial^2 v}{\partial t^2} \tag{9}$$

Rearranging the above equations, we obtain the weak form as:

$$-\int_\Omega g_1\left(\rho\frac{\partial^2 u}{\partial t^2}\right)d\Omega - \int_\Omega g_2\left(\rho\frac{\partial^2 v}{\partial t^2}\right)d\Omega + \int_\Omega g_1\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right)d\Omega \tag{10}$$

$$+\int_\Omega g_2\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right)d\Omega = \int_\Omega g_1 b_x\,d\Omega + \int_\Omega g_2 b_y\,d\Omega \tag{11}$$

Applying the Gauss-Divergence theorem, the above weak form can be rewritten as:

$$\underbrace{\int_\Omega \left(g_1 \rho \frac{\partial^2 u}{\partial t^2} + g_2 \rho \frac{\partial^2 v}{\partial t^2} \right) d\Omega}_{\text{mass term}} + \underbrace{\int_\Omega \left(\sigma_x \frac{\partial g_1}{\partial x} + \sigma_{xy} \frac{\partial g_1}{\partial y} \right) d\Omega}_{\text{bilinear term}} + \underbrace{\int_\Omega \left(\sigma_{xy} \frac{\partial g_2}{\partial x} + \sigma_y \frac{\partial g_2}{\partial y} \right) d\Omega}_{\text{bilinear term}}$$
$$= \underbrace{\int_\Omega (g_1 b_x + g_2 b_y)\, d\Omega}_{\text{linear term/force term}} + \underbrace{\int_\Gamma (g_1(\sigma_x n_x + \sigma_{xy} n_y))\, d\Gamma}_{\text{boundary term}} + \underbrace{\int_\Gamma (g_2(\sigma_{xy} n_x + \sigma_y n_y))\, d\Gamma}_{\text{boundary term}}$$

n_x and n_y are unit normals along x and y directions. This has to hold true for all $g_1(x, y, t), g_2(x, y, t)$. We represent the arbitrary functions as a linear combination of Lagrange basis functions. Similarly, both $u(x, y, t)$ and $v(x, y, t)$ can also be represented as a linear combination of Lagrange basis functions.

$$u(x, y, t) = \sum_j \Phi_j u_j(t) \tag{12}$$

$$v(x, y, t) = \sum_j \Phi_j v_j(t) \tag{13}$$

$$g_1(x, y, t) = \sum_j \Phi_j g_1(t) \tag{14}$$

$$g_2(x, y, t) = \sum_j \Phi_j g_2(t) \tag{15}$$

We choose the arbitrary functions g_1, g_2, and u, v to be of the same degree (referred to as Bubnov-Galerkin) as it enables the formation of a square matrix, providing a unique solution. It is important to note that the coefficients u_j and v_j are time-dependent while the Φ_j are time-independent.

Substituting the Lagrange relations and simplifying, the weak form boils down to the following equation:

$$M \frac{d^2 u}{dt^2} + K u = F \tag{16}$$

$$
\begin{aligned}
\text{where:} \quad \mathbf{M} &= \int_\Omega \Phi^T \rho \Phi \, d\Omega, \\
\mathbf{K} &= \int_\Omega B^T C B \, d\Omega, \\
\mathbf{F} &= \int_\Gamma \Phi \, \sigma \cdot n \, d\Gamma + \int_\Omega \Phi^T b \, d\Omega.
\end{aligned}
$$

$$
B = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} & 0 & \frac{\partial \phi_2}{\partial x} & 0 & \frac{\partial \phi_3}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1}{\partial y} & 0 & \frac{\partial \phi_2}{\partial y} & 0 & \frac{\partial \phi_3}{\partial y} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_3}{\partial y} & \frac{\partial \phi_3}{\partial x} \end{bmatrix}, \quad b = \begin{bmatrix} b_x \\ b_y \end{bmatrix},
$$

$$
\Phi = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix}, \quad n = \begin{bmatrix} n_x \\ n_y \end{bmatrix}.
$$

$\tilde{u}$ represents that it holds for both u and v. $\mathbf{K}$ and $\mathbf{M}$ are both 2 nodes $\times$ 2 nodes matrices, while $\mathbf{F}$ is a 2 nodes $\times$ 1 vector.

The above problem that we are solving is a two-degree-of-freedom problem (2-DOF). u, v are the **primary variables** and $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ are the **secondary variables**. The boundary condition imposed on the primary variables is called **Dirichlet**, and the boundary condition imposed on the secondary variable is called Neumann. We impose the Dirichlet boundary condition on the bottom points that are in contact with the ground by setting $u, v = 0$. We impose the Neumann boundary condition everywhere else, that is,$\sigma \cdot n = 0$ or traction $= 0$.

1.4 Materials Used and their Properties

Table 1: Material Properties

Material	**Young's Modulus (GPa) (E)**	**Density (kg/m³) (ρ)**	**Poisson's Ratio (ν)**
Concrete	30	2400	0.2
Aluminium	70	2700	0.3
Water	2.2	1000	0.5

References

[1] `https://cse.guilan.ac.ir/article_5650.html`, Properties of Concrete.

[2] `https://www.mit.edu/~6.777/matprops/aluminum.htm`, Properties of Aluminium.

[3] `https://www.researchgate.net/post/What-is-the-Youngs-or-E-modulus-of-water`, Properties of Water.

In our analysis, we treat both tanks as constructed from concrete. When the helicopter is positioned on top of one of the tanks, we create a new mesh that incorporates both the tank and the helicopter. This scenario necessitates the inclusion of the helicopter's material properties as well. For our calculations, we assume that the helicopter is primarily made of aluminum, so we will utilize the relevant property values for aluminum. Since the tank is filled with water, it is essential to appropriately assign the material properties for the water region within the mesh. The following code snippet is responsible for identifying the different regions of the mesh and assigning the corresponding values for Young's modulus (E), density (ρ), and Poisson's ratio (ν).

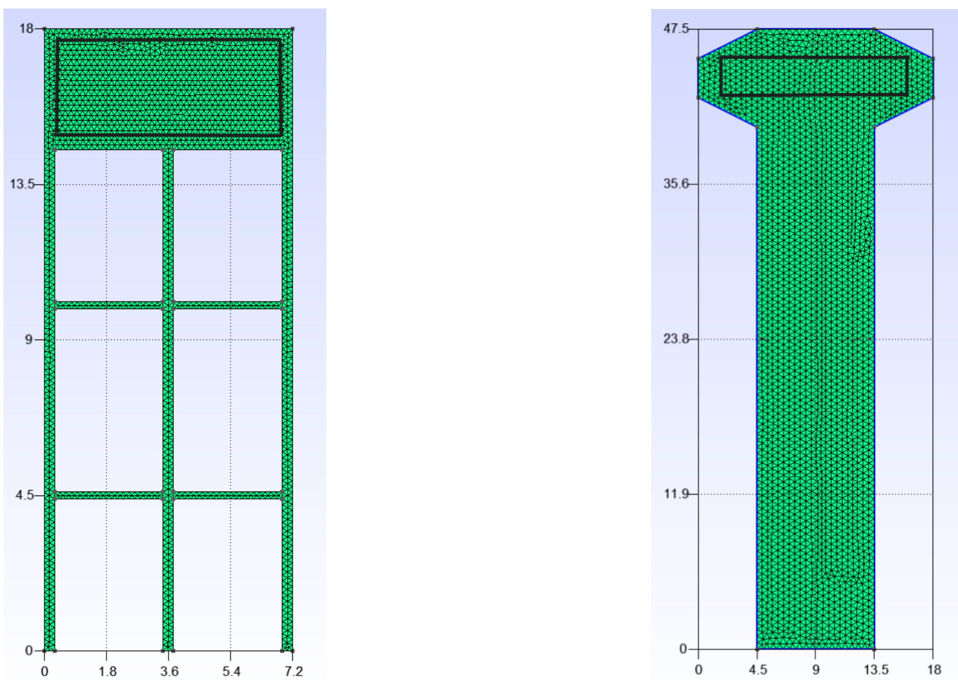
```
def properties_tank1(x, y):
    if 0.3 < x < 6.9 and 14.8 < y < 17.7:
        return 2.2e9, 1000, 0.5 # E, pho, mu
    elif y > 18:
        return 70e9, 2700, 0.3
    else:
        return 30e9, 2400, 0.2


def properties_tank2(x,y):
    if 1 < x < 17 and 42.25 < y < 45.25:
        return 2.2e9, 1000, 0.5 # E, pho, mu
    elif y > 47.5:
        return 70e9, 2700, 0.3
    else:
        return 30e9, 2400, 0.2
```

1.5 Geometry and Mesh Details

The dimensions of both tanks were determined through a combination of engineering estimates and practical measurements. For the Gymkhana tank, we calculated the radius by measuring footsteps and converting that count into meters. The heights of both tanks were assessed using the Inclinometer app on mobile devices alongside Google Earth for reference.

Regarding the helicopter, which is a Chetak model, we obtained its dimensions from online resources. To simplify the modeling process, we represent the helicopter as a cuboid, where the height matches that of the actual helicopter and both the width and depth correspond to the base dimensions of the real aircraft. The Chetak helicopter has a maximum gross weight of 2200 kg.

For the tank construction, we assume an inner wall thickness of 0.3 m for the Gymkhana tank and 1 m for the MSB tank, with the latter being thicker due to its larger size. Additionally, we assume that both tanks are completely filled with water.



(a) Gymkhana Tank - with water region (b) MSB Tank - with water region

Figure 1: Meshes for both tanks without helicopter

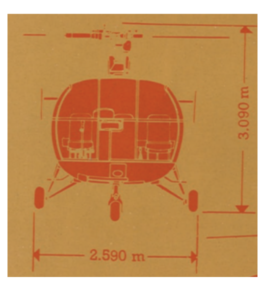


Figure 2: Chetak Helicopter Dimensions

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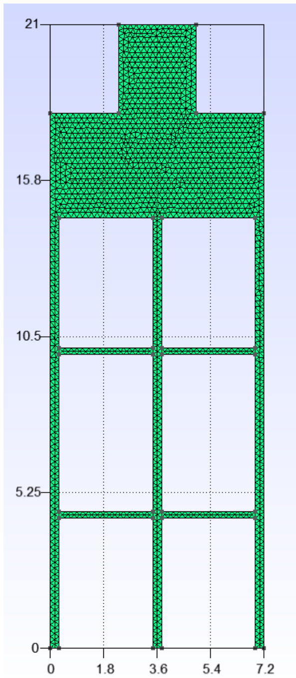
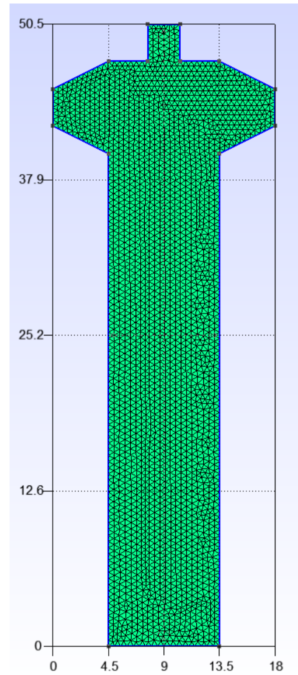


(a) Gymkhana Tank



(b) MSB Tank

Figure 3: Meshes for both tanks with helicopter

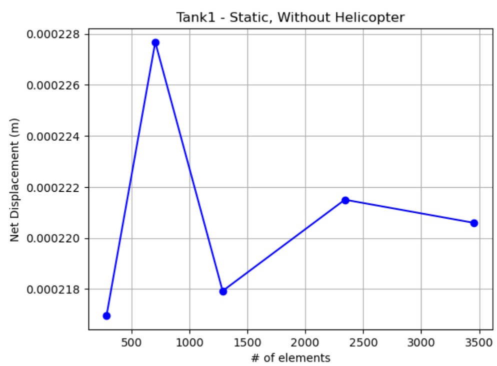
1.6 Assumptions

- The helicopter is represented as a cuboid, with dimensions corresponding to those of the Chetak model. The helicopter is also modeled as an axisymmetric object.
- The tank is considered to be filled with water. Modeling the water is essential, as it helps to support the stress imposed by the weight of both the tank and the helicopter.
- Sloshing effects in the water are neglected in this analysis. We assume that the water remains static and does not exhibit fluid behavior, simplifying calculations.
- In the analysis of the free vibration problem, the helicopter is incorporated into the mesh of the tank and water. This inclusion is essential, as the entire system undergoes

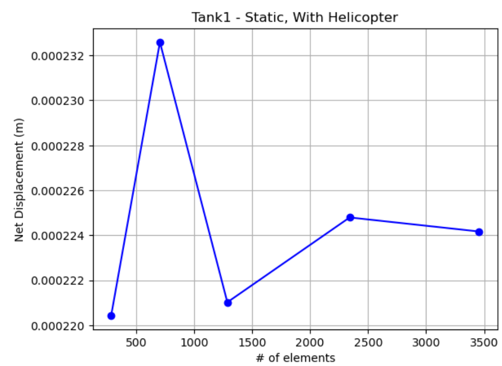
oscillations, necessitating the consideration of the helicopter being part of the mesh

- For the static analysis, the helicopter is treated merely as an additional force acting on the system. Consequently, only the force term is adjusted accordingly. We assume that only the nodes in direct contact with the helicopter, along with a few layers beneath to a certain depth, are affected and the other nodes remain unaffected

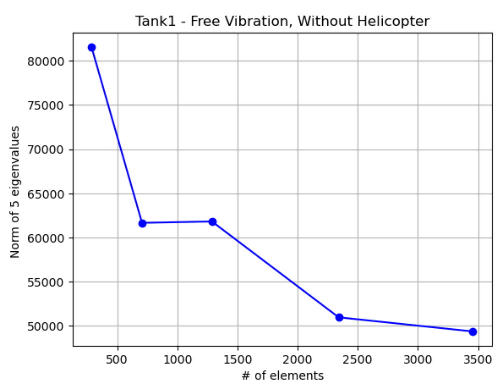
1.7 Mesh Convergence Test



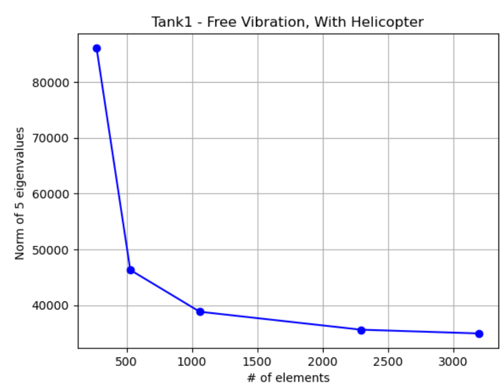
(a) static without helicopter



(b) static with helicopter



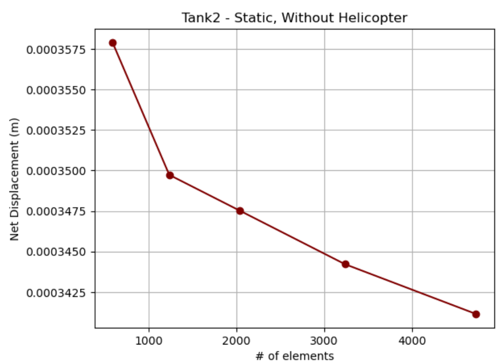
(c) free vibration without helicopter



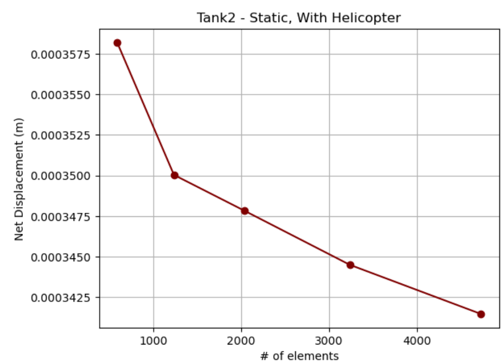
(d) free vibration with helicopter

Figure 4: Convergence Plots - Gymkhana Tank

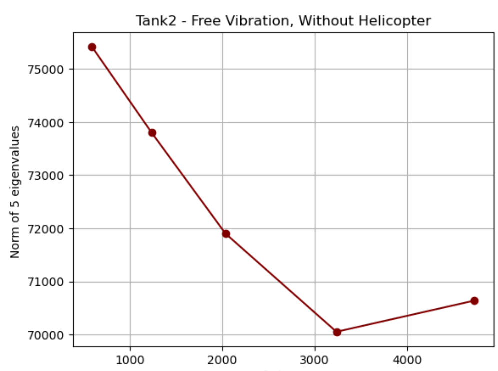
It can observed that the net displacement converges to a value as the mesh gets finer, which implies our convergence test is successful for the static problem. For the free vibration problem, we check the first eigenvalues and choose the meshes for which the eigenvalues become constant. Hence, based on the above plots, we choose a mesh with of elements = 2400 for the Gymkhana tank and of elements = 3300 for the MSB tank for further calculations.



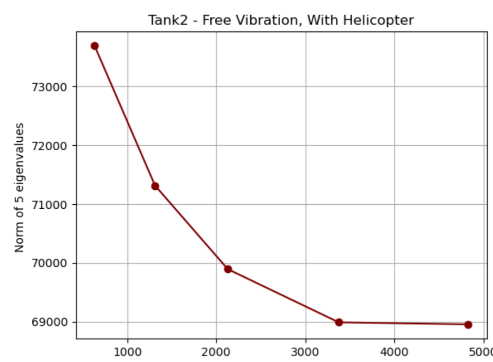
(a) static without helicopter



(b) static with helicopter



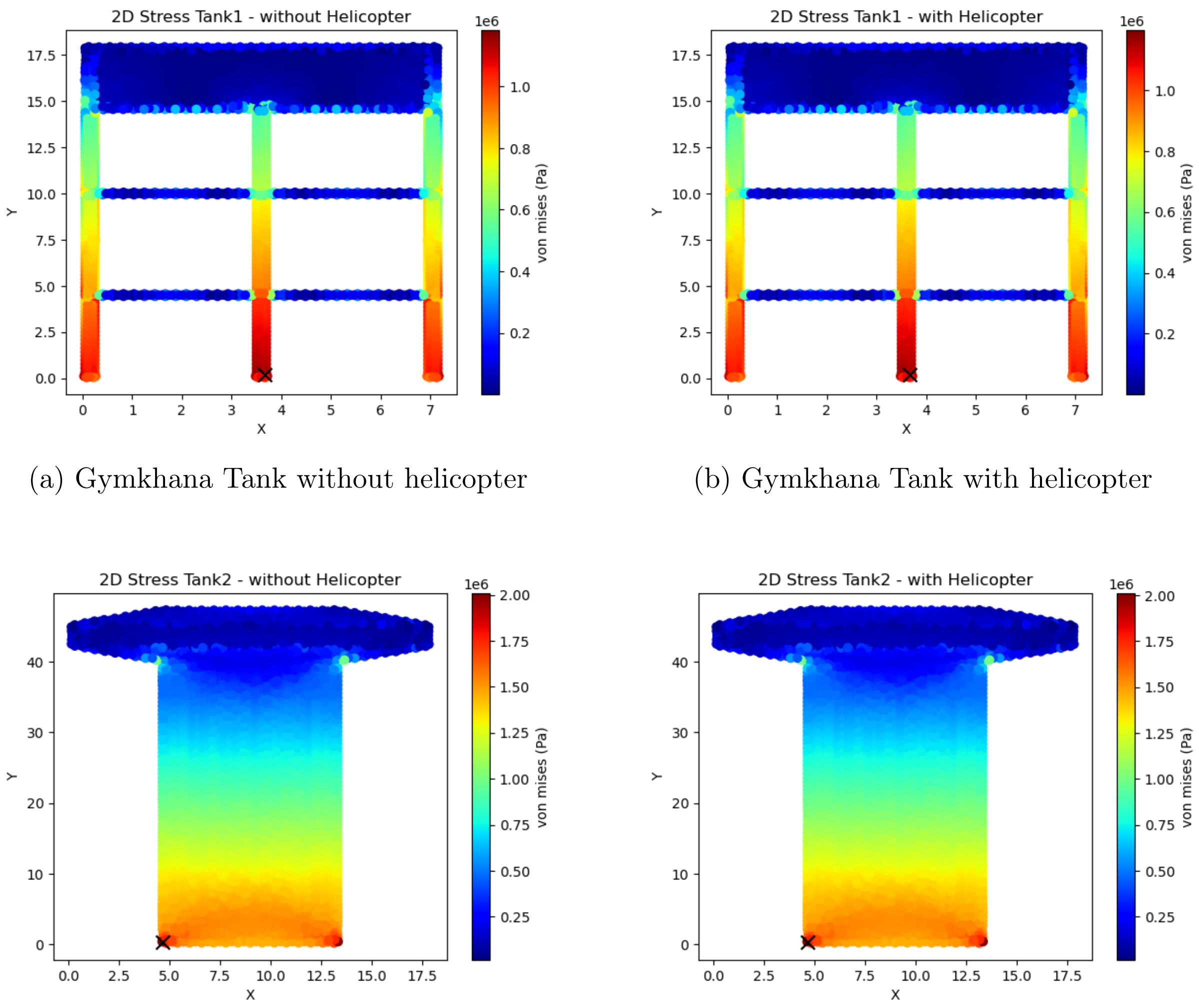
(c) free vibration without helicopter



(d) free vibration with helicopter

Figure 5: Convergence Plots - MSB Tank

1.8 Maximum Stress and Fundamental Frequencies



(a) Gymkhana Tank without helicopter (b) Gymkhana Tank with helicopter

(c) MSB Tank without helicopter (d) MSB Tank with helicopter

Figure 6: Maximum Stress indicated by '×'

As we can see, the maximum stress in both the tanks appears to happen at the lower nodes.

Table 2: Gymkhana Tank Maximum Pressure Data

Helicopter	Coordinate	Pressure value (Pa)
No	(3.67, 0.21)	1.182996×10^6
Yes	(3.67, 0.21)	1.197502×10^6

Table 3: MSB Tank Maximum Pressure Data

Helicopter	Coordinate	Pressure value (Pa)
No	(4.65, 0.34)	2.007955 $\times 10^6$
Yes	(4.65, 0.34)	2.009444 $\times 10^6$

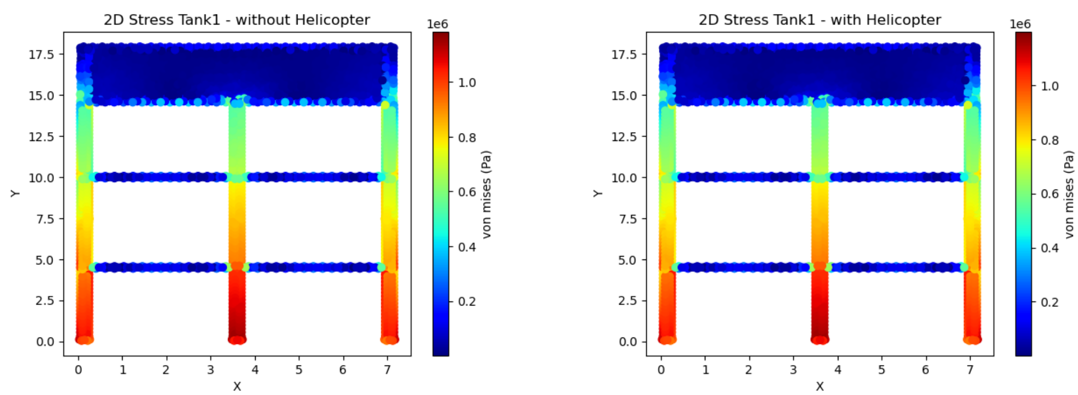
Table 4: Gymkhana Tank 5 fundamental Frequencies(ω^2)

Helicopter	Freq. 1	Freq. 2	Freq. 3	Freq. 4	Freq. 5
No	88	2266	8912	34279	36580
Yes	65	2256	9165	23198	25277

Table 5: MSB Tank 5 fundamental Frequencies (ω^2)

Helicopter	Freq. 1	Freq. 2	Freq. 3	Freq. 4	Freq. 5
No	171	4837	12226	22996	64852
Yes	158	4499	11635	21946	64202

1.9 Stress and Displacement Plots



(a) without helicopter

(b) with helicopter

Figure 7: Von Mises Stress Distribution For Gymkhana Tank

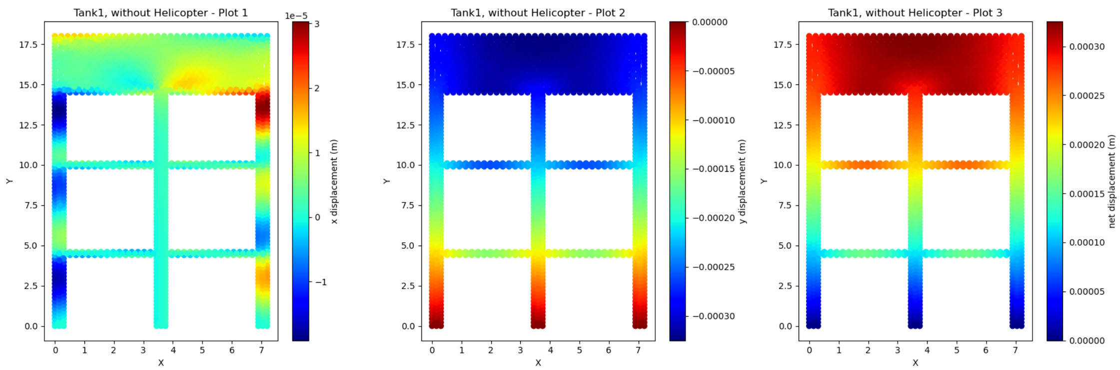


Figure 8: Gymkhana Tank without helicopter - x, y and total displacement

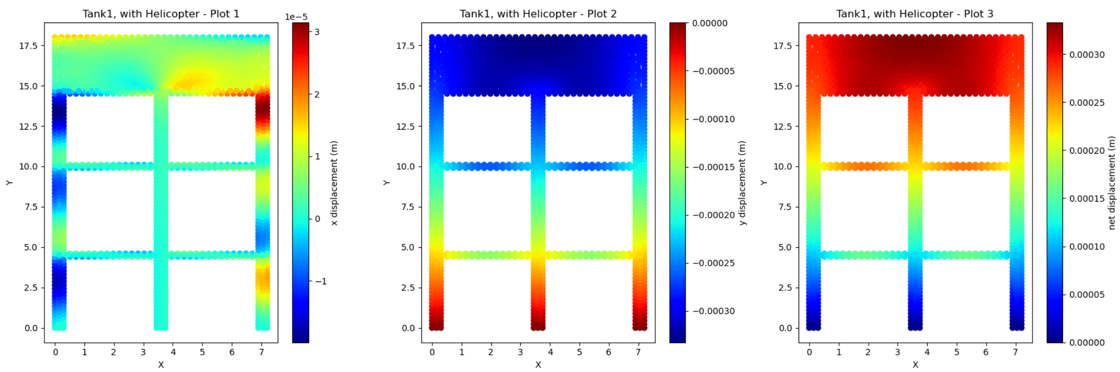
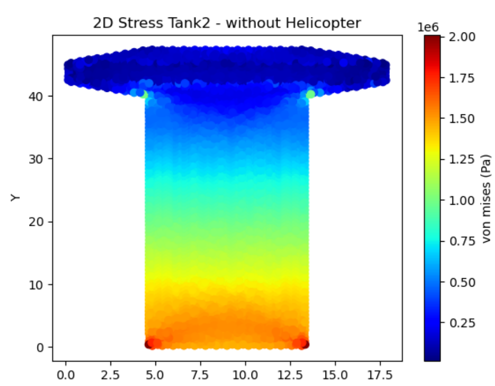
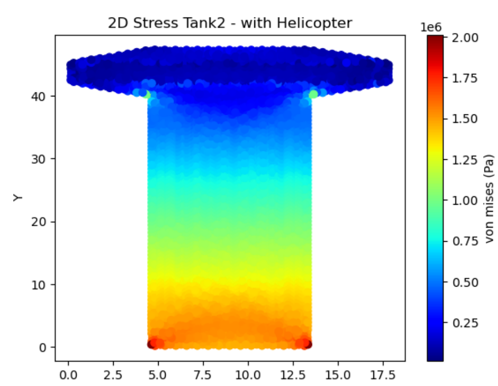


Figure 9: Gymkhana Tank with helicopter - x, y and total displacement



(a) without helicopter



(b) with helicopter

Figure 10: Von Mises Stress Distribution For MSB Tank

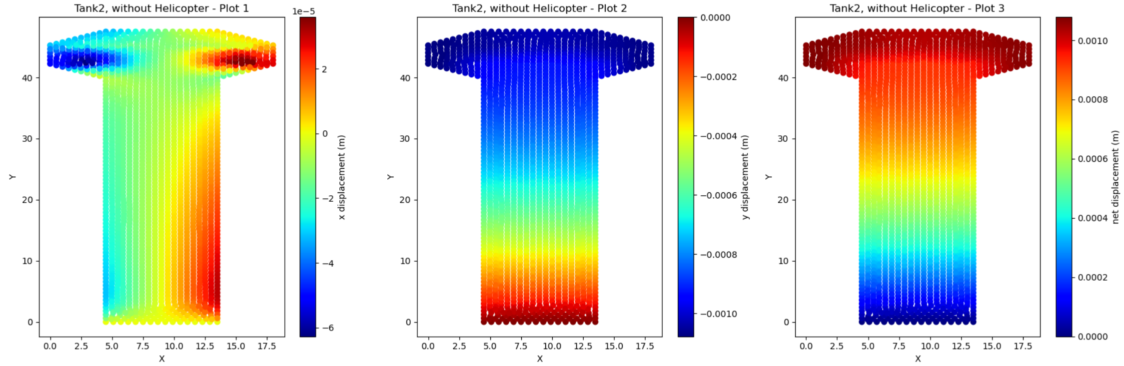


Figure 11: MSB Tank without helicopter - x, y and total displacement

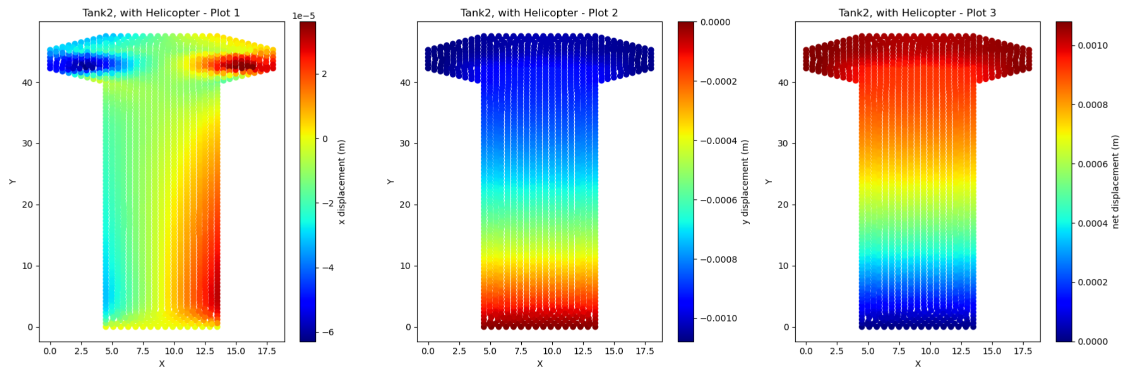


Figure 12: MSB Tank with helicopter - x, y and total displacement

1.10 Mode Shapes

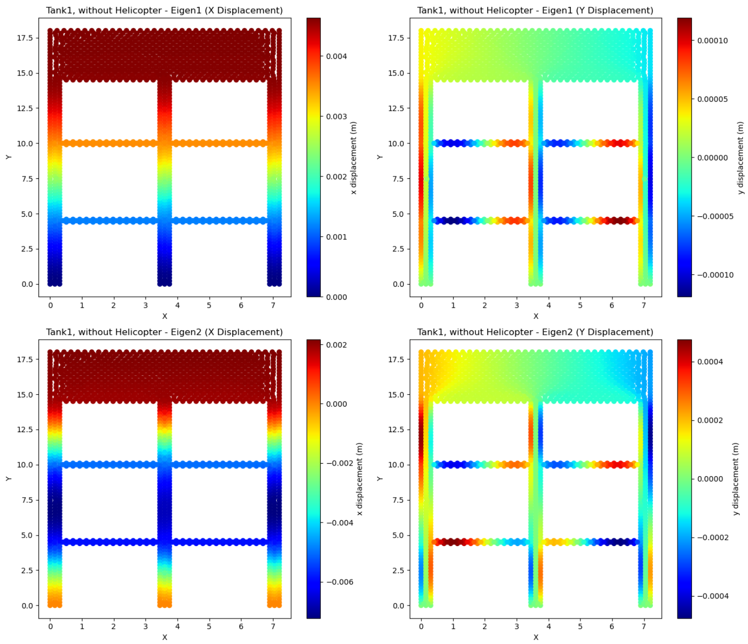


Figure 13: Gymkhana Tank without helicopter - x, y mode shapes

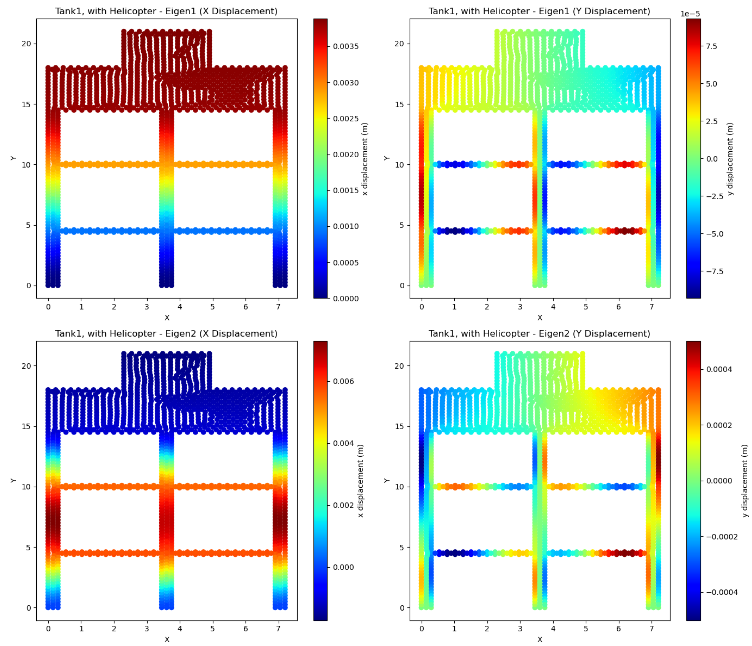


Figure 14: Gymkhana Tank with helicopter - x, y mode shapes

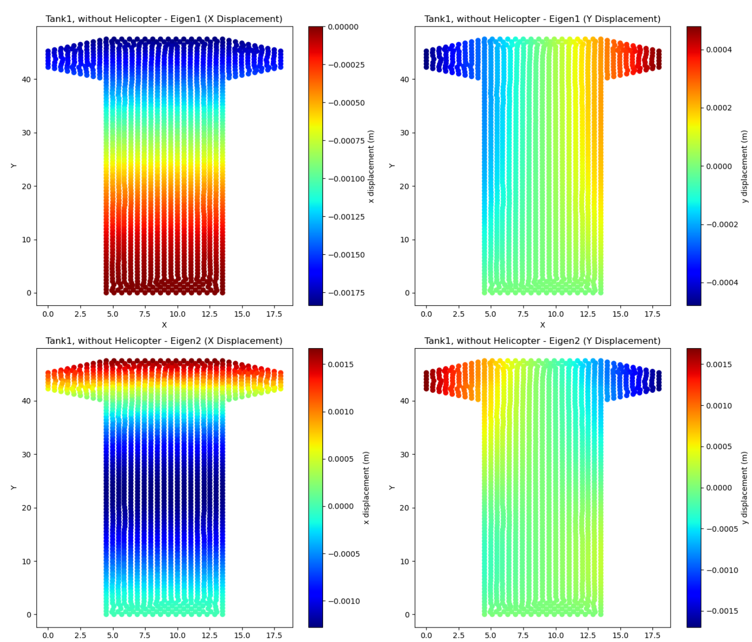


Figure 15: MSB Tank without helicopter - x, y mode shapes

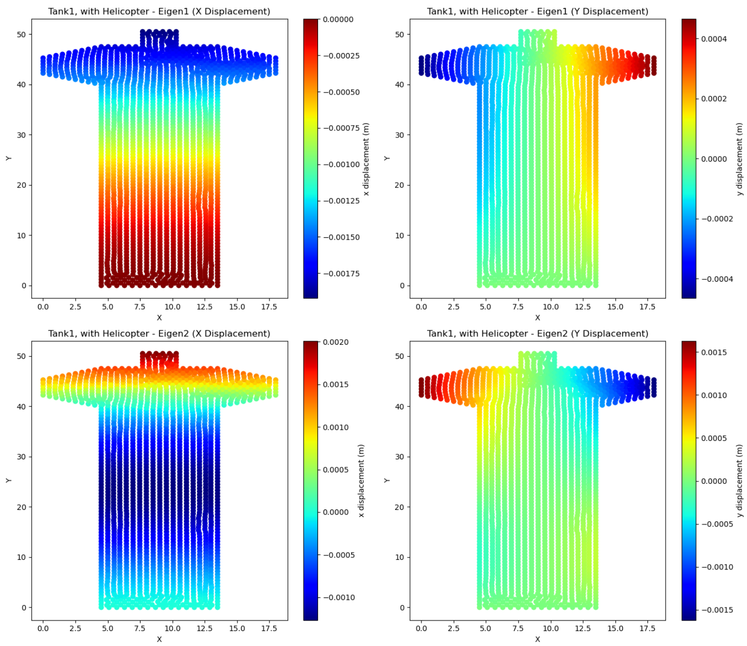


Figure 16: MSB Tank with helicopter - x, y mode shapes

1.11 Impact on Stress, Frequency and Mode Shape

The Magnitude of the maximum stress increases as the helicopter lands on top of the tank. The location of maximum stress does not change though and even the change is in the order of the weight of the helicopter

First Fundamental Frequency decreases due to the helicopter and there is a slight change in mode shape for the 2nd eigenvector. For the 1st eigenvector the shape remains relatively same but it suffers a change in magnitude.

1.12 Choice of WaterTank-

I feel like the Gymkhana water is a better water tank to land on because: 1) The maximum stress value experienced by it is way lesser than MSB Tank, making it more safer 2) In terms of accessability to all the zones, the gymkhana tank is better positioned than the MSB tank making it a better landing spot.

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