# ME5204 Finite Element Analysis

Assignment 2

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ME21B145

# Contents

# 1 Problem Statement

To calculate the area of IITM using numerical integration. Use elements from the .msh file to get coordinates of triangles and perform transformation to a reference triangle to enable numerical integration. To calculate area moment of inertia w.r.t the centroid of IITM and Gajendra Circle

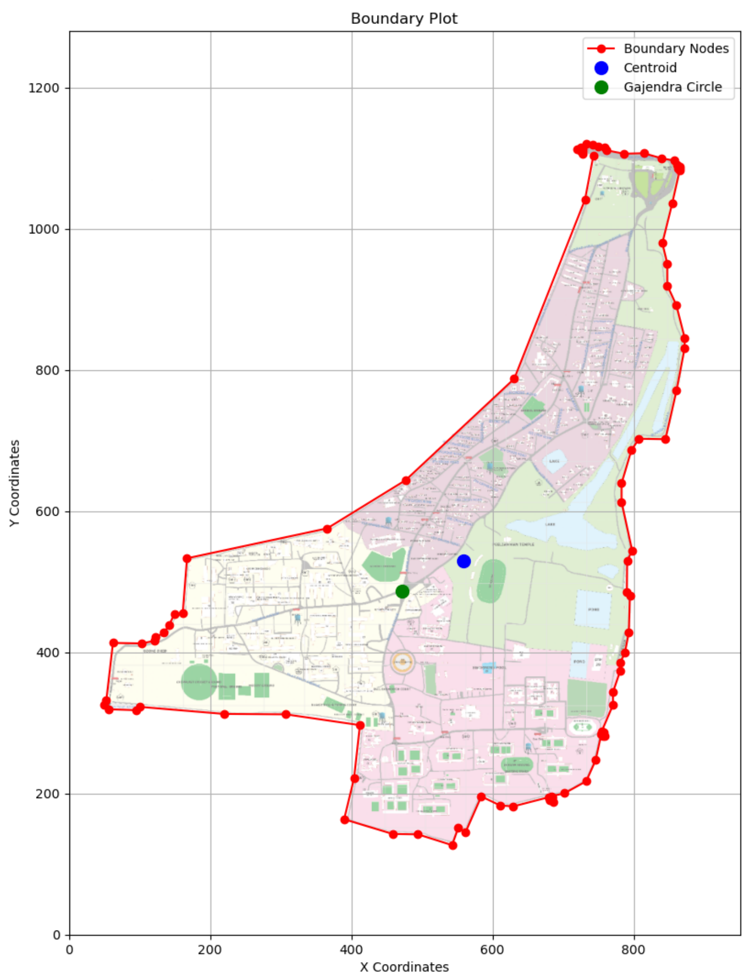



Figure 1: 88 Boundary points, Location of Centroid and GC.

# 2 Solution

## 2.1 Defining Functions

- `extract_first_88_nodes(gmsh_file)`: This function reads a Gmsh file and extracts the coordinates of the first 88 nodes. It identifies the `$Nodes` section in the file and then iterates through the node data to collect the x and y coordinates, which are stored in a list and returned.

- `plot_nodes(node_coordinates)`: This function plots the extracted node coordinates on top of an image of the IITM map. It loads the image, adjusts the figure size to match the image dimensions, and plots the node coordinates. The centroid of the boundary and a specific point for Gajendra Circle are also plotted. The image's opacity is controlled using the `alpha` parameter.

- `triangle_jacobian(x1, y1, x2, y2, x3, y3)`: This function calculates the Jacobian determinant for a triangular element defined by its three vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$. The Jacobian is essential in transforming coordinates and is used in numerical integration over the triangle.

- `numerical_integration(func, x1, y1, x2, y2, x3, y3)`: This function performs numerical integration over a triangular element using a 3-point Gaussian quadrature. The function `func` is evaluated at the integration points, and the results are weighted and summed to approximate the integral over the triangle.

- `unit_func(x, y, x1, y1, x2, y2, x3, y3)`: This is a simple function that always returns 1. It can be used as a placeholder or a test function for numerical integration, representing a constant function over the triangle.

- `radial_func(x, y, x1, y1, x2, y2, x3, y3)`: This function calculates a radial function at a point $(x, y)$ within a triangle defined by the vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$. This function called when calculating the area moment of inertia of IITM.

## 2.2 Scaling Factor

The scale of the map formed by the boundary points is unknown. The IITM map present on the internet has a scale. Using this as a reference, we can find the scale of the map formed using the boundary points.

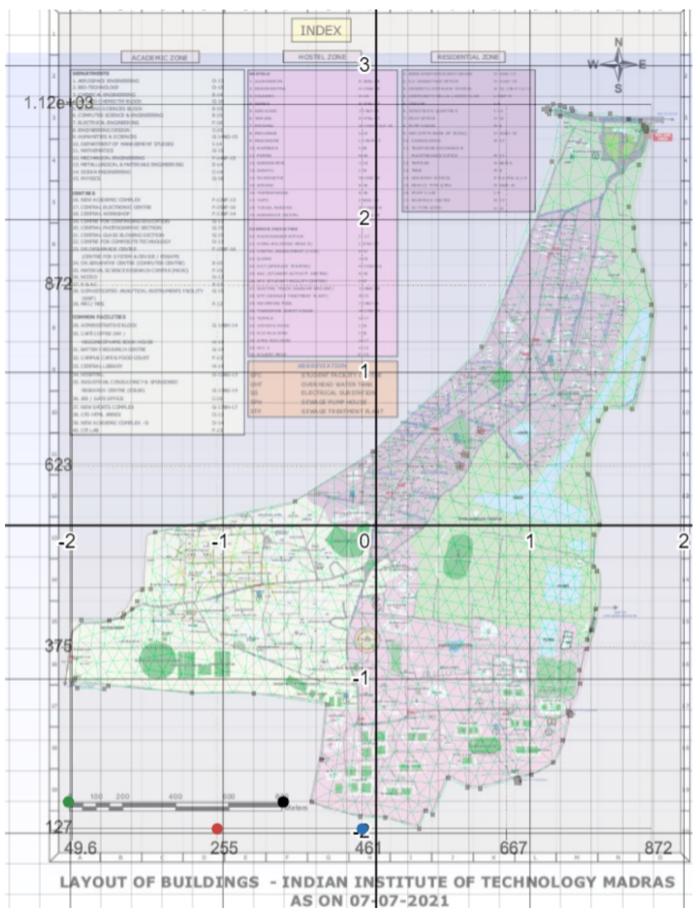
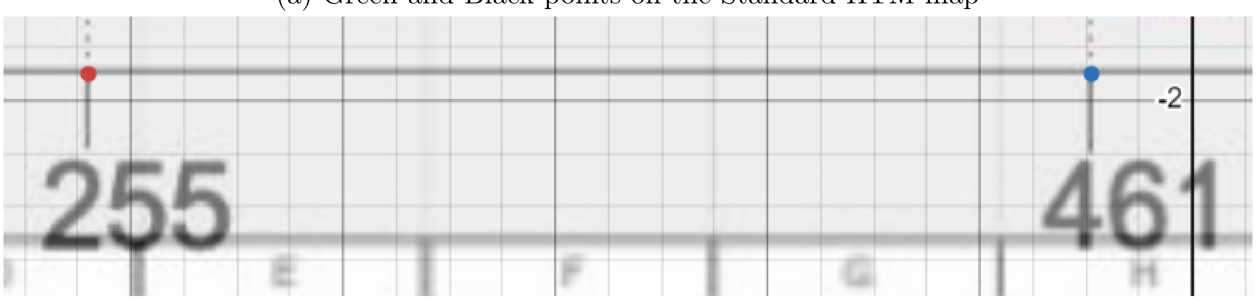



Figure 2: Overlapping standard IITM map and my map formed by 88 boundary nodes.

Steps:

- Open images of standard IITM map and map constructed using boundary nodes on desmos
- Overlap the image of the standard IITM map with the scale and the map constructed using the boundary nodes, such that they match in size.
- Mark the end points of the scale present on the standard IITM map. The total length of the scale corresponds to 800 metres.
- Similar for the map constructed using the boundary nodes, mark one point corresponding to 255 and one corresponding to 461. These numbers correspond to length in the .geo file.
- Using this we can find the scaling factor for the map with the boundary points


(a) Green and Black points on the Standard IITM map




(b) Red and Blue points on the map constructed using Boundary Points

Figure 3: Scaling

**Standard IITM Map:**

- Coordinate of green point: $\mathbf{G} = [-2.01, -1.8]$
- Coordinate of black point: $\mathbf{B} = [-0.61, -1.8]$

**Boundary Constructed Map:**

- Coordinate of red point: $\mathbf{R} = [-1.07, -1.98]$
- Coordinate of blue point: $\mathbf{U} = [-0.095, -1.98]$

**Scaling Factor:**

$$\text{SF} = \frac{800 \times (U_x - R_x)}{(461 - 255) \times (B_x - G_x)}$$

$$\text{SF} = 2.70$$

## 2.3 Area of IITM

The area of the IITM map is calculated by numerical integration. The number of triangles in the mesh is **1758**. We find area of each triangle using numerical integration and sum all of them to give total area. Gauss-Legendre quadrature is used with 3 quadrature points.

### 2.3.1 Gauss - Legendre Quadrature

The Gauss-Legendre quadrature for a triangular domain involves specific points (also called nodes) and associated weights. For the standard right angled triangle, the 3 integration points and their corresponding weights are:

- Points:

$$\left(\frac{1}{6}, \frac{2}{3}\right), \quad \left(\frac{1}{6}, \frac{1}{6}\right), \quad \left(\frac{2}{3}, \frac{1}{6}\right)$$

- Weights:

$$\frac{1}{6}, \quad \frac{1}{6}, \quad \frac{1}{6}$$

The integral of a function $f(x, y)$ over a triangular domain $\Omega$ can be approximated using Gauss-Legendre quadrature as follows:

$$\int_\Omega f(x, y)\, d\Omega \approx \sum_{i=1}^{n} w_i\, f(x_i, y_i)\ \det(J)$$

where:

- $(x_i, y_i)$ are the quadrature points (nodes),
- $w_i$ are the corresponding weights,
- $\det(J)$ is the determinant of the Jacobian matrix, accounting for the transformation from the reference triangle to the actual triangle,
- $n$ is the number of quadrature points.

For the specific case with the given points:

$$\int_\Omega f(x, y)\, d\Omega \approx \det(J)\left[\frac{1}{6} f\left(\frac{1}{6}, \frac{2}{3}\right) + \frac{1}{6} f\left(\frac{1}{6}, \frac{1}{6}\right) + \frac{1}{6} f\left(\frac{2}{3}, \frac{1}{6}\right)\right]$$

This equation shows that the integral over the triangular domain is approximated by summing the values of the function $f(x, y)$ at the specified points, each multiplied by their respective weights and the determinant of the Jacobian.

### 2.3.2 Calculation

**To find the area of each triangle:**

The function $f(x, y)$ is set to 1 over the domain $\Omega$:

$$f(x, y) = 1 \quad \text{over } \Omega$$

**Determinant of the Jacobian ($\det(J)$) for a Triangle:**

Given the vertices of the triangle $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$, the determinant of the Jacobian is calculated using the following equation:

$$\det(J) = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

IITM Area Scaling Table

| IITM Area Unscaled($sq.unit$) | Scaling Factor (SF) | IITM Area Scaled($m^2$) |
|---|---|---|
| $3.3795 \times 10^5$ | 2.70 | $2.4637 \times 10^6$ |

**Scaling Equation:**

$$\text{IITM Area Unscaled} \times \text{SF}^2 = \text{IITM Area Scaled}$$

## 2.4 Area moment of Inertia

### 2.4.1 Coordinates of Centroid

Let the map be divided into $n$ triangles. The x-coordinate and y-coordinate of the centroid of the composite figure can be calculated as follows:

**X-coordinate of Centroid:**

$$\bar{x} = \frac{1}{A}\sum_{i=1}^{n}(\bar{x}_i \cdot A_i)$$

where:

- $\bar{x}$ is the x-coordinate of the centroid of the composite figure,
- $\bar{x}_i$ is the x-coordinate of the centroid of the $i$-th triangle,
- $A_i$ is the area of the $i$-th triangle,
- $A$ is the total area of the composite figure,
- $\sum_{i=1}^{n}(\bar{x}_i \cdot A_i)$ represents the sum of the products of the centroid coordinates and areas of each triangle.

**Y-coordinate of Centroid:**

$$\bar{y} = \frac{1}{A}\sum_{i=1}^{n}(\bar{y}_i \cdot A_i)$$

where:

- $\bar{y}$ is the y-coordinate of the centroid of the composite figure,
- $\bar{y}_i$ is the y-coordinate of the centroid of the $i$-th triangle,
- $A_i$ is the area of the $i$-th triangle,
- $A$ is the total area of the composite figure,
- $\sum_{i=1}^{n}(\bar{y}_i \cdot A_i)$ represents the sum of the products of the centroid coordinates and areas of each triangle.

**Centroid of the Map**

Using the above formula the centroid of the map is located at:

$$\text{Centroid} = [559.4, 529.3]$$

### 2.4.2 MOI about the centroid

The area moment of inertia $I$ about an axis is given by:

$$I = \int_A f(x, y)\, dA$$

where $f(x, y) = x^2 + y^2$ is the function to be integrated over the area $A$.

The transformation from the actual triangle to the reference triangle is given by:

$$x = (1 - \xi - \eta)x_1 + \xi x_2 + \eta x_3$$

$$y = (1 - \xi - \eta)y_1 + \xi y_2 + \eta y_3$$

where:

- $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are the coordinates of the vertices of the actual triangle.
- $\xi$ and $\eta$ are the reference coordinates within the triangle.

we substitute $x$ and $y$ as functions of $\xi$ and $\eta$ such that

$$f(\xi, \eta) = ((1 - \xi - \eta)x_1 + \xi x_2 + \eta x_3)^2 + ((1 - \xi - \eta)y_1 + \xi y_2 + \eta y_3)^2$$

$$I_{\text{origin}} = \sum_{i=1}^{n} f(\xi_i, \eta_i) \cdot w_i \cdot \det(J), \quad \text{where } i \text{ is the quadrature point}$$

Using Gauss - Legendre quadrature , the area moment of inertia about the origin (according to the coordinates given by gmsh) $= 2.2997 \times 10^{11}\ unit^4$

**Parallel Axis Theorem**

The moment of inertia $I_{\text{origin}}$ about an axis through the origin is related to the moment of inertia $I_{\text{centroid}}$ about an axis through the centroid by:

$$I_{\text{origin}} = I_{\text{centroid}} + A\left(\bar{x}^2 + \bar{y}^2\right)$$

where:

- $I_{\text{origin}}$ is the moment of inertia about the axis through the origin,
- $I_{\text{centroid}}$ is the moment of inertia about the axis through the centroid,
- $A$ is the area of the figure,
- $\bar{x}$ is the x-coordinate of the centroid,
- $\bar{y}$ is the y-coordinate of the centroid.

$I_{\text{centroid}} = = 2.29547 \times 10^{10}\ unit^4$

$$I_{\text{centroid}}(scaled) = I_{\text{centroid}} * SF^4$$

$I_{\text{centroid}}$ (scaled) $= 1.5702 \times 10^{12}\ m^4$

### 2.4.3 MOI about Gajendra Circle

The coordinates of Gajendra circle was found using gmsh. The coordinates are as follows : [472, 486.7]

$$I_{\mathrm{GC}} = I_{\mathrm{centroid}} + A\left((x_1 - x_2)^2 + (y_1 - y_2)^2\right)$$

where $(x_1, y_1)$ are the coordinates of the centroid, and $(x_2, y_2)$ are the coordinates of GC (Gajendra Circle).

$I_{\mathrm{GC}} = 3.274 \times 10^{10}\ unit^4$

$$I_{\mathrm{GC}}(scaled) = I_{\mathrm{GC}} * SF^4$$

$I_{\mathrm{GC}}\ (\text{scaled}) = 1.7400 \times 10^{12}\ m^4$

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