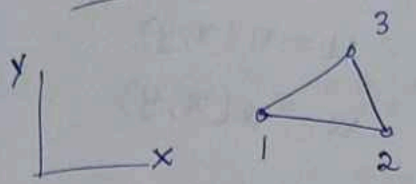
21st Oct

Two D Vector



Each node to have 2 dofs
↳ Elasticity (mechanical)

- thermo - mechanical
- chemo - mechanical
- electro - mechanical
- magneto - mechanical

} 3 dofs / node

Elasticity

- small deformation
- linear elastic
- cauchy stress (σ), small strain (ε)
- displacements (u) — u in x-direction; v in y-direction

3D $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xz} \\ \sigma_{zx} \\ \sigma_{yz} \\ \sigma_{zy} \\ \sigma_{xy} \\ \sigma_{yx} \end{Bmatrix}$ 9

Stress components: 81 constants

Moment equilibrium $\begin{cases} 9 \text{ comp} \rightarrow 6 \text{ comp} \\ 36 \text{ constants} \end{cases}$

Mtl. symmetry: $\begin{cases} \text{orthotropic} \\ \text{Transvesly isotropic} \\ \text{isotropic} \end{cases}$ (E, ν, G, K)

E ↑ Youngs, ν ↑ Poiss, G ↓ shear, K ↓ Bulk

3D ⟶ 2D

consider one of the dimensions (constant) invariant

plane stress: $\sigma_z = 0$; $\sigma_{zx}, \sigma_{zy} = 0$

plane strain: $\varepsilon_z = 0$; $\nu_{zx}, \nu_{zy} = 0$

$$\underbrace{\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}}_{3\times1} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}}_{3\times3} \underbrace{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}}_{3\times1}$$

→ bcz. of material symmetry

← constitutive relation

Elasticity matrix

Material matrix

$f(E, \nu, G, K)$

"strain - displacement" relation

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

approximate u, v

$u = u(x,y)$

$v = v(x,y)$

$$u = \sum \phi_I(x,y)\, u_I$$

$$v = \sum \underbrace{\phi_I(x,y)}_{\text{Lagrange shape functions}}\, v_I$$

← Nodal values

<u>2D</u>

Force Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

ignore body load & inertia

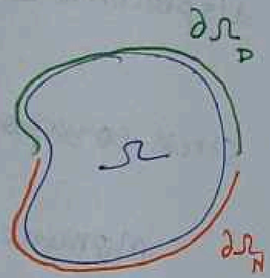
(we have 2 eqs & 3 variables) cannot be solved →

$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$ ← unknowns

$\boxed{\text{constitutive relation}}$ convert this to strain & mtl. matrix

$\boxed{\text{strain-displacement relation}}$

convert to 2 eqns with 2 unknowns (displacements) ←



" displacement based formulation"

↓

displacements becomes the primary unknowns

strain & stress : secondary unknowns

Boundary conditions < Dirichlet → on primary variable ; Neumann → secondary variable

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$

$\phi = \partial\Omega_N \cap \partial\Omega_D$

on Dirichlet, $(\partial\Omega_D)$

$\underset{\sim}{u} = \underset{\sim}{\bar{u}}$

on Neumann, $(\partial\Omega_N)$

$\sigma \cdot n = \hat{t}$ → traction

↑ line normal (2D) / surface normal (3D)

$$\int_{\Omega} \Big(\underbrace{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}}_{\text{eq}^n \,①} + \underbrace{\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}}_{\text{eqn } ②} \Big) d\Omega = 0$$

$$u = \sum \phi_I u_I$$
$$v = \sum \phi_I v_I$$
$$g_1 = \sum \phi_I g_{1I}$$
$$g_2 = \sum \phi_I g_{2I}$$

multiply

$\left.\begin{array}{ll} \text{eq}^n ① \times g_1 & g_1 \\ \text{eq}^n ② \times g_2 & g_2 \end{array}\right\}$ arbitrary

$$\Rightarrow \int_{\Omega} \Big\{ \Big(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \Big) g_1 + \Big(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \Big) g_2 \Big\} d\Omega = 0$$

$$\Rightarrow \boxed{\int_{\partial\Omega} (g_1 \sigma_x n_x + g_1 \sigma_{xy} n_y)\, d\Omega - \int_{\Omega} \Big(\frac{\partial g_1}{\partial x} \sigma_x + \frac{\partial g_1}{\partial y} \sigma_{xy} \Big) d\Omega}$$

$$+ \boxed{\int_{\partial\Omega} (g_2 \sigma_{xy} n_x + g_2 \sigma_y n_y)\, d\Omega - \int_{\Omega} \Big(\frac{\partial g_2}{\partial x} \sigma_{xy} + \frac{\partial g_2}{\partial y} \sigma_y \Big) d\Omega} = 0$$

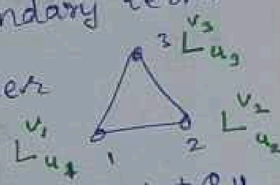
combine,

$$\int_{\Omega} \Big(\frac{\partial g_1}{\partial x} \sigma_x + \frac{\partial g_1}{\partial y} \sigma_{xy} + \frac{\partial g_2}{\partial x} \sigma_{xy} + \frac{\partial g_2}{\partial y} \sigma_y \Big) d\Omega = \int_{\partial\Omega} (g_1 \sigma_x n_x + g_1 \sigma_{xy} n_y + g_2 \sigma_{xy} n_x + g_2 \sigma_y n_y)$$

$$\underbrace{\int_{\Omega} \Big(\frac{\partial g_1}{\partial x} \quad \frac{\partial g_2}{\partial y} \quad \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} \Big) \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} d\Omega}_{\text{bilinear term}} = \underbrace{\int_{\Omega} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}^{-1} \begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} d\Omega}_{\text{boundary term}}$$

order can be changed

We consider



$u = \phi_1 u_1 + \phi_2 u_2 + \phi_3 u_3$

$v = \phi_1 v_1 + \phi_2 v_2 + \phi_3 v_3$

using constitutive relation, & "strain displacement" relation

$$\int_{\Omega} \Big(\frac{\partial g_1}{\partial x} \quad \frac{\partial g_2}{\partial y} \quad \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} \Big)\, \mathbb{C} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} d\Omega$$

~~Constitutive~~ Elasticity matrix

can be used for M matrix formulation

$$\underset{2\times 1}{\begin{Bmatrix} u \\ v \end{Bmatrix}} = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Can be also written as

$$\checkmark \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

preferred in all commoricial packages

$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 & u_3 \\ v_2 & v_3 \end{Bmatrix}$

$$\begin{array}{cccc} & \text{node 1} & \text{node 2} & \text{node 3} \end{array}$$

$$\varepsilon = \underbrace{\underset{3\times 6}{\begin{bmatrix} \frac{\partial \phi_1}{\partial x} & 0 & \frac{\partial \phi_2}{\partial x} & 0 & \frac{\partial \phi_3}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1}{\partial y} & 0 & \frac{\partial \phi_2}{\partial y} & 0 & \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_3}{\partial y} & \frac{\partial \phi_3}{\partial x} \end{bmatrix}}}_{B \text{ strain displacement matrix}} \underset{6\times 1}{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}}$$

$$\underbrace{[\leftarrow g \text{ vector} \rightarrow]}_{1\times 6} \int_{\Omega} \underbrace{\underset{6\times 3}{B^T} \, \underset{3\times 3}{\mathbb{C}} \, \underset{3\times 6}{B}}_{6\times 6} \, d\Omega \underset{6\times 1}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}} \quad \forall g$$

2D vector programming.

Plane stress

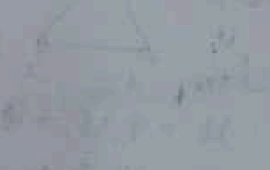
$$\mathbb{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane strain

$$\mathbb{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

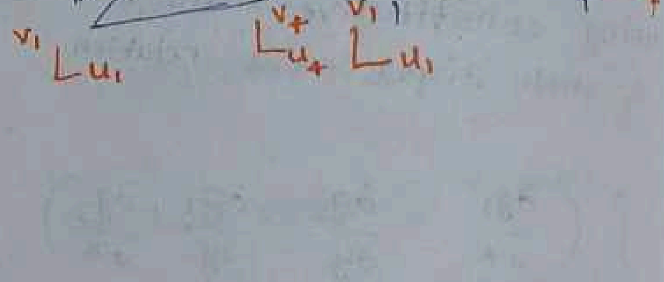
Elemental bilinear form

$$\int_{\Omega^e} B^T \mathbb{C} B \, d\Omega \qquad B = \begin{bmatrix} \frac{\partial \phi_I}{\partial x} & 0 \\ 0 & \frac{\partial \phi_I}{\partial y} \\ \frac{\partial \phi_I}{\partial y} & \frac{\partial \phi_I}{\partial x} \end{bmatrix} \quad I = 1, 2 \ldots n$$



- 3 noded Δ^{le} element
CST (constant strain Δ^{le})

↳ "only" reproduce a constant strain field



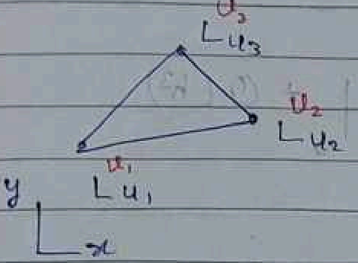
Oct 25^{th}

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y &= \rho \frac{\partial^2 v}{\partial t^2}\end{aligned}\right\}$$

$$\int_\Omega g_1 \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x - \rho\frac{\partial^2 u}{\partial t^2} \right) + g_2 \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y - \rho\frac{\partial^2 v}{\partial t^2} \right) d\Omega$$

custom body load $\quad \int_\Omega g_1 b_x \, d\Omega + \int_\Omega g_2 b_y \, d\Omega \ \forall g_1, g_2 \quad = 0$

$g_1 = \sum \varphi_I g_{I1}$; $g_2 = \sum \varphi_I g_{I2}$; g_{I1}, g_{I2} are arbitrary



$$\underbrace{\int_{\Omega^e} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} b_x}_{\text{body force in X-direction}} + \underbrace{\int_{\Omega^e} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} b_y}_{\text{body force in Y-direction}}$$

$$\int_{\Omega^e} \begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 \end{bmatrix}^T \begin{bmatrix} b_x \\ b_y \end{bmatrix} d\Omega \rightarrow \text{size } 6\times1 \text{ vector}$$

if $\begin{bmatrix} b_x \\ b_y \end{bmatrix} = f(x,y)$

Inertia

$$- \int_\Omega g_1 \rho \frac{\partial^2 u}{\partial t^2} d\Omega - \int_\Omega g_2 \rho \frac{\partial^2 v}{\partial t^2} d\Omega$$

$u = \sum \varphi_I (x,y) u_I(t) \qquad v = \sum \varphi_I (x,y) v_I(t)$

$$- \int_\Omega \underbrace{\overbrace{\begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 \end{bmatrix}^T}^{g_1 \,\&\, g_2}_{6\times2} \rho \underbrace{\begin{bmatrix} \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 & 0 \\ 0 & \varphi_1 & 0 & \varphi_2 & 0 & \varphi_3 \end{bmatrix}}_{2\times6} d\Omega}_{M:\ \text{Mass matrix } (6\times6)} \underbrace{\begin{bmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 v_1}{\partial t^2} \\ \vdots \\ \frac{\partial^2 u_3}{\partial t^2} \\ \frac{\partial^2 v_3}{\partial t^2} \end{bmatrix}}_{6\times1}$$

General form:-

$$M \frac{\partial^2 \underset{\sim}{u}}{\partial t^2} + K \underset{\sim}{u} = F \longrightarrow \text{body load : } \int_\Omega \phi^T b \, d\Omega$$

M: $\int_\Omega \varphi^T \rho \varphi \, d\Omega$

K: $\int_\Omega B^T C B \, d\Omega$

$F \longrightarrow$ BC : $\int_{\partial\Omega} \varphi \, \sigma \cdot n \, d\Omega$

To approximate : $\frac{\partial^2 \underset{\sim}{u}}{\partial t^2}$ we use central scheme

$$f(x_0+h) = f(x_0) + \cancel{h\frac{\partial f}{\partial x}\Big|_{x_0}} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_0} + O(h^3)$$

$$+\quad f(x_0-h) = f(x_0) - \cancel{h\frac{\partial f}{\partial x}\Big|_{x_0}} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_0} + O(h^3)$$

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2\frac{\partial^2 f}{\partial x^2}\Big|_{x_0} + O(h^3)$$

Transient Analysis

$$u = u_0 e^{i\omega t} \;;\; v = v_0 e^{i\omega t}$$

$u_0 \to u_0(x,y)$; $v_0 \to v_0(x,y)$

Assume harmonic solution for $\underset{\sim}{u}$

$$M\ddot{u} + Ku = F \qquad \ddot{u} = \frac{\partial^2 u}{\partial t^2}$$

$$(i\omega)^2 M \underset{\sim}{u_0} e^{i\omega t} + K\underset{\sim}{u_0} e^{i\omega t} = F$$

$$(-\omega^2 M + K)\underset{\sim}{u_0} e^{i\omega t} = F$$

Free Vibration : $F = 0$ No external excitation

$$(-\omega^2 M + K) u_0 = 0$$

Non-trivial soln.

$$\det(-\omega^2 M + K) = 0$$

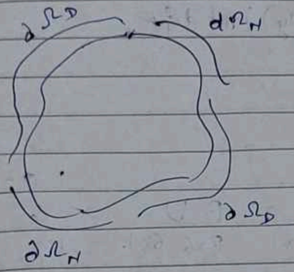
eigen value problem

ω - frequencies - Natural frequency

u_0 - eigenvectors - mode shapes

Oct 28th

$$\int_{\Omega} \phi^T \rho \phi \, d\Omega + \int_{\Omega} \begin{bmatrix} \frac{\partial \phi_I}{\partial x} & 0 \\ 0 & \frac{\partial \phi_I}{\partial y} \\ \frac{\partial \phi_I}{\partial y} & \frac{\partial \phi_I}{\partial x} \end{bmatrix}_J C \begin{bmatrix} \frac{\partial \phi_I}{\partial x} & 0 \\ 0 & \frac{\partial \phi_I}{\partial y} \end{bmatrix} d\Omega$$



$\partial\Omega_N \quad \sigma \cdot n = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → traction

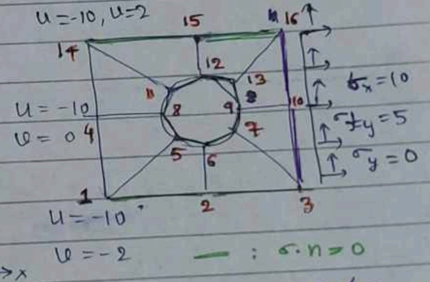
$\sigma \cdot n = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

$\left.\begin{matrix} \text{when } \sigma \cdot n = 0 \\ q = -K\nabla T \quad q \cdot n = 0 \end{matrix}\right\}$ – we apply "ZERO" Neumann on the boundary

if $\sigma \cdot n \neq 0$ either traction is given (or) state of stress is given $\begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} \cdot \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$

$\sigma_x \cdot n_x + \sigma_{xy} \cdot n_y = t_x$

$\sigma_{xy} \cdot n_x + \sigma_y \cdot n_y = t_y$



— : $\sigma \cdot n = 0$

— : $\sigma \cdot n \neq 0$

Ex:- Edge information

1 2
2 3
3 10
10 16
16 15
15 14
14 4
4 1

$\int \phi \, \sigma n \, d\Omega$ = Non zero only in $\int_3^{10} + \int_{10}^{16}$

remaining everywhere = 0 bcz $\sigma \cdot n = 0$

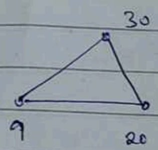
$$\int_{3-10} \varphi^T \begin{bmatrix} \sigma_x & \tau_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} d\Omega = \int_{-1}^{+1} \underbrace{?}_{4\times 2} \begin{bmatrix} 10 & 5 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} |J| \, d\xi$$

Compute contribution along x $\quad \sigma_x n_x + \sigma_{xy} n_y$

$$\begin{Bmatrix} 5 \\ 19 \end{Bmatrix} = \int_{-1}^{+1} \begin{Bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{Bmatrix} (\sigma_x n_x + \sigma_{xy} n_y) \, |J| \, d\xi$$

for the y

$$\begin{Bmatrix} 6 \\ 20 \end{Bmatrix} = \int_{-1}^{+1} \begin{Bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{Bmatrix} (\tau_{xy} n_x + \sigma_y n_y) \, |J| \, d\xi .$$



$$\begin{Bmatrix} \vdots \\ u_9 \to 17 \\ v_9 \to 18 \\ \vdots \\ u_{20} \to 39 \\ v_{20} \to 40 \\ \vdots \\ u_{30} \to 59 \\ v_{30} \to 60 \end{Bmatrix} \leftarrow \text{full displacement vector}$$

$1-\varphi = \xi$

ξ

η

Strain of element $= B \, U$
$\quad 3\times 1 \quad 3\times 6 \quad 6\times 1$

$$= \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & & \\ -1 & -1 & 1 & 1 & & \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Stress

constitutive relationship

$$\underset{3\times 1}{\sigma} = \underset{3\times 3}{C} \, (\underset{3\times 6}{B} \, \underset{6\times 1}{U})$$