PROPOSITIONAL CALCULUS

INTRODUCTION

Logic is the basis of all mathematical reasoning, and of all automated reasoning. The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

PROPOSITIONAL LOGIC (PROPOSITIONAL CALCULUS)

A **Proposition (**or a **Statement)** is the basic building block of logic. It is defined as a declarative sentence that is either True or False, but not both. The Truth Value of a proposition is True(denoted as T) if it is a true statement, and False(denoted as F) if it is a false statement. To represent propositions, propositional variables are used. By Convention, these variables are represented by capital letters such as P, Q, R, S, etc.

For Example,

P: The sun rises in the East and sets in the West.

Q: 1 + 1 = 2

R: 'b' is a vowel.

All of the above sentences are propositions, where the P and Q are valid(True) and R is invalid(False). Some sentences that do not have a truth value or may have more than one truth value are not propositions.

For Example,

S: What time is it?

T: Go out and play.

U: x + 1 = 2.

The above sentences are *not propositions* as the S and T do not have a truth value, and U may be true or false.

The area of logic which deals with propositions is called **Propositional Calculus** or **Propositional Logic**. A proposition consisting of only a single propositional variable is called an **Atomic Proposition**. It also includes producing new propositions using existing ones. Propositions constructed using one or more propositions are called **Compound Propositions**. The propositions are combined together using **Logical Connectives** or **Logical Operators**.

LOGICAL CONNECTIVES (LOGICAL OPERATORS)

In logic, a logical connective is a symbol or word used to connect two or more sentences in a meaningful way. Propositional logic provides mainly five types of connectives -

1. Conjunction / AND ($\wedge$)
2. Disjunction / OR ($\vee$)
3. Negation / NOT ($\neg$)
4. Implication / if-then ($\rightarrow$)
5. Double Implication /if and only if ($\Leftrightarrow$)

Conjunction (AND)

If P and Q are two propositions, then the conjunction of P and Q (read as P and Q) is a proposition (denoted by P∧Q) whose truth values are as given in table. This new proposition is true exactly when both P and Q are true.

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Example

Let

P: The sun rises in the east (T)

Q: The sun revolves around the earth (F)

Then the sentence

P∧Q: The sun rises in the east and the sun revolves around the earth (F)

Here P∧Q is false since one of the component sentences, Q, is false.

Disjunction (OR**)**

If P and Q are two propositions, then the disjunction of P and Q (read as P or Q) is a proposition (denoted by P∨Q) whose truth values are as given in table. This new proposition is true when P is true, or Q is true, or both.

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Example

Let

P: The sun rises in the east (T)

Q: The sun revolves around the earth (F)

Then the sentence

P∨Q: The sun rises in the east or the sun revolves around the earth (T)

Here P∨Q is true since at least one of the sentences P and Q is true.

Negation (NOT)

If P is a proposition, then the negation of P is denoted by ~ P or ¬P, which when translated to simple English means "*It is not the case that P*" or simply "*not P*". The truth value of ~ p is the opposite of the truth value of p. The truth table of ~P is as follows.

p	~p
T	F
F	T

Example

The negation of

P: It is raining today

is

~P: It is not the case that is raining today

or simply

~P: It is not raining today.

CONDITIONAL AND BI-CONDITIONAL STATEMENTS

Conditional Statement or Implication (if … then)

For any two propositions P and Q, the statement “if P then Q” is called an implication and it is denoted by P→Q. In the implication P→Q, P is called the hypothesis or antecedent or premise and Q is called the conclusion or consequence. The implication is P→Q is also called a **conditional statement**. The implication is false when P is true and Q is false otherwise it is true. The truth table of P→Q is-

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

From the above truth table, we have

$$\boxed{P \to Q \equiv {\sim}P \vee Q}$$

Example

Let

P: Today is Friday

Q: It is raining today

Then

P→Q: If it is Friday then it is raining today

The above proposition is true if it is not Friday (premise is false) or if it is Friday and it is raining, and it is false when it is Friday but it is not raining.

Conditional statements play a very important role in mathematical reasoning; thus a variety of terminology is used to express P→Q, some of which are listed below.

- "if P, then Q"
- " P is sufficient for Q"
- " Q when P"
- "a necessary condition for P is Q"
- "P only if Q"
- "Q unless ~ P"
- "Q follows from P"

Bi-conditional or Double Implication (if and only if)

For any two propositions P and Q, the statement “P if and only if (iff) Q” is called a biconditional and it is denoted by P ⇔ Q. The statement P ⇔ Q is also called a bi-implication. The implication is true when P and Q have same truth values, and is false otherwise. The truth table of P⇔ Q is:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \Leftrightarrow Q$ can be also represents in the following ways.

$$P \Leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$$
$$P \Leftrightarrow Q \equiv (\sim P \vee Q) \wedge (\sim Q \vee P)$$
$$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

Example:

Let

P: Today is Friday

Q: It is raining today

Then

$P \Leftrightarrow Q$: It is raining today if and only if it is Friday today.

The above proposition is true if it is not Friday and it is not raining or if it is Friday and it is raining, and it is false when it is not Friday or it is not raining.

Some other common ways of expressing $P \Leftrightarrow Q$ are-

- "P is necessary and sufficient for Q "
- "if P then Q, and conversely"
- " P iff Q "

Precedence of Operators

As a way of reducing the number of necessary parentheses, one may introduce precedence rules: $\neg$ has higher precedence than $\wedge$, $\wedge$ higher than $\vee$, and $\vee$ higher than $\to$. Here is a table that shows a commonly used precedence of logical operators.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4
$\leftrightarrow$	5

Example:

$\neg P \wedge Q$ means $(\neg P) \wedge Q$

$P \wedge Q \to R$ means $(P \wedge Q) \to R$

Functional Completeness

A set of logical connectives is called functionally complete if every propositional statement is equivalent to one involving only these connectives.

- The set $\{\neg, \vee, \wedge\}$ is functionally complete.
- The sets $\{\neg, \vee\}$ and $\{\neg, \wedge\}$ are functionally complete.
- The set $\{\neg, \to\}$ is functionally complete.
- {NAND} and {NOR} are functionally complete.

Duality of Compound Proposition

The dual of compound proposition that contains only logical operators $\vee$, $\wedge$,$\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T.

Example

1. Dual of the proposition $\sim(P \wedge Q)$ is $\sim(P \vee Q)$
2. Dual of the proposition $(P \vee Q) \wedge S$ is $(P \wedge Q) \vee S$
3. Dual of the proposition $P \vee \sim P \equiv T$ is $P \wedge \sim P \equiv F$

Problems

1. If P represents 'This book is good' and Q represents 'This book is cheap', write the following sentences in symbolic form:

(a) This book is good and cheap.
(b) This book is not good but cheap.
(c) This book is costly but good.
(d) This book is neither good nor cheap.
(e) This book is either good or cheap.

Solution

Given

P: This book is good
Q: This book is cheap

(a) $P \wedge Q$
(b) $\sim P \wedge Q$
(c) $\sim Q \wedge P$
(d) $\sim P \wedge \sim Q$
(e) $P \vee Q$

2. Translate the following sentences into propositional forms:

(a) If it is not raining and I have the time then I will go to a movie.
(b) It is raining and I will not go to a movie.
(c) It is not raining.
(d) I will not go to a movie.
(e) I will go to a movie only if it is not raining.

Solution

Let

P: It is raining
Q: I have the time
R: I will go to a movie

Then

(a) $(\sim P \wedge Q) \rightarrow R$
(b) $P \wedge \sim R$
(c) $\sim P$
(d) $\sim R$
(e) $R \rightarrow \sim P$

3. If P, Q, R are the propositions as given in problem 2, write the sentences in English corresponding to the following propositional forms:

(a) $(\sim P \wedge Q) \Leftrightarrow R$

(b) $(Q \rightarrow R) \wedge (R \rightarrow Q)$

(c) $\sim(Q \vee R)$

(d) $R \rightarrow \sim P \wedge Q$

Solution

(a) I will go to a movie if and only if it is not raining and I have the time.

(b) I will go to a movie if and only if I have the time.

(c) It is not the case that I have the time or I will go to a movie.

(d) I will go to a movie, only if it is not raining and I have the time.

4. Let p and q be the propositions “Swimming at the New Jersey shore is allowed” and “Sharks have been spotted near the shore,” respectively. Express each of these compound propositions as an English sentence.

a) $\neg q$

b) $p \wedge q$

c) $\neg p \vee q$

d) $p \rightarrow \neg q$

e) $\neg q \rightarrow p$

f) $\neg p \rightarrow \neg q$

g) $p \leftrightarrow \neg q$

h) $\neg p \wedge (p \vee \neg q)$

Solution

a) The proposition $\neg q$ as an English sentence is "Sharks have not been spotted near the shore"

b) The proposition $p \wedge q$ as an English sentence is "Swimming at the New Jersey shore is allowed and sharks have been spotted near the shore"

c) The proposition $\neg p \vee q$ as an English sentence is "Swimming at the New Jersey shore is not allowed or sharks have been spotted near the shore"

d) The proposition $p \rightarrow \neg q$ as an English sentence is "If swimming at the New Jersey shore is allowed then sharks have not been spotted near the shore"

e) The proposition $\neg q \rightarrow p$ as an English sentence is "If sharks have not been spotted near the shore then swimming at the New Jersey shore is allowed"

f) The proposition $\neg p \rightarrow \neg q$ as an English sentence is "If swimming at the New Jersey shore is not allowed then sharks have not been spotted near the shore"

g) The proposition $p \leftrightarrow \neg q$ as an English sentence is "Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore"

h) The proposition $\neg p \wedge (p \vee \neg q)$ as an English sentence is "Swimming at the New Jersey shore is not allowed, and swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore"

5. Translate the following English sentence into propositional statement.

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

Solution

The above statement could be considered as a single proposition but it would be more useful to break it down into simpler propositions. That would make it easier to analyze its meaning and to reason with it.

The above sentence could be broken down into three propositions,

P : "You can access the Internet from campus."

Q : "You are a computer science major."

R : "You are a freshman."

Using logical connectives we can join the above-mentioned propositions to get a logical expression of the given statement.

$$P \rightarrow (Q \lor \sim R)$$

Problems

Qn1. Express the following statements in Propositional Logic.

a) If he campaigns hard, he will be elected.

b) If the humidity is high, it will rain either today or tomorrow.

c) Cancer will not be cured unless its cause is determined and a new drug for cancer is found.

d) It requires courage and skills to climb a mountain.

Qn2. Let

P : He needs a doctor, Q : He needs a lawyer,

R : He has an accident, S : He is sick,

U : He is injured.

State the following formulas in English.

a) $(S \rightarrow P) \land (R \rightarrow Q)$ b) $P \rightarrow (S \lor U)$

c) $(P \land Q) \rightarrow R$ d) $(P \land Q) \leftrightarrow (S \land U)$

TRUTH TABLES

Truth table a diagram in rows and columns showing how the truth or false of a proposition varies with that of its components. A truth table has one column for each input variable (for example, P and Q), and one final column showing all of the possible results of the logical operation that the table represents (for example, P Ʌ Q). Each row of the truth table contains one possible combination of the input variables (for instance, P=true Q=false), and the result of the operation for those values.

Since each input variable has two possible values (True or False), a truth table will have 2^n rows, where n is the number of input variables.

Example 1:

Construct a truth table for the formula ~P Ʌ (P →Q).

P	Q	$\sim P$	$P \to Q$	$\sim P \wedge (P \to Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 2:

Construct a truth table for (P→Q) Ʌ (Q→R)

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Example 3:

The truth table for the propositional form ((¬ P) ∨ Q) Ʌ (¬ R) is

P	Q	R	$(\neg P)$	$(\neg P) \vee Q$	$(\neg R)$	$((\neg P) \vee Q) \wedge (\neg R)$
T	T	T	F	T	F	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	T	T	T	T

Problems

1. Obtain the truth table for $(P \vee Q)$ A $(P \rightarrow Q)$ A $(Q \rightarrow P)$.

2. Construct the truth table for $(P \vee Q) \rightarrow ((P \vee R) \rightarrow (R \vee Q))$.

WELL-FORMED FORMULAS

A well-formed formula (wff) is defined recursively as follows:

(i) If P is a propositional variable, then it is a wff.

(ii) If A is a wff, then ~A is a wff.

(iii) If A and B are well-formed formulas, then (AAB), $(A \vee B)$, $(A \rightarrow B)$, $(A \Leftrightarrow B)$ are well-formed formulas.

(iv) A string of symbols is a wff if and only if it is obtained by a finite number of applications of (i)-(iii).

Examples

Here are some examples of well-formed formulas, along with brief explanations how these formulas are formed in accordance with the three rules of syntax:

Well-formed formula	Explanation
1. A	by rule 1
2. ~A	by rule 2, since A is a wff
3. ~ ~A	by rule 2 again, since ~A is a wff
4. (~A ∧ B)	by rule 3, joining ~A and B
5. ((~A ∧ B) ∨ ~ ~C)	by rule 3, joining (~A ∧ B) and ~ ~C
6. ~ ((~A ∧ B) ∨ ~ ~C)	by rule 2, since ((~A ∧ B) ∨ ~ ~C) is a wff

In contrast, here are a few formulas that are not well-formed:

Non-wff	Explanation
1. A~	the ~ belongs on the left side of the negated proposition
2. (A)	( ) are only introduced when joining two wffs with ∧, ∨, →, or ⇔
3. (A ∧)	there's no wff on the right side of the ∧
4. (A ∧ B) → C)	missing parenthesis on the left side
5. (A ∧ → B)	cannot be formed by the rules of syntax

TAUTOLOGY, CONTRADICTION, AND CONTINGENCY

Tautology

A Tautology is a formula which is always true for every value of its propositional variables. To check whether a given logic is a tautology or not, you can use truth table method or equivalence rules. A proposition P is a tautology if and only if ~P is a contradiction.

Example

1. Show that P ∨ (¬ P) is a tautology.

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

The last column contains only T's. Therefore, P ∨ (¬ P) is a tautology.

2. Show that (P → Q) V (Q →P) is a tautology.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last column contains only T's. Therefore, the formula is a tautology.

3. Show that (P → Q) V (Q →P) is a tautology using equivalence rules.

Contradiction

A Contradiction is a formula which is always false for every value of its propositional variables. To check whether a given logic is a contradiction or not, you can use truth table method or equivalence rules. A proposition P is a contradiction if and only if ~P is a tautology.

Example

1. Show that P A (¬ P) is a contradiction

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

The last column contains only F's. P A (¬ P) is a contradiction

2. Show that ¬ (P → Q) A ¬ (Q →P) is a contradiction by using truth table.

3. Show that ¬ (P → Q) A ¬ (Q →P) is a contradiction by using equivalence rules.

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables. A statement is contingent if it is neither tautologous nor self-contradictory. In other words, it is logically possible for the statement to be true and it is also logically possible for it to be false.

Example

1. Show that P V Q is a contingency.

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Here P V Q has both some true and some false values. Therefore, P V Q is a contingency.

2. Show that ~P A (P →Q)is a contingency.

Valid Formula

A formula is said to be **valid** if and only if it is true under all its interpretations. A formula is said to be **invalid** if and only if it is not true under at least one interpretation. A valid formula is also called a Tautology.

Example

Inconsistent (Unsatisfiable) Formula

A formula is said to be **inconsistent (or unsatisfiable)** if and only if it is False under all its interpretations. A formula is said to be **consistent or satisfiable** if and only if it is not inconsistent.

Example

Problems

1. Show that $(P \lor Q) \lor ((\neg P) \land (\neg Q))$ is a tautology.

2. Show that $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ is a tautology.

3. Show that $(P \land Q) \land ((\neg P) \lor (\neg Q))$ is a contradiction.

4. Show that $(P \land Q) \lor (\sim P \land \sim Q)$ is a contingency.

5. For each of the following formulas, determine whether it is valid, inconsistent, consistent or some combination of these.

(i) ~ (~A) → B

(ii) (A → B) → (~ B → ~ A)

(iii) (A ∨ ~ A) → (A A B) A (~ A)

(iv) (A A B) A (~ A) → (B ∨ ~ B)

EQUIVALENCE OF FORMULAS (LOGICAL EQUIVALENCES)

Two statements A and B are logically equivalent if any of the following two conditions hold.

1. The truth tables of each statement have the same truth values.
2. The bi-conditional statement A⇔B is a tautology.

When A and B are equivalent, we write A≡B.

Example

1. Show that ¬ (PΛQ) and (¬ P) ∨ (¬ Q) are logically equivalent.

p	q	$p \wedge q$	$\neg (p \wedge q)$	$(\neg p)$	$(\neg q)$	$(\neg p) \vee (\neg q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Here truth values of both ¬ (PΛQ) and (¬ P) ∨ (¬ Q) are same. So they are equivalent.

Therefore, ¬ (P Λ Q) ≡ (¬ P) ∨ (¬ Q).

Some basic established logical equivalences are tabulated below-

Equivalence	Name
$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}, \quad p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$	Complement laws

Logical Equivalences involving Conditional Statements

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences involving Bi-conditional Statements

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Examples

1. Show the following propositions are logical equivalent using equivalent rules.

$\neg(P \lor (\neg P \land Q))$ and $\neg P \land \neg Q$.

Solution

$\neg(P \lor (\neg P \land Q))$	$\equiv \neg P \land \neg(\neg P \land Q)$	De-Morgan's law
	$\equiv \neg P \land (P \lor \neg Q)$	De-Morgan's law
	$\equiv (\neg P \land P) \lor (\neg P \land \neg Q)$	Distributive law
	$\equiv F \lor (\neg P \land \neg Q)$	Because $\neg P \land P \equiv F$
	$\equiv \neg P \land \neg Q$	Because $F \lor P \equiv P$ for any P
Therefore $\neg (P \lor (\neg P \land Q))$	$\equiv \neg P \land \neg Q$	

2. Show the following proposition is a tautology without using truth tables.

$(P \land Q) \rightarrow (P \lor Q)$

Solution

$(P \land Q) \rightarrow (P \lor Q)$	$\equiv \neg (P \land Q) \lor (P \lor Q)$	Substitution for $\rightarrow$
	$\equiv (\neg P \lor \neg Q) \lor (P \lor Q)$	De-Morgan's law
	$\equiv (\neg P \lor P) \lor (\neg Q \lor Q)$	Commutative and Associative
	$\equiv T \lor T$	Because $\neg P \lor P \equiv T$
	$\equiv T$	

Since $(P \land Q) \rightarrow (P \lor Q) \equiv T$, $(P \land Q) \rightarrow (P \lor Q)$ is a tautology.

Problems

1. Prove that: $[\neg P \land (P \lor Q)] \rightarrow Q$ is a tautology.
   - By using truth table.
   - By using logic equivalence laws.

2. Prove that: $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow [P \rightarrow R]$ is a tautology.

- By using truth table.
- By using logic equivalence laws.

3. Prove De-Morgan’s law using truth tables.

4. Show that (P AQ) V (P A ~ Q) ≡ P by using logic equivalence laws.

5. Prove Distributive law using truth tables.

6. Show that (P → (Q V R) ≡ ((P →Q) V (P →R) using truth table.

7. Show that (P → Q) A (R → Q) ≡ (P V R) ~ Q by using logic equivalence laws.

8. Check whether the following statements are equivalent or not.

(i) (A→ B) → (A A B) ≡ (~ A → B) A (B → A)

(ii) A A B A (~A ∨ ~ B) ≡~ A A ~ B A (A ∨ B)

Inverse, Converse, and Contra-positive of a Conditional Statement

The conditional statement, $P \rightarrow Q$ has two parts -

Hypothesis, P

Conclusion, Q

Example:

$P \rightarrow Q$: "If you eat health food, you will not be sick."

For this statement, hypothesis is 'you eat health food' and conclusion is 'you will not be sick'.

Inverse

The negation of both hypothesis and conclusion is known as Inverse of the conditional statement. If the statement is "If P, then Q", the inverse will be "If not P, then not Q". Thus the inverse of $P \rightarrow Q$ is $\mathbf{\sim P \rightarrow \sim Q}$.

Example:

The inverse for the above example is "If you do not eat health food, you will be sick."

Converse

The interchange of hypothesis and conclusion is known as Converse of the conditional statement. If the statement is "If P, then Q", the converse will be "If Q, then P". The converse of $P \rightarrow Q$ is $\mathbf{Q \rightarrow P}$.

Example:-

The converse for the example is "You will not be sick only if you eat health food".

Contra-positive

The interchange of hypothesis and conclusion of inverse statement is Contra-positive. If the statement is "If p, then q", the contra-positive will be "If not q, then not p". The contra-positive of $P \rightarrow Q$ is $\mathbf{\sim Q \rightarrow \sim P}$.

Example:-

The Contra-positive for the example is "You will be sick only if you do not eat health food".

Truth table for Inverse, Converse, and Contra-positive of a Conditional Statement

P	**Q**	**~P**	**~Q**	**P → Q**	**~ P → ~Q**	**Q → P**	**~Q → ~P**
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

From this truth table it is clear that

1) a conditional statement has the same truth value as its contrapositive. So, these two statements are logically equivalent.

   $P \rightarrow Q \equiv \sim Q \rightarrow \sim P$

2) inverse of a conditional statement has the same truth value as its converse. So, these two statements are logically equivalent.

   $\sim P \rightarrow \sim Q \equiv Q \rightarrow P$

TAUTOLOGICAL IMPLICATION (Rules of Inference)

Tautological implication describes a logical relationship between two propositions in such a way that if the first statement is true, the second statement must also be true, without any exceptions or conditions. This relationship is often denoted using symbols like "⇒" (double arrow). If statement A tautologically implies statement B, it is typically written as: A ⇒ B.

This relationship is sometimes referred to as a "tautology" because it always leads to a true conclusion when the premise is true. If P, Q, and R are propositional statements, then the most commonly used Rules of Inference are given below:

1. Modus Ponens

[(P→Q) AP] ⇒Q

In words:

If P implies Q, and if P is true, then Q must be true.

In symbols:

P→Q

P

∴ Q

Example

If I love math, then I will pass this course.

I love math

Therefore, I will pass this course.

2) Modus Tollens

[(P→Q) A ~Q] ⇒ ~P

In words:

If P implies Q, and if Q is false, then P is also false.

In symbols:

P→Q

~Q

∴ ~P

Example

If I love math, then I will pass this course.

I will fail the course

Therefore, I do not love math.

3) Hypothetical Syllogism (Chain Rule or Transitivity)

[(P→Q) A (Q→R)] ⇒ P→R

In words:

If P implies Q, and if Q implies R, then P implies R.

In symbols:

P→Q

Q→R

∴ P→R

Example

if it rains then the ground gets muddy.

if the ground is muddy then my shoes get dirty.

Therefore, if it rains then my shoes get dirty.

4) Simplification

$P \wedge Q \Rightarrow P$

and also

$P \wedge Q \Rightarrow Q$

In words, the first says: If both P and Q are true, then, in particular, P is true.

In symbols:

$P \wedge Q$

$\therefore P$

Example

The sky is blue and the moon is round.

Therefore, the sky is blue.

5) Addition

$P \Rightarrow P \vee Q$

and also,

$Q \Rightarrow P \vee Q$

In words, the first says: If P is true, then we know that either P or Q is true.

In symbols:

P

$\therefore P \vee Q$

Example

The sky is blue.

Therefore, the sky is blue or the moon is round.

6) Disjunctive Syllogism

$[(P \vee Q) \wedge (\sim P)] \Rightarrow Q$

and also,

$[(P \vee Q) \wedge (\sim Q)] \Rightarrow P$

Example

The sky is blue or the moon is round

The sky is not blue

Therefore, the moon is round.

NORMAL FORMS

A method of reducing a given formula to an equivalent form called the 'normal form'. A clause is a disjunction of literals. For example, $(E \vee \sim F \vee \sim G)$ is a clause. But $(E \vee \sim F \wedge \sim G)$ is not a clause. A literal is either an atom, say A, or its negation, say ~ A.

Every propositional statement (truth table) can be written as:

1. Disjunctive Normal Form (DNF)
2. Conjunctive Normal Form (CNF)

Disjunctive Normal Form (DNF)/ Sum of Products Form

A formula E is said to be in Disjunctive Normal Form (DNF) if and only if E has the form $E: E1 \vee E2 \vee \ldots \vee En$, where each Ei is a conjunction of literals. For example, $P \vee (Q \wedge R)$

and P ∨ (~ Q ∧ R) are in disjunctive normal form. P ∧ (Q ∨ R), (~ A ∨ B) ∨ (A ∧ ~ B ∨ C) are not in disjunctive normal form.

The steps for conversion to DNF are as follows

Step 1: Use the equivalences to remove the logical operators '↔' and '→': (i) E ↔ G = (E → G) ∧ (G → E) (ii) E → G = ~ E ∨ G Step 2: Remove ~'s, if occur consecutively more than once, using (iii) ~ (~E) = E (iv) Use De Morgan's laws to take '~' nearest to atoms ~(E ∨ G) = ~ E ∧ ~ G ~(E ∧ G) = ~ E ∨ ~ G Step 3: Use the distributive laws repeatedly (G ∨ H) ∧ E = (G ∧ E) ∨ (H ∧ E) E ∧ (G ∨ H) = (E ∧ G) ∨ (E ∧ H)

Example

1. Obtain a disjunctive normal form for the formula ~ (A → (~ B ∧ C)).

Consider A → (~B ∧ C)	≡ ~ A ∨ (~B ∧ C)	(Using Implication law)
Hence, ~ (A → (~ B ∧ C))	≡ ~ (~ A ∨ (~B ∧ C))	
	≡ ~ (~ A) ∧ (~ (~ B ∧ C))	(Using De-Morgan's law)
	≡A ∧ (~ (~ B ∧ C))	(Using Double negation law)
	≡A ∧ (B ∨ ~ C)	(Using De-Morgan's law)
	≡ (A ∧ B) ∨ (A ∧ ~C)	(Using distributive law)

Minterms

A minterm is a conjunction of literals in which each variable is represented exactly once. If a Boolean function (truth table) has the variables(A,B,C) then A ∧ ¬B ∧ C is a minterm but A∧¬B is not.

Each minterm is true for exactly one assignment. A ∧ ¬B ∧ C is true if A is True, B is False and C is True. Any deviation from this assignment would make this particular minterm false.

A disjunction of minterms is true only if at least one of its constituents minterms is true. The truth table for minterms for three variables A, B, and C.

A	B	C	Minterm	Designation
0	0	0	A'B'C'	m_0
0	0	1	A'B'C	m_1
0	1	0	A'BC'	m_2
0	1	1	A'BC	m_3
1	0	0	AB'C'	m_4
1	0	1	AB'C	m_5
1	1	0	ABC'	m_6
1	1	1	ABC	m_7

Principal Disjunctive Normal Form (PDNF)/ Sum of Products Normal Form

For a given formula, an equivalent formula consisting of disjunctions of minterms only is called PDNF. Minterms of P and Q are $P \wedge Q$, $\sim P \wedge Q$, $P \wedge \sim Q$, $\sim P \wedge \sim Q$; i.e. each variable occurs either complemented or uncomplemented but not both occur together in the conjunction.

The steps for conversion to PDNF are as follows

Step 1: Obtain a Disjunctive Normal Form.
Step 2: Drop the elementary products which are contradictions such as $(P \wedge \sim P)$
Step 3: If P or ~ P are missing in an elementary product Ei replace Ei by $(Ei \wedge P) \vee (Ei \wedge \sim P)$
Step 4: Repeat step 3 until all the elementary products are reduced to sum of minterms.
Use the idempotent laws to avoid repetition of minterms.

Example

1. Obtain PDNF for $P \rightarrow ((P \rightarrow Q \wedge \sim (\sim Q \vee \sim P)))$.

Solution:

$P \rightarrow ((P \rightarrow Q \wedge \sim (\sim Q \vee \sim P)))$

$\equiv P \rightarrow ((P \rightarrow Q \wedge (P \wedge Q)))$ (Using De-Morgan's law)

$\equiv P \rightarrow ((P \rightarrow P \wedge Q))$ (Using $Q \wedge Q = Q$)

$\equiv P \rightarrow (\sim P \vee (P \wedge Q))$ (Using Implication law)

$\equiv \sim P \vee (\sim P \vee (P \wedge Q))$ (Using Implication law)

$\equiv \sim P \vee (P \wedge Q)$ (Using associative law)

$\equiv (\sim P \wedge (Q \vee \sim Q)) \vee (P \wedge Q)$ (Using $Q \vee \sim Q \equiv T$)

$\equiv (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) \vee (P \wedge Q)$ (Using distributive law)

$\equiv (\sim P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (P \wedge Q)$

$\equiv \Sigma (m_0, m_1, m_3)$

2. Find the PDNF of $P \rightarrow Q$.

Solution:

$P \rightarrow Q \equiv \sim P \vee Q$ [Using implication rule]

$\equiv (\sim P \wedge (Q \vee \sim Q)) \vee ((P \vee \sim P) \wedge Q)$

$\equiv (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)) \vee (P \wedge Q) \vee (\sim P \wedge Q)$ [Using Distributive law]

$\equiv (\sim P \wedge \sim Q)) \vee(\sim P \wedge Q) \vee (P \wedge Q)$ [Using Idempotent law]

$\equiv \sum (m_0, m_1, m_3)$

Conjunctive Normal Form (CNF) / Product of Sums Form

A formula E is said to be in Conjunctive Normal Form (CNF) if and only if E has the form E: E1 ∧ E2 ∧…. ∧En, where each Ei is a disjunction of literals. For example, P ∧ (Q ∨ R) and P ∧ (~Q ∨ R) are in disjunctive normal form. P ∨ (Q ∧ R), (~ A ∧ B) ∨ (A ∧ ~ B ∨ C) are not in conjunctive normal form.

The steps for conversion to CNF are as follows

Step 1: Use the equivalences to remove the logical operators '↔' and '→': (i) E ↔ G = (E → G) ∧ (G → E) (ii) E → G = ~ E ∨ G Step 2: Remove ~'s, if occur consecutively more than once, using (i) ~ (~E) = E (ii)Use De Morgan's laws to take '~' nearest to atoms ~(E ∨ G) = ~ E ∧ ~ G ~(E ∧ G) = ~ E ∨ ~ G Step 3: Use the distributive laws repeatedly (vii) E ∨ (G ∧ H) = (E ∨ G) ∧ (E ∨ H) (viii) (G ∧ H) ∨ E = (G ∨ E) ∧ (H ∨ E)

Example:

Obtain conjunctive Normal Form (CNF) for the formula: D → (A → (B ∧ C))

Consider

$D \rightarrow (A\rightarrow (B \wedge C))$

$\equiv D \rightarrow (\sim A \vee (B \wedge C))$ (using E → F = ~ E ∨ F for the inner implication)

$\equiv \sim D \vee (\sim A \vee (B \wedge C))$ (using E → F = ~ E ∨ F for the outer implication)

$\equiv (\sim D \vee \sim A) \vee (B \wedge C)$ (using associative law)

$\equiv ((\sim D \vee \sim A \vee B) \wedge (\sim D \vee \sim A \vee C)$ (using disjunctive law)

Maxterm

A maxterm is a disjunction of literals in which each variable is represented exactly once. If a Boolean function (truth table) has the variables(A,B,C) then A V ¬B V C is a maxterm but AV¬B is not.

The truth table for maxterm s for three variables A, B, and C.

A	B	C	Maxterm	Designation
0	0	0	A+B+C	M_0
0	0	1	A+B+C'	M_1
0	1	0	A+B'+C	M_2
0	1	1	A+B'+C'	M_3
1	0	0	A'+B+C	M_4
1	0	1	A'+B+C'	M_5
1	1	0	A'+B'+C	M_6
1	1	1	A'+B'+C'	M_7

Principal Conjunctive Normal Form (PCNF)/ Product of Sums Normal Form

For a given formula, an equivalent formula consisting of conjunctions of maxterms only is called PCNF. Maxterms of P and Q are $P \vee Q$, $\sim P \vee Q$, $P \vee \sim Q$, $\sim P \vee \sim Q$; i.e. each variable occurs either complemented or uncomplemented but not both occur together in the disjunction.

The steps for conversion to PCNF are as follows

Step 1: Obtain a Conjunctive Normal Form.
Step 2: Drop the elementary sums which are contradictions such as $(P \vee \sim P)$
Step 3: If P and ~P are missing in an elementary sum Ei replace Ei by $(Ei \vee P) \wedge (Ei \vee \sim P)$
Step 4: Repeat step 3 until all the elementary sums are reduced to product of maxterms. Use the idempotent laws to avoid repetition of maxterms.

Example

1. Find the PCNF of $\sim (P \Leftrightarrow Q)$

$\equiv \sim ((P \rightarrow Q) \wedge (Q \rightarrow P))$	[Using double implication rule]
$\equiv \sim ((\sim P \vee Q) \wedge (\sim Q \vee P))$	[Using implication rule]
$\equiv \sim (\sim P \vee Q) \vee \sim (\sim Q \vee P)$	[Using De-Morgan's law]
$\equiv (P \wedge \sim Q) \vee (Q \wedge \sim P)$	[Using De-Morgan's law]
$\equiv (P \vee Q) \wedge (P \vee \sim P) \wedge (Q \vee \sim Q) \wedge (\sim P \vee \sim Q)$	[Using Distributive law]
$\equiv (P \vee Q) \wedge (\sim P \vee \sim Q)$	[Drop $(Q \vee \sim Q)$ and $(P \vee \sim P)$]
$\equiv \pi\,(M_0, M_3)$	

Problems

1. Transform the following into disjunctive normal forms.
   (i) $\sim (A \vee \sim B) \wedge (C \rightarrow D)$
   (ii) $(A \rightarrow B) \rightarrow C$

2. Transform the following into conjunctive normal forms.

(i) $(A \rightarrow B) \rightarrow C$
(ii) $(\sim A \wedge B) \vee (A \wedge \sim B)$

3. Obtain a PDNF of

i. $P \vee (\sim P \rightarrow (Q \vee (Q \rightarrow \sim R)))$
ii. $(P \wedge \sim (Q \wedge R)) \vee (P \rightarrow Q)$

4. Obtain the PCNF of

  i. (~P→ R) A (Q → P).
  ii. (~P→ R) A ((Q→ P) A (P→ Q)).

PREDICATE CALCULUS

INTRODUCTION

There are some kinds of human reasoning that we can't do in propositional logic. For example

- Every person likes ice cream.
- For every number there is a prime larger than that number.
- The set of all real numbers is larger than the set of all natural numbers.

The above statements cannot be adequately expressed using only propositional logic. The problem in trying to do so is that propositional logic is not expressive enough to deal with quantified variables (for all, there exist, etc.). It would have been easier if the statement were referring to a specific person or object. But since it is not the case and the statement applies to all people or object, we are stuck. Therefore, we need a more powerful type of logic. In order to overcome this limitation predicate logic were introduced. **Predicate Logic or Predicate Calculus** is an extension of propositional logic. It adds the concept of predicates and quantifiers to better capture the meaning of statements that cannot be adequately expressed by propositional logic.

PREDICATES

Consider the statement, "x is greater than 3″. It has two parts.

- The first part, the variable x, is the subject of the statement.
- The second part, "is greater than 3", is the predicate. It refers to a property that the subject of the statement can have.

The statement "x is greater than 3″ can be denoted by P(x) where P denotes the predicate "is greater than 3" and x is the variable.

The predicate P can be considered as a function. It tells the truth value of the statement P(x) at x. Once a value has been assigned to the variable x, the statement P(x) becomes a proposition and has a True or False value.

In general, a statement involving n variables x_1, x_2, x_3,.. , x_n can be denoted by $P(x_1,x_2,x_3,.. , x_n)$. Here P is also referred to as n-place predicate or a n-ary predicate.

Example 1

Let P(x) denote the statement "x > 10″. What are the truth values of P(11) and P(5)?

Solution:

P(11) is equivalent to the statement 11 > 10, which is True.

P(5) is equivalent to the statement 5 > 10, which is False.

Example 2

Let R(x,y) denote the statement "x = y + 1''. What is the truth value of the propositions R(1,3) and R(2,1)?

Solution:

R(1,3) is the statement 1 = 3 + 1, which is False.

R(2,1) is the statement 2 = 1 + 1, which is True.

STATEMENT FUNCTIONS

Let the predicate

P: is mortal

and the names

a: Ram

b: Rose

c: Shirt

then P(a), P(b) and P(c) all denote statements. These statements have a common form.

If we write

P(x): x is mortal

then P(a), P(b), and P(c) can be obtained from P(x) by replacing x by an appropriate name. P(x) is not a statement but it results in a statement when x is replaced by the name of an object.

A simple **statement function** of one variable is defined to be an expression consisting of a predicate symbol and an individual variable. This statement function becomes statement when the variable is replaced by any object. The replacement is called a *substitution instance* of the statement function.

Let

P(x): x is mortal

M(x): x is a man

then we can form **compound statement functions** such as

$P(x) \wedge M(x)$, $P(x) \vee M(x)$, $M(x) \to P(x)$, $\sim P(x)$, $M(x) \vee \sim P(x)$, etc.

Consider the statement function of two variables:

H(x, y) : x is taller than y.

If both x and y are replaced by the name of objects, we get a statement. If i represents Ramu and j represents Binu, then we have,

H(i, j) : Ramu is taller than Binu and

H(j, i) : Binu is taller than Ramu.

VARIABLES AND QUANTIFIERS

Quantifiers are expressions that indicate the scope of the term to which they are attached. The variable of predicates is quantified by quantifiers. There are two types of quantifiers in predicate logic:

1. Universal Quantifier
2. Existential Quantifier

Universal Quantifier

Mathematical statements sometimes assert that a property is true for all the values of a variable in a particular domain, called the domain of discourse. Such a statement is expressed using universal quantification.

Formally, the universal quantification of P(x) is the statement "P(x) for all values of x in the domain". The notation $\forall x P(x)$ denotes the universal quantification of P(x). Here $\forall$ is called the universal quantifier. $\forall x P(x)$ is read as "for all x P(x)". The universal quantification of P(x) for a particular domain is the proposition that asserts that P(x) is true for all values of x in this domain.

Example 1

Let

M(x): x is matter

then the expression

✯xM(x)

means “for every x, x is matter”, or simply, “everything is matter”.

Example 2

Symbolize the statement “everything is matter or energy” in predicate logic.

Let

M(x): x is matter

E(x): x is energy

Then the statement can be written using universal quantifier as

✯x(M(x) ∨ E(x))

Example 3

Let P(x) be the statement “x + 2 > x”. What is the truth value of the statement ✯xP(x)?

Solution:

As x+2 is greater than x for any real number, so P(x) ≡ **T** for all x or ✯xP(x) ≡ **T**

Remarks

The common logical phrase “All P are Q” may be formulated using a universal quantifier and a conditional statement:

✯x(P(x) → Q()x))

This reads as “For every x, if x is a P, then x is a Q”.

Existential Quantifier

Some mathematical statements assert that there is an element with a certain property. Such statements are expressed by existential quantification.

Formally, the existential quantification of P(x) is the statement "There exists an element x in the domain such that P(x)". The notation ∃xP(x) denotes the existential quantification of P(x). Here ∃ is called the existential quantifier. ∃xP(x) is read as "There is at least one such x such that P(x)". Existential quantification can be used to form a proposition that is true if and only if P(x) is true for at least one value of x in the domain.

Example 1

Let

B(x) : x is a ball

Then we may formalize an expression using a existential quantifier:

∃x B(x)

maens “there is an x such that x is a ball”, or more simply, “there is a ball”.

Example 2

Let P(x) be the statement “x > 5″. What is the truth value of the statement ∃xP(x) ?

Solution:

P(x) is true for all real numbers greater than 5 and false for all real numbers less than 5. So ∃xP(x) ≡ **T**.

Remarks

The expression “Some P are Q” can be formulized using an existential quantifier and a conjunction:

∃x(Px A Qx)

This reads as “There exists an x such that x is a P and x is a Q”, which may be literally as “at least one P is a Q”, or more generally, “some P are Q”.

Truth Value of Quantified Statements

Statement	**True**	**False**
∀xP(x)	P(x) is true for every x.	There is one x for which P(x) is false.
∃xP(x)	There is one x for which P(x) is true.	P(x) is false for every x.

Nested Quantifiers (Multiple/Compound Quantifiers)

Two quantifiers are nested if one is within the scope of the other.

Example

Symbolize the statement “Every rabbit is faster than some tortoise.”

Let P(x, y): Rabbit x is faster than tortoise y

Then the statement can be represented in predicate logic as

✯x∃y P(x, y)

Negation of Quantified Statements

The negation of a statement with predicates and quantifiers are carried out as follows.

~∀xP(x) ≡ ∃x~P(x)
~∃xP(x) ≡ ∀x~P(x)

Examples

Negate each of the following propositions:

1.✯x P(x)A ∃ y Q(y)

Sol:

~ (✯x P(x) A ∃yQ(y))

≡~✯x P(x) ∨ ~∃yQ(y)   (∴~(PAQ) ≡~P∨~Q)

≡ ∃x ~P(x) ∨ ✯y~Q(y)2.

∃x(x+6=25)

Sol:

~∃x(x+6 =25)

≡ ✯x~((x+6) =25)

≡ (✯x) (x+6) ≠25

3. ∃x P(x) ∨ ✯yQ(y)

Sol:

~ (∃x P(x) ∨ ✯yQ(y))

≡ ~∃x P(x) A ~✯y Q(y)   (∴~ (P∨Q) = ~PA~Q)

≡ ✯x~P(x) A ∃y~Q(y))

Problems

Symbolize the following sentences using predicates and quantifiers.

1. All men are mortal.

Let

M(x) : x is a man.

H(x) : x is mortal.

Then

✯x(M(x) → H(x))

2. Every apple is red.

Let

A(x) : x is an apple.

R(x) : x is red.

Then

✯x(A(x) → R(x))

3. Any integer is either positive or negative.

Let

I(x) : x is an integer.

P(x) : x is positive integer.

N(x) : x is negative integer.

Then

✯x(I(x) → (P(x)VN(x))

4. There exists a man.

Let

M(x) : x is a man.

Then

∃x M(x)

5. Some men are clever.

Let

M(x) : x is a man.

C(x) : x is clever.

Then

∃ x (M(x) Ʌ C(x))

6. Some real numbers are rational.

Let

R1(x) : x is a real number.

R2(x) : x is rational.

Then

∃x(R1(x)Ʌ R2(x))

PREDICATE FORMULAS

Consider n-place predicate formulas

P(x1,x2,…,xn)

The letter P is an n-place predicate and x1,x2, … , xn are individual variables.

In general P(x1,x2,…,xn) is called an **atomic formula of predicate calculus**. Some examples of atomic formulas:

P, Q(x), R(x, y), A(x, y, z), B(a, y), C(x, a, z).

A **well-formed formula of predicate calculus** is obtained by using the following rules.

1. An atomic formula is a well-formed formula.
2. If A is a wff, then ~A is a wff.
3. If A and B are wffs, then (A ∧ B), (A ∨ B), (A → B), and (A⇔B) are also wffs.
4. If A is a wff and x is any variable, then (✯x)A and (∃x)A are wffs.
5. Only those formulas obtained by using rules (1) to (4) are wffs.

FREE & BOUND VARIABLES

Variables play two different roles in predicate logic. Given a formula containing a part of the form (✯x)P(x) or (∃x)P(x), such a part is called an **x-bound part of the formula**. Any occurrence of x in an x-bound part of a formula is called a **bound occurrence of x**, while any occurrence of x, or of any variable, that is not bound is called a **free occurrence**.

The formula P(x) either in (✯x) P(x) or in (∃x)P(x) is described as the scope of the quantifier. If the scope is an atomic formula, then no parentheses are used to enclose the formula.

Examples

1. (✯x)P(x,y).

Here P(x, y) is the scope of the quantifiers, and occurrence of x is bound occurrence, while the occurrence of y is a free occurrence.

2. (✯x) (P(x) → Q(x)).

Here the scope of the universal quantifier is P(x) → Q(x), and all occurrences of x are bound.

3. (✯x)(P(x)→ (∃y)R(x, y)).

Here the scope of (✯x) is P(x) → (∃y)R(x, y), while the scope of (∃y) is R(x, y). All occurrences of both x and y are bound occurrences.

4. (✯x)(P(x)→ R(x)) ∨ (✯x)(P(x)→ Q(x)).

Here the scope of the first quantifier is P(x) → R(x), and the scope of the second is P(x)→Q(x). All occurrences of x are bound occurrences.

5. (∃x)((P(x)∧Q(x)).

Here the scope of (∃x) is P(x)∧Q(x).

6. (∃x)P(x) ∧ Q(x).

Here the scope of (∃x) is P(x), and the last occurrence of x in Q(x) is free.

UNIVERSE OF DISCOURSE (DOMAIN OF DISCOURSE/UNIVERSE)

The domain of a predicate variable is the collection of all possible values that the variable may take. When we symbolize a statement some simplification can be introduced by limiting the class of individuals or objects. The limitation means that the variables which are quantified stand for only those objects which are members of a particular set or class. Such a restricted class is called the universe of discourse or the domain of individuals or simply the universe. If it refers to human beings only, then the universe of discourse is the class of human beings.

Examples

Translate the following statement into logical expression.

1. If a person is a student and is computer science major, then this person takes a course in mathematics.

Let

S(x): x is a student.

C(x): x is a computer science major.

T(x,y): x takes a course y.

Domain of x: all people

Domain of y: all courses in mathematics

Translate the sentence into logical expression as

✯x ((S(x) Ʌ C(x)) → ∃yT(x,y))

2. Everyone has exactly one best friend.

Let

B(x,y): y is the best friend of x.

Domain of x, y and z: all people

Express the English statement using variable and individual propositional function as:

“For all x, there is y who is the best friend of x and for every person z, if person z is not person y, then z is not the best friend of x”.

Translate the sentence into logical expression as:

✯x∃y (B(x,y) Ʌ ✯z ((z ≠ y) ¬B(x,z))

3. “Everyone has exactly one best friend. ”

Let

B(x,y): y is the best friend of x.

Domain of x, y and z: all people

Express the English statement using variable and individual propositional function as

“For all x, there is y who is the best friend of x and for every person z, if person z is not person y, then z is not the best friend of x”.

Translate the sentence into logical expression as:

✯x∃y✯z ((B(x,y) Ʌ B(x,z)) → (y = z))

4. There is a person who has taken a flight on every airline in the world.

Let

F(x,f): x has taken flight f.

A(f,a): flight f is on airline a.

Domain of x: all people

Domain of f: all flights

Domain of a: all airlines

Translate the sentence into logical expression as:

∃x✯a∃f (F(x,f) Ʌ A(f,a))

OR

Let

R(x,f,a): x has taken flight f on airline a.

Domain of x: all people

Domain of f: all flights

Domain of a: all airlines

Translate the sentence into logical expression as:

∃x✯a∃f R(x,f,a)

5. Let

P(x) : x is person.

F(x, y) : x is the father of y.

M(x, y) : x is the mother of y.

Write the predicate " x is the father of the mother of y".

Solution:

In order to symbolize the predicate, we name a person called z as the mother of y. That is we want to say that x is the father of z and z the mother of y. It is assumed that such a person z exists.

We symbolize the predicate as $(\exists z)(P(z) \land F(x, z) \land M(z, y))$.

6. Symbolize the expression. "All the world loves a lover".

Solution:

It means that everybody loves a lover.

Let P(x): x is person.

L(x): x is a lover.

R(x. y): x loves y.

The required expression is (✯x)(P(x) → (✯y)(P(y) ∧ L(y) → R(x,y))).

Problems

Express the following statements as logical expressions

1. If somebody is a female and is a parent, then this person is someone’s mother.
2. Some student in this class has visited Mexico.
3. Every student in this class has visited either Canada or Mexico.
4. All lions are fierce.
5. No large birds live on honey.