

# **Intro to Deep Learning**

## **67822**

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Lecture Notes

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Based on the lectures given by Raanan Fattal, and the recitations given by Matan Halfon.

Quick note before we start - the course does not intend to be formal, and nor does this summary. Having said that, I may elaborate on some topics I find important, and those cases will be marked with a (†) sign.

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# Contents

<b>1</b>	<b>Intro to Information Theory</b>	<b>5</b>
1.1	Entropy and Cross-Entropy	5
1.1.1	minimizing the CE = maximizing the Likelihood	6
<b>2</b>	<b>Neural Network Model</b>	<b>7</b>
2.1	Perceptron	7
2.2	Multi Layer Perceptron (MLP)	8
2.3	Activation Function	9
2.3.1	Types	9
2.4	Loss Function	10
2.4.1	Regression Loss	10
2.4.2	Classification Loss	10
2.5	Network Training	11
2.5.1	Gradient Descent Optimization	11
2.5.2	Mini Batching and SGD	12
2.5.3	Back-propagation	12
<b>3</b>	<b>Deep Neural Network Theory</b>	<b>15</b>
3.1	Universal Approximation Theorem	15
3.2	Shallow compared to Deep NN	16
3.3	Training a NN is NP-Complete	17

<b>4</b>	<b>Time/Translation Invariance .....</b>	<b>19</b>
<b>4.1</b>	<b>Notion of Time-Space</b>	<b>19</b>
4.1.1	Translation/Time Invariant Systems .....	19
4.1.2	Linear Translation/Time Invariant Systems (LTI) .....	19
<b>4.2</b>	<b>Convolutional NN</b>	<b>21</b>
4.2.1	the outline .....	21
4.2.2	pooling layer and strides .....	21
4.2.3	the architecture .....	22
<b>4.3</b>	<b>Other Invariances</b>	<b>22</b>
<b>5</b>	<b>Recurrent Neural Networks (RNN) .....</b>	<b>25</b>
<b>5.1</b>	<b>Sequence signals</b>	<b>25</b>
<b>5.2</b>	<b>Recurrent NN</b>	<b>25</b>
5.2.1	RNN architectures .....	26
<b>5.3</b>	<b>Back Propagation Through Time (BPTT)</b>	<b>26</b>
<b>5.4</b>	<b>Vanishing/Exploding gradient problem</b>	<b>27</b>
5.4.1	First solution: Long-Short-Term-Memory (LSTM) .....	28
5.4.2	Second solution: Gated Recurrent Unit (GRU) .....	30
<b>6</b>	<b>Attention Layers .....</b>	<b>31</b>
<b>6.1</b>	<b>The problem with encoder-decoder models</b>	<b>31</b>
<b>6.2</b>	<b>The solution - Attention Layer</b>	<b>32</b>
6.2.1	Image Captioning .....	32
<b>6.3</b>	<b>Neural Machine Translation</b>	<b>34</b>

# 1. Intro to Information Theory

Suppose we are given with data points  $x \in X$ , which are sampled from some unknown probability space  $P$ . One might ask - how would I find a somewhat similar probability space  $Q$ , that represents  $P$  well enough? in other words, what metric should I define to distinguish if  $Q$  resembles  $P$ ? lucky for us, this has been studied long before neural networks walked the earth, and is related to the theory of information.

## 1.1 Entropy and Cross-Entropy

**Definition 1.1.1 — Shannon's Entropy.** for a discrete probability distribution  $p$ , we define the (Shannon) Entropy as

$$H(P) = -\sum_{x \in X} p(x) \log_2(p(x)) = \mathbb{E}_{x \in X} \log(1/p(x))$$

A "hand-wavy" definition to Entropy would be the amount of uncertainty a system has, that is - Entropy is an amount that represents how our data is unexpected when sampled.

■ **Example 1.1** as a simple example, consider three boxes with 4 balls in each. denoting  $b_1 = \{R, R, R, R\}$ ,  $b_2 = \{R, R, R, B\}$  and  $b_3 = \{R, R, B, B\}$ . in  $b_1$ , it is easy to guess the outcome of sampling one ball, and in  $b_3$  - not so much. when it is "easy" to guess the outcome, the entropy is low and vice versa. Notice how this relates to the definition of uncertainty - in  $b_1$ , everything is certain as all balls have the same color  $R$  (therefore the entropy is low). In  $b_3$  on the other hand the entropy is high, as the uncertainty is high (maximal in fact, as there is 50% chance for either  $R$  or  $B$ ). ■

going past the basic definition, the entropy also bounds from below the minimal number of bits that are needed to represent our data. for example, if our data is comprised of images, every image (sample) can only be compressed to a size that is exactly the entropy of our sample. What is the entropy of an image you ask? Well, we can define a probability function on some pixel to provide the chance to be equal to some grey-scale value. the summation of all the pixels would give the entropy of the image (but that's a topic for another time).

it is important to notice that as of now, we haven't answered the question that was presented in

the opening section. More specifically, the entropy is a function of a single probability space, and doesn't provide us with some way to compare two probability spaces as we wished for. For that we will define a new method.

**Definition 1.1.2 — Cross Entropy (CE).** for two discrete probability distributions  $p$  and  $q$ , we define the cross Entropy as

$$H(p, q) = -\sum_{x \in X} p(x) \log_2(q(x)) = \mathbb{E}_{x \sim p(x)} \log(1/q(x))$$

This definition measures the average number of bits needed to identify an event drawn from the set if a coding scheme used for the set is optimized for an estimated probability distribution  $q$ , rather than the true distribution  $p$ . some notes that are worth mentioning:

- we can refer to  $p$  as the target probability (as in the probabilities that are associated with the true labels), and  $q$  would be our approximation of  $p$ .
- the cross entropy does not obtain the value 0 when  $p = q$ , like we would have wanted. For that reason (and others) we define the Kullback Leibler (KL) divergence  $D(p||q)$ , that does achieve value 0 when the probabilities are the same:

$$D(p||q) = \sum_{x \in X} p(x) \log(p(x)/q(x)) = \mathbb{E}_{x \sim p(x)} \log(p(x)/q(x))$$

the cross entropy defines in many cases the loss function which we wish to minimize, and this fact is strongly related the the principle of maximum likelihood (denoted  $\mathcal{L}$ , where we try to find a set of parameters  $w$  that are correlated to the highest probabilities

### 1.1.1 minimizing the CE = maximizing the Likelihood

this section is denoted ( $\dagger$ ).

as we clean the dust from our basic ML course, a bird may whisper to you that given parameters  $w$ , a sample matrix  $X$  and true labels  $y$ , the maximum likelihood principle wishes to find the most probable  $w$ 's for such  $X, y$ . in a more formal way, we have

$$\mathcal{L}(w|X, y) = \Pr[y_1, \dots, y_m | X, w] \stackrel{x \sim iid}{=} \prod_i \Pr[y_i | x_i, w]$$

as we wish to maximize the term, applying  $\log$  wouldn't matter, and so does multiplying by some constant. so we denote  $\mathfrak{l} = \frac{1}{|X|} \log(\mathcal{L})$ , and from here:

$$\mathfrak{l}(w|X, y) = \frac{1}{m} \sum_{i=1}^m \log \Pr(y_i | x_i, w) = \mathbb{E}_{x \sim p(x)} \log \Pr(y_i | x_i, w)$$

the latter may be a bit complex, but the meaning of it is that given some true label  $y_i$ , we ask how probable is the that we could represent it given our sampled  $x \sim p(x)$ , and provided the currently initialized parameters  $w$ . this formally mean that we can write the above as

$$\mathfrak{l}(w|X, y) = \mathbb{E}_{x \sim p(x)} \log(q(x))$$

now our goal is to find some  $\hat{w}$  such that  $\hat{w} = \operatorname{argmax}_w \mathfrak{l}(w|X, y)$ , or in other words:

$$\hat{w} = \operatorname{argmin}_w -\mathbb{E}_{x \sim p(x)} \log(1/q(x)) = H(p, q)$$

which is what we wanted to show.

## 2. Neural Network Model

before we dive to the model of a neural network (aka a multi layer Perceptron), we will first remind ourselves what the Perceptron exactly is

### 2.1 Perceptron

this section is denoted ( $\dagger$ )

the Perceptron is a supervised learning algorithm for binary classification, that finds a hyper-plane which splits a data-set  $S$  into two distinct sections/clusters. to formally describe to approach, let us first define some useful notations

**Definition 2.1.1 — half-plane.** we define a half-plane to be the group of points  $x$  as followed

$$\{x : \langle w, x \rangle = b, x \in \mathbb{R}^d\}$$

which is equivalent to

$$\{x : \text{sign}(\langle w, x \rangle - b) \geq 0, x \in \mathbb{R}^d\}$$

■ **Example 2.1** consider the weights  $w = \{1, 2, 3\}$  and the bias  $b = 0$ . Those define the plane  $\langle w, x \rangle = x_1 + 2x_2 + 3x_3 = 0$ , or perhaps with relation to 3D space  $x_1 + 2x_2 + 3x_3 = 0$ . This defines a plane that intersects with the point  $(0, 0, 0)$ . we can ask ourselves - given the points  $a = (1, 2, 3)$  and  $b = (-1, -2, -1)$ , how are they positioned in relation to the plane defined by  $w$  and  $b$ . it is easy to verify that  $a$  is "above" the plane (+1), and  $b$  is "below" it (-1), as figure 2.1 portrays ■

let  $\mathbb{H}_{half} = \{h_{w,b}(x) = \text{sign}(x^T w + b) : w \in \mathbb{R}^n, b \in \mathbb{R}\}$  be an hypothesis class that represents all half-planes. we would like to find some  $h \in \mathbb{H}_{half}$  that classifies every sample  $x$  from our data-set  $S$  to some value  $\in \{+1, -1\}$  (note that this might not always be possible). The main approach uses the fact that when ever there's a "mismatch" between the signs of a true label  $y_i$  and  $h(x_i)$  (for any  $i$ ), which is the same as saying that  $y_i \langle x_i, w \rangle \leq 0$ , the weights are altered in a manner that decreases the chance for such mismatch to occur in later iterations. This can be formalized as followed: we initialize the weights to  $w_0 = (0, 0, \dots, 0)$ . Then, if  $\exists i$  s.t  $y_i \langle x_i, w_t \rangle \leq 0$  we update  $w_{t+1} = w_t + y_i x_i$ .

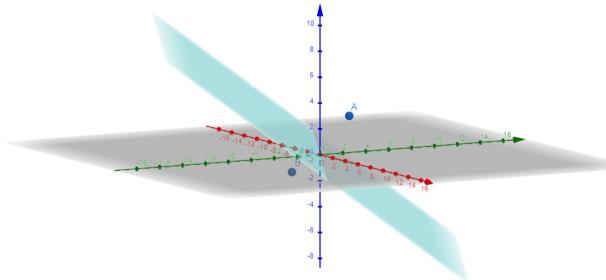


Figure 2.1: A hyper-plane in 3D

notice that  $y_i \langle x_i, w_{t+1} \rangle = y_i \langle x_i, w_t + y_i x_i \rangle = y_i \langle x_i, w_t \rangle + y_i^2 \|x_i\|$ . that is, in every iteration we add an element  $\|x_i\| \geq 0$ , which guarantees there exists some  $t$  for which the if condition doesn't hold, and in that case we return  $w_t$ .

## 2.2 Multi Layer Perceptron (MLP)

The term multi-layer Perceptron refers to a number of single Perceptrons that are organized to a layer. In addition, the MLP has more "freedom" to it, as it is equipped with a non-linearity function that enhances the expressibility of the model, and in general can also be used in regression problems (as opposed to the original Perceptron). In most books, the model of a single "neuron" is represented with the diagram in figure 2.2

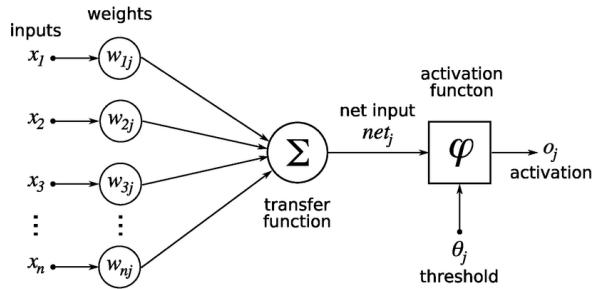


Figure 2.2: A single neuron

which indicates that given some input vector with  $n$  features  $x = (x_1, x_2, \dots, x_n)$ , every value  $x_i$  is multiplied with some weight  $w_{ij}$ , that should in theory represent how valuable the  $i^{th}$  neuron to the learning process, such that the sum of all of those is the input to some activation function  $\phi$ , to which we also add a threshold  $\theta_j$  (that somewhat resembles to the activation notion of an actual neuron). This long sentence could be summarized to a single formula to the activation  $o_j$  of the  $j^{th}$  neuron (which will resemble the notation for the Perceptron of the previous section):

$$o_j = \phi(\sum_{i=1}^n w_{ij} x_i + \theta_j) = \phi(\langle w_j, x \rangle + \theta_j)$$

where  $w_j$  in the single neuron model is a vector that represents the weight associated our neuron ( $j$ ) and all of the other neurons in the next layer.

the generalization of the above to many neurons is straight forward, as the  $(i+1)^{th}$  layer  $x_{i+1}$  is represented using the previous layer  $x_i$  and a now weights matrix  $W$ , such that  $w_{ij} = [W]_{ij}$  indicates the weight associated with the edge  $(i, j)$ . formally we can write  $x_{i+1} = \phi(W_i x_i + \theta_i)$  (note that though  $\phi$  here is the same for every neuron, this is not obligatory). Before moving on, notice that

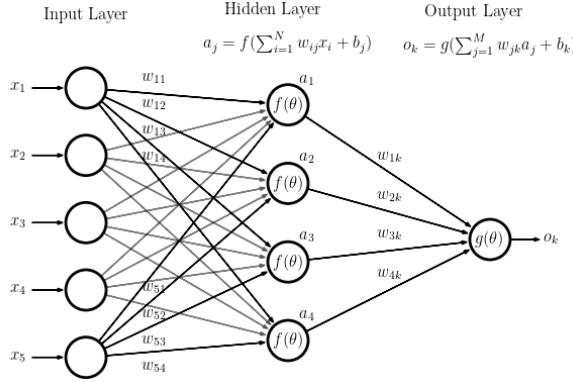


Figure 2.3: A Neural Network diagram.

as much as the NN model diagrams are very cool, all they represent eventually is a series of affine transformations (with the addition of the non-linearity). That is, the output of a MLP is simply a set of affine transformations + non-linearities, and that doesn't require any diagram.

## 2.3 Activation Function

an activation function  $\phi$  is a function (mostly non linear one) that is given the activation of a neuron as an input, and outputs some number. The fact that those function are mostly non-linear is an important one, as it is what distinguishes the neural network model from a basic model, only capable of predicting linear phenomena from a much more capable one. In other words, the non-linear activation function allows the model to predict/classify various range of cases, that are not necessarily linear in nature (this will be formulated later on). Lastly, in relation to the previous sentence, notice that when using linear activation functions, the outcome is a linear component, and when summed over all the layers network, we are left with an output that can be represented as  $x_{out} = \mathbf{w}x_{in} + \mathbf{b}$ , where  $\mathbf{w}$  and  $\mathbf{b}$  are some combinations of the original  $\{W_i, b_i\}_{i=1}^n$ . This simply indicates that when using a linear model we lose the significance of the layer formation (as well as strict ourselves only to linear cases, as stated before).

### 2.3.1 Types

- sigmoid: also called logistic. defined as

$$\phi(x) = \frac{1}{1+e^{-x}}$$

this one maps all values  $x$  to some normalized value  $\in (0, 1)$

- relu: which stands for rectified linear unit.

$$\phi(x) = \max(0, x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

obviously, many more options exist, but those are the basic (and most known for) ones. Furthermore, every function may be better or worse depending on the task in hand. for example, sigmoid will not describe magnitudes, but is valuable when the output should resemble distribution (as it is normalizing the values). relu is useful in many cases, but may cause problems if we are dealing with negative activations, and in that case the function outputs 0, which may cause problem in SGD process (small or 0 grad may affect convergence)

## 2.4 Loss Function

The following step to building a NN model would be to come up with a meaningful loss function, that will somehow provide us with a number that represents how "good" our prediction is. This loss is sometimes called the "Empirical Risk", and is defined as followed:

**Definition 2.4.1 — Empirical Loss.** given algorithm  $h$  (in our case, the algorithm is the network), and  $m$  samples  $x_i \in \mathbb{R}^d$  the ER is

$$ER(h) = \frac{1}{m} \sum_{i=1}^m l(h(x_i), y_i)$$

where  $l : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

remember that our "goal" is to find  $\hat{h}$  such that  $\hat{h} = \operatorname{argmin}_{h \in H} ER(h)$ , and  $H$  is the hypothesis class. this is the empirical risk minimization (ERM) rule. The method to achieve such minima will be discussed later on.

### 2.4.1 Regression Loss

when a specific number is to be predicted (for example - predicting housing prices, weight percentage, etc..), a good rule of thumb would be that a small/big deviation from the "true label" would yield small/big loss respectively. One option to do that is using  $l_2$  norm loss. Of course, other forms of functions  $l$  could work, depending on the problem. In formal notation, if we denote out hypothesis as  $h = N_\theta$ , where  $\theta$  represents the network parameters, and use  $l_2$  norm we have

$$ER(N_\theta) = \frac{1}{m} \sum_{i=1}^m \|N_\theta(x_i) - y_i\|_2^2$$

### 2.4.2 Classification Loss

When the task is hand is to predict finite number of classes, a naive approach could use a "misclassification" loss (sometimes called " $l_0$ " norm, though isn't a norm per say, or "0 – 1" loss).

**Definition 2.4.2 — Misclassification (0,1) Loss.** given algorithm  $h$ , and  $m$  samples  $x_i \in \mathbb{R}^d$  the  $0,1$  loss is

$$L_{0,1}(h) = \frac{1}{m} \sum_{i=1}^m \mathbb{1}(h(x_i) = y_i)$$

where  $\mathbb{1}$  is the indicator function

the misclassification loss may be a bad idea in cases where it is important to distinguish Type1 or Type2 errors. Another option is to use Cross Entropy, which we have already seen before. For this to have any meaning though, we must first "cast" the network output to some bounded values, that would represent some distribution  $p$ . Two common metrics are (for a network output vector  $\hat{y}_i$ ):

- logistic:  $\hat{y}_i \mapsto 1/(1 + e^{-\hat{y}_i})$  suited for binary classification

- softmax:  $\hat{y}_i \mapsto e^{\hat{y}_i} / (\sum_i e^{-\hat{y}_i})$ . mostly suited for multi class classification after this was done, the cross entropy would give an estimation to the network's "capabilities", by comparing the true distribution of the data with the network's output distribution:

**Definition 2.4.3 — Cross Entropy in Multi Class Classification.**  $H(p, q) = -\sum_{c=1}^C \sum_{i=1}^m \hat{y}_i \log(h(x_i))$  where  $C$  is the number of classes.

specifically for binary classification, we have

$$H(p, q) = -\sum_{i=1}^m [\hat{y}_i \log(h(x_i)) + (1 - \hat{y}_i) \log((1 - h(x_i)))]$$

where  $h(x_i)$  are the logistic values of the output labels.

## 2.5 Network Training

We have structured a neural network model  $N_\theta$ , and chose a meaningful loss function  $L$ . Now its time to train the model, that is, find all the parameters  $\theta$  that would minimize  $L$ . In other words, we wish to calculate  $\min_\theta L$ , and a key component to this is the fact that we can find the (local) minimum iteratively, by taking "steps" in the direction of the maximum descent, aka the minus gradient direction  $-\nabla L$

### 2.5.1 Gradient Descent Optimization

We would like to "move" in a direction  $\Delta x \in \mathbb{R}^n$  such that  $f(x_t + \Delta x) < f(x_t)$  (where  $t$  denotes the current iteration and  $f$  is the loss). usually we pick  $\Delta x = -\eta \nabla f|_{(x_t)}$ , where  $\eta$  is a step size (predefined hyper parameter). This choice is not arbitrary - if  $f$  is differentiable around  $x_t$  (if its not we use the sub-differential, though we will not linger on this) we can look at its Taylor approximation

$$f(x_t + \Delta x) = f(x_t) + \Delta x^T \nabla f|_{(x_t)} + O(||\Delta x||^2)$$

as we wish to maximize the difference in every step, our problem is

$$\max_{||\Delta x||=\epsilon} f(x_t + \Delta x) - f(x_t) \approx \max_{||\Delta x||=\epsilon} \Delta x^T \nabla f|_{(x_t)}$$

(for some small  $\epsilon$ ). using Lagrange multipliers for  $h = \Delta x^T \nabla f|_{(x_t)}$  and  $g = ||\Delta x|| - \epsilon = (\Delta x)^T \Delta x - \epsilon$ :

$$\begin{aligned} \nabla_{(\Delta x)^T} h &= \lambda \nabla_{(\Delta x)^T} g \rightarrow \nabla_{(\Delta x)^T} (\Delta x^T \nabla f|_{(x_t)}) = \lambda \nabla_{(\Delta x)^T} ((\Delta x)^T \Delta x - \epsilon) \\ &\rightarrow \nabla f|_{(x_t)} = \lambda \Delta x \end{aligned}$$

What this really tells us, is that if we wish to maximize the objective, we should pick  $\Delta x$  that is in the direction of the gradient  $\nabla f$  (which in general correlates with that we know about the gradient - direction of steepest ascend). the above can be summarized to an iteration loop which is sometimes referred to as gradient descend process:

**Definition 2.5.1 — Gradient Descend Process.** given a neural network model  $N_\theta$  with parameters  $\theta$  (weights,biases,etc..), step size  $\eta$ , true labels  $y$ , samples  $X$  and a loss function  $L$ , a gradient descend process over the parameters  $\theta$  is

$$\theta_{i+1} = \theta_i - \eta \nabla_\theta L(N_{\theta_i}, X, y)$$

some important things to keep in mind

- the convergence isn't guaranteed to be towards a global maximum.
- the step size should be small enough to justify linear approximation, though not too small so we wont converge

### 2.5.2 Mini Batching and SGD

as our data becomes larger, it may be less feasible to calculate the gradient of the loss function with respect to every sample in every iteration. For that reason we may choose a subset of samples randomly, and find the gradients with respect to those samples. Notice that this does not come without risks, as our data may not be consistent, which may cause problems when trying to converge. Nevertheless, if we are sufficiently confident that every random sample can represent our data properly, the calculated gradient in this approach gives an approximation to the actual gradient, which in theory should bring us closer to a (local) minima.

formally speaking, if  $B \subset X$  is a mini batch of samples that is randomly generated from our data  $X$ , the gradient of the loss function over all  $X$  is approximated by the gradient over all  $B$ :

$$L = \frac{1}{|X|} \sum_{i \in X} l(N_\theta(x_i), y_i) \Rightarrow \nabla L \approx \frac{1}{|B|} \sum_{i \in B} \nabla l(N_\theta(x_i), y_i)$$

from the section above, we can now formulate an algorithm

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#### Algorithm 1 Stochastic Gradient Descent

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**Require:** learning rate  $\eta$ , number of epochs  $T$ , batch size  $m$ , NN model  $N_\theta$

```

 $\theta \leftarrow (0, 0, \dots, 0)$ 
for  $t = 1, 2, \dots, T$  do
    sample a mini-batch:  $B = \{\mathbf{x}_i, y_i\}_{i=1}^m$ 
     $g \leftarrow \frac{1}{m} \sum_{i \in B} \nabla l(N_\theta(\mathbf{x}_i), y_i)$ 
     $\theta \leftarrow \theta - \eta g$ 
end for

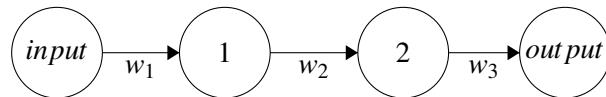
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### 2.5.3 Back-propagation

this section is (mostly) denoted ( $\dagger$ )

It turns out that the calculation of the gradient is in itself a quite daunting task. In this section we will see how to calculate it efficiently using what is known as the back propagation algorithm. lets start with a basic neural network - input node, two hidden nodes and one output node



in this network the loss is  $L = L(w_1, b_1, w_2, b_2, w_3, b_3)$ , and we wish to find what are the weights and biases that provide the minimal  $L$ . In the section below, we will seclude ourselves to  $l_2$  loss (though this is generally true for every loss), the networks error over the  $0'th$  sample (say, the first image out of 20,000) is

$$L_0 = (a^{(L)} - y)^2$$

where  $a^{(L)}$  is the activation of the  $L'th$  layer (the last one), and  $y$  is our true labels vector. more specifically, the activation is (as stated at the beginning of the chapter) is

$$a^{(L)} = \sigma(w^{(L)} a^{(L-1)} + b^{(L)}) := \sigma(z^{(L)})$$

our goal is to understand how the loss  $L$  changes with respect to  $w^{(L)}$ , which can be written as its partial derivative

$$\frac{\partial L_0}{\partial w^{(L)}} = \frac{\partial z^{(L)}}{\partial w^{(L)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial L_0}{\partial a^{(L)}}$$

we know the exact forms of  $L_0$ ,  $a^{(L)}$  and  $z^{(L)}$ , therefore

$$\begin{aligned}\frac{\partial L_0}{\partial a^{(L)}} &= 2(a^{(L)} - y) \\ \frac{\partial a^{(L)}}{\partial z^{(L)}} &= \sigma'(z^{(L)}) \\ \frac{\partial z^{(L)}}{\partial w^{(L)}} &= a^{(L-1)}\end{aligned}$$

which means

$$\frac{\partial L_0}{\partial w^{(L)}} = a^{(L-1)} \sigma'(z^{(L)}) 2(a^{(L)} - y)$$

note that this is only for the  $0'th$  sample, and for the entire data-set we take the average:

$$\frac{\partial L}{\partial w^{(L)}} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial L_k}{\partial w^{(L)}}$$

now we repeat the process for the biases  $b$ :

$$\frac{\partial L_0}{\partial b^{(L)}} = \frac{\partial z^{(L)}}{\partial b^{(L)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial L_0}{\partial a^{(L)}} = 1 \sigma'(z^{(L)}) 2(a^{(L)} - y)$$

and when averaging:

$$\frac{\partial L}{\partial b^{(L)}} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\partial L_k}{\partial b^{(L)}}$$

Great! but those were just two elements in the vector  $\nabla L$  we wish to calculate, as we remember that

$$\nabla L = \left( \frac{\partial L}{\partial w^{(0)}} \quad \frac{\partial L}{\partial b^{(0)}} \quad \dots \frac{\partial L}{\partial w^{(L)}} \quad \frac{\partial L}{\partial b^{(L)}} \right)^T$$

so now we have to look at every layer  $l \leq L$ . we start with  $l = L-1$ :

$$\frac{\partial L_0}{\partial w^{(L-1)}} = \frac{\partial z^{(L-1)}}{\partial w^{(L-1)}} \frac{\partial a^{(L-1)}}{\partial z^{(L-1)}} \frac{\partial L_0}{\partial a^{(L-1)}}$$

the first two elements are calculated by hand:

$$\begin{aligned}\frac{\partial z^{(L-1)}}{\partial w^{(L-1)}} &= a^{(L-2)} \\ \frac{\partial a^{(L-1)}}{\partial z^{(L-1)}} &= \sigma'(z^{(L-1)})\end{aligned}$$

for the last element though, notice that When using chain rule again we get

$$\frac{\partial L_0}{\partial a^{(L-1)}} = \frac{\partial z^{(L)}}{\partial a^{(L-1)}} \frac{\partial a^{(L)}}{\partial z^{(L)}} \frac{\partial L_0}{\partial a^{(L)}}$$

aside from  $\frac{\partial z^{(L)}}{\partial a^{(L-1)}} = w^{(L)}$ , we already have the two right most elements, as they were calculated in the previous part (where  $l = L$ ), so we could write

$$\frac{\partial L_0}{\partial w^{(L-1)}} = a^{(L-2)} \sigma'(z^{(L-1)}) w^{(L)} \sigma'(z^{(L)}) 2(a^{(L)} - y)$$

in other words, to find  $\frac{\partial L_0}{\partial w^{(L-1)}}$  we needed  $\frac{\partial L_0}{\partial a^{(L-1)}}$ , but the latter is already known from previous iterations. This would give us a recursive definition for  $\frac{\partial L_0}{\partial w^{(l)}}$ . for any  $l \leq L$ , but before that - lets generalize for a multi layered network:

we will add a sub-script that indicates which neuron are we referring to in the specific layer. for example:  $a_k^{(l)}$  is the activation of the  $k'th$  neuron of the  $l'th$  layer. Now,  $L_0$  is simply the sum over all neurons. suppose the  $L'th$  layer has  $n_L$  neurons, so

$$L_0 = \sum_{j=0}^{n_L} (a_j^{(L)} - y_j)^2$$

lets denote the weight between  $node_k^{L-1}$  and  $node_j^L$  as  $w_{jk}^L$ . the activation would now be

$$a_j^{(L)} = \sigma(w_{j,0}^{(L)} a_0^{(L-1)} + \dots + w_{j,n_L-1} a_{n_L-1}^{(L-1)} + b_j^{(L)}) = \sigma(\mathbf{w}_j^{(L)} \cdot \mathbf{a}^{(L-1)}) := \sigma(z_j^{(L)})$$

or simply

$$z_j^{(L)} = \sum_{k=0}^{n_{L-1}-1} w_{jk}^{(L)} a_k^{(L-1)} + b_j^{(L)}$$

we now have everything we need to find the derivatives

$$\frac{\partial L_0}{\partial w_{jk}^{(L)}} = \frac{\partial z_j^{(L)}}{\partial w_{jk}^{(L)}} \frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} \frac{\partial L_0}{\partial a_j^{(L)}} = a_k^{(L-1)} \sigma'(z_j^{(L)}) 2(a_j^{(L)} - y_j)$$

where as usual, the entire loss is an average

$$\frac{\partial L}{\partial w_{jk}^{(L)}} = \frac{1}{n} \sum_{i=0}^{n-1} \frac{\partial L_i}{\partial w_{jk}^{(L)}}$$

repeating the process for the bias:

$$\frac{\partial L_0}{\partial b_j^{(L)}} = \frac{\partial z_j^{(L)}}{\partial b_j^{(L)}} \frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} \frac{\partial L_0}{\partial a_j^{(L)}} = 1 \sigma'(z_j^{(L)}) 2(a_j^{(L)} - y_j)$$

and the average

$$\frac{\partial L}{\partial b_j^{(L)}} = \frac{1}{n} \sum_{i=0}^{n-1} \frac{\partial L_i}{\partial b_j^{(L)}}$$

following the thread of the single layer network, we would also wish to find the derivatives according to the activations. This is where the multi layer network differs from the single layer one. The main difference is that while for the single layer model there was only one activation, in the multi layer model every node has its own activation, and they all should be summed together (as the loss  $L$  is a function of them all). this can be formally written as

$$\frac{\partial L_0}{\partial a_k^{(L-1)}} = \sum_{j=0}^{n_L-1} \frac{\partial z_j^{(L)}}{\partial a_k^{(L-1)}} \frac{\partial a_j^{(L)}}{\partial z_j^{(L)}} \frac{\partial L_0}{\partial a_j^{(L)}}$$

This is pretty much all we need - so we can write it all together:

$$\begin{aligned} \frac{\partial L_0}{\partial w_{jk}^{(l)}} &= a_k^{(l-1)} \sigma'(z_j^{(l)}) \frac{\partial L_0}{\partial a_j^{(l)}} \\ \frac{\partial L_0}{\partial b_j^{(l)}} &= \sigma'(z_j^{(l)}) \frac{\partial L_0}{\partial a_j^{(l)}} \end{aligned}$$

where

$$\frac{\partial L_0}{\partial a_j^{(l)}} = \begin{cases} \sum_{j=0}^{n_{l+1}-1} w_{jk}^{l+1} \sigma'(z_j^{l+1}) \frac{\partial L_0}{\partial a_j^{(l+1)}}, & l < L \\ 2(a_j^{(l)} - y_j), & l = L \end{cases}$$

finally, the elements of the vector  $\nabla L$  are

$$\begin{aligned} \frac{\partial L}{\partial w_{jk}^{(L)}} &= \frac{1}{n} \sum_{i=0}^{n-1} \frac{\partial L_i}{\partial w_{jk}^{(L)}} \\ \frac{\partial L}{\partial b_j^{(L)}} &= \frac{1}{n} \sum_{i=0}^{n-1} \frac{\partial L_i}{\partial b_j^{(L)}} \end{aligned}$$

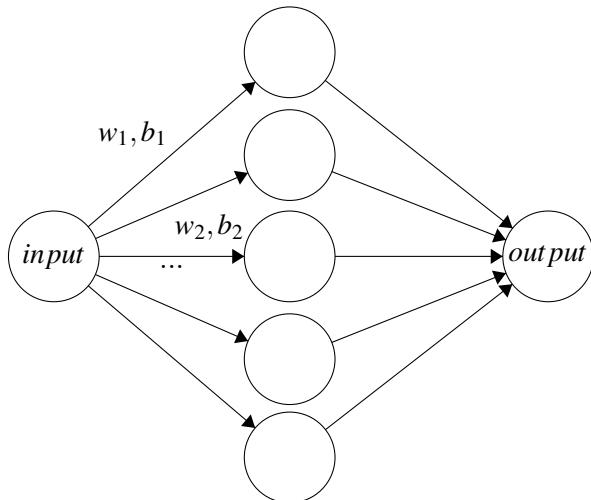
## 3. Deep Neural Network Theory

### 3.1 Universal Approximation Theorem

let us deal with the question of how "expressive" a neural network can get. More specifically, a network with only one hidden layer.

**Theorem 3.1.1 — Cybenko 89.** for a continuous monotone function  $\sigma$  with  $\sigma(-\infty) = 0$  and  $\sigma(\infty) = 1$ , the family of function  $f(x) = \sum_i \alpha_i \sigma(w_i x + b_i)$  is dense in  $C([0, 1])$  w.t.y the supremum norm  $d(f, g) = \sup |f(x) - g(x)|$

In a less formal form, Cybenko's theorem states that we can approximate any function  $g$  that is continuous and defined on  $[0, 1]$  (sigmoid for example) with arbitrarily small precision using a function  $f$  that represents a single hidden layer neural network. the network itself can be represented as below



notice that both input and output are scalars, and it that matter the network is "as simple as it gets".

A proof is not provided here, but intuitively speaking - the model can represent any stretch  $w_i$ , shift  $b_i$  and weight  $\alpha_i$  to the input, and this is what gives the model its power or expressiveness. with respect to Cybenko's theorem, few notes are worth taking:

- the work of Cybenko was extended by Hornik 2 years later, and in his theorem the statement was provided for every bounded  $\sigma$ , which is a relaxation to the claim that  $\sigma$  must be monotone.
- the above does not contradict the fact that one can also use  $\sigma \equiv \text{relu}$  function, even though  $\text{relu}$  is not bounded. this is because the following composition:

$$f(x) = \text{relu}(Wx + b_1) - \text{relu}(Wx + b_2)$$

is bounded, and for a specific choice of  $b_1, b_2$ , the function  $f$  is sigmoidal ("S" shaped, and bonded)

### 3.2 Shallow compared to Deep NN

A legitimate question that may arise from Cybenko's Theorem is - if a shallow layer can represent every function with small error, why should we go the extra mile by defining a deep network (with 2 or more hidden layers)? To answer this question, let us begin with a short example

■ **Example 3.1** let us define the t-saw-tooth function as a piece-wise affine function with  $t$  pieces. Also, we define the hat function as  $\hat{h}(x) = \text{relu}(2\text{relu}(x) - 4\text{relu}(x - 0.5))$ . note that  $\hat{h}$  is a 4-saw-tooth function by definition. We can use  $\hat{h}$  to concatenate saw-tooth functions - note that  $t(x) = \hat{h}(x) + \hat{h}(x - 1)$  comprises of two hats, and is a 6-saw-tooth function, and the composition  $t(t(x))$  is a 10-saw-tooth,  $t(t(t(x))))$  is a 18-saw-tooth, and in general, a composition of  $T$  functions is a  $(2 + 2^{T+1})$ -saw-tooth, namely, the composition is comprised of  $2^T$  hats. How can we represent this composition using a shallow network? remember that every neuron in the (single) hidden layer "represents" a single  $\text{relu}$ , therefore two neurons can represent a single  $\hat{h}$ . As there are exponentially many hats, a shallow network will have to use exponentially many neurons ( $2^{T+1}$  to be exact). On the other hand, a deep neural network would only need  $\Theta(T)$  (if, for example, we use  $T$  layers and 2 nodes per layer). This is the case because the deep network  $i^{th}$  layer receives as input the values of the  $(i-1)^{th}$  layer, and for that matter is represents the composition of  $t(x)$  simply as a product of its architectural design. ■

the example above demonstrates an exponential gap (in terms of size) between the shallow and deep NN architectures, and can be summarized to the following lemma:

**Lemma 3.2.1** if  $f$  is a-saw-tooth and  $g$  is b-saw-tooth,  $f + g$  is at most  $(a+b)$ -saw-tooth and  $f(g)$  is at most  $ab$ -saw-tooth

"proof". as for  $f + g$ , the "worst-case-scenario" is when the edges of  $f$  and  $g$  doesn't intersect, and in that case  $f + g$  will inherit all the discontinuities that  $f$  and  $g$  had. For  $f(g)$ , the image of each piece of  $g$  can contain the entire range in which  $f$  has all its pieces, which means that for every one of the  $b$  affine section of  $g$  we provide an input (to  $f$ ) that generates  $a$  affine sections, to a total sum of  $a \cdot b$ . ■

the above statements could also be asked in reverse order - we have claimed that a deep layer with  $T$  layers and  $O(T)$  neurons can be represented using a shallow layer with  $2^T$  neurons, and now we claim that a shallow network with  $n$  neurons could be represented with a deep network, using  $\Theta(n)$  neurons. As an example, we go back to our shallow network model  $f(x) = \sum_i \alpha_i \sigma(w_i x + b_i)$  under the assumption  $\alpha_i > 0$  and construct a deep network with  $n$  layers and  $O(1)$  neurons per layer

as followed: the network is built from three main "branches"  $h_1, h_2$  and  $h_3$ .  $h_1$  will be used to "carry" the input  $x$  as it was initially provided,  $h_2$  is for the actual calculation of the affine transformation on the input and  $h_3$  sums the values of  $h_2$  note that the first value of  $h_3$  is  $\sigma(x+L)$ , where  $L$  is a shift

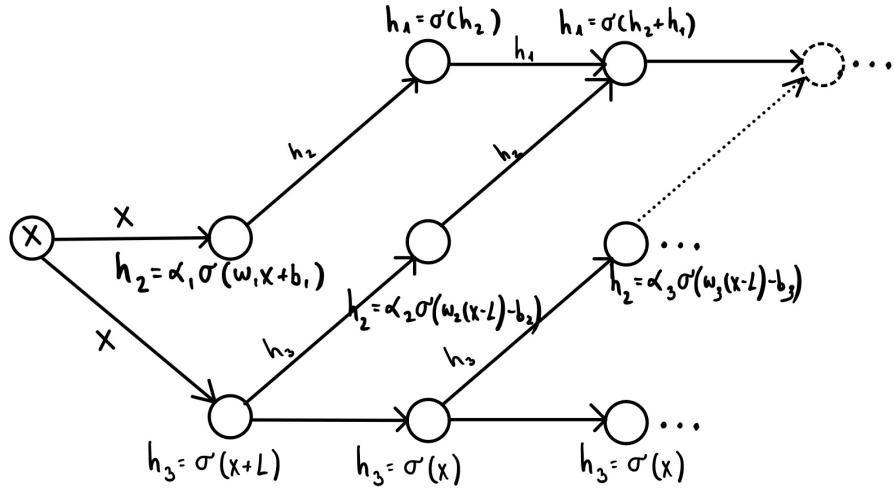


Figure 3.1: Shallow NN to Deep NN conversion.

we make to ensure positive values. This is needed so we wont lose any negative value edges under the `relu` function. In the following parts of  $h_2$  we shift back to the original location  $x - L$ . Also note that as  $\alpha_i > 0$ , the values of  $h_3$  are always positive, that is  $\sigma(h_1 + h_2) = h_1 + h_2$ . the network provided in the figure has  $\Theta(n)$  layers with 3 neurons in each, which is what we intended to show ( $O(n)$  neurons). All in all, we can summarize the above with the following diagrams

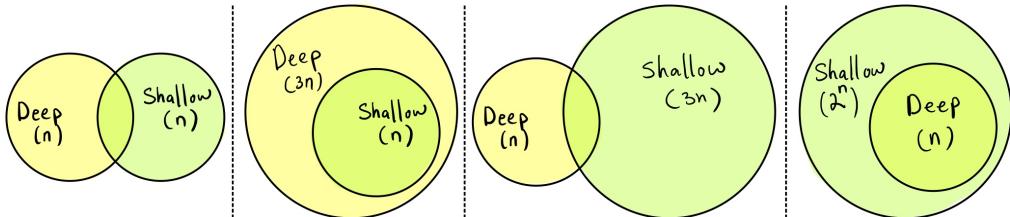
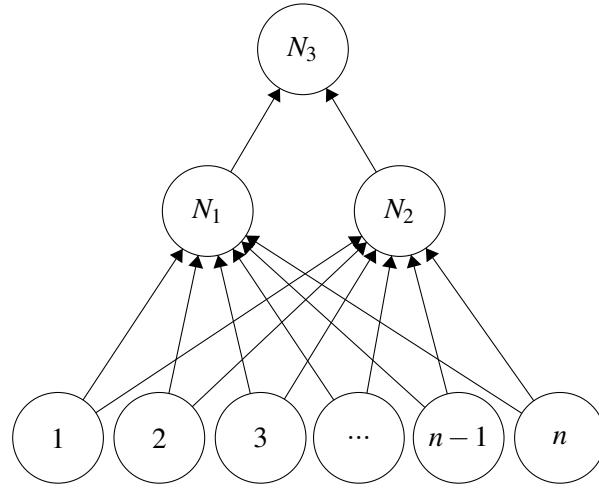


Figure 3.2: Venn Diagram of NN models with ( $\cdot$ ) neurons

### 3.3 Training a NN is NP-Complete

In this section we will prove that the training process of a NN is considered a NP-complete problem. The model we will use consists of Three neurons and is fully connected, where the input is of length  $n$ .



Moreover, the neurons  $N_1, N_2$  and  $N_3$  performs Heaviside activation on its input, as followed - given input  $x = (x_1, \dots, x_n)$ , weights  $(a_1^{(1)}, a_2^{(1)}, \dots, a_n^{(1)}, a_1^{(2)}, a_2^{(2)}, \dots, a_n^{(2)}, a_1^{(3)}, a_2^{(3)})$  and biases  $(b_1, b_2, b_3)$

$$f_{i=1,2}(x) = \begin{cases} 1, & \sum_{j=1}^n a_j^{(i)} x_i + b_i > 0 \\ -1, & \text{else} \end{cases}$$

$$f_{i=3}(x) = \begin{cases} 1, & a_1^{(3)} N_1 + a_2^{(3)} N_2 + b_3 > 0 \\ -1, & \text{else} \end{cases}$$

where  $N_1, N_2$  in  $f_{i=3}$  indicates the outputs of neurons  $N_1, N_2$ , which are provided as input to  $N_3$ .

**Corollary 3.3.1 — The Trainability Problem.** given a set of  $O(n)$  samples  $\in \{0, 1\}^n$ , finding the Networks' parameters and threshold functions for which it produces outputs consistent with the training set is NP-complete

*Proof.* soon to be ■

## 4. Time/Translation Invariance

### 4.1 Notion of Time-Space

In this section we will discuss systems that are Invariant - either in time or in space translations. Such prior knowledge on our data will allow us to define more suited NN models that could potentially provide precise solutions using less parameters, and with great efficiency.

#### 4.1.1 Translation/Time Invariant Systems

Systems that does not depend on time or spatial coordinated are called Translation/Time Invariant. This formally means that a shift in the input leads to a shift in the output, which for some operator  $L$  translates to:

$$L[f(x+t)](y) = L[f(x)](y+t)$$

where  $f$  is our input (image/audio for example). for example - our face detector algorithm should operate the same for all faces that are in different offsets in the image.

from here on we will refer to  $f(x)$  as discrete samples of our signal, that is

$$f(x) = \sum_s \delta(x-s)f(s)$$

where  $f(s)$  are scalars and  $\delta$  is Dirac delta function

#### 4.1.2 Linear Translation/Time Invariant Systems (LTI)

Let us also consider the linear and Time/Translation Invariant (TI) case, in which  $L$  performing on the sum of samples is as if we sum the operation of  $L$  on every sample by itself. This setup provides a convenient conclusion:

$$\begin{aligned} L[f(x)](y) &= L[\sum_s \delta(x-s)f(s)] \xrightarrow[TI]{linearity} \\ &\sum_s f(s)L[\delta(x-s)](y) \end{aligned}$$

$$\sum_s f(s) L[\delta(x)](y-s) \stackrel{g(x):=L[\delta(x)]}{=} \sum_s f(s) g(y-s) = f * g$$

In other words, if  $L$  represents a LTI system, it can be written as convolution of the input  $f$  with some filter/kernel  $g = L[\delta]$  (that can be chosen to our likelihood).

It is also worthy mentioning that every linear transformation can be expressed by a matrix. In our case

$$(f(x) * g(x))(y) = \sum_s f(s)g(y-s) = \langle f(x), g(y-x) \rangle = G\hat{f}$$

Where we restricted ourselves to finite space (sum is from  $s = 0$  to  $n - 1$ ) and cyclic convolution, where  $n$  signifies the number of elements in the signal, and there are infinitely many cycles of repeating  $n$  such elements. This is for

$$G = \begin{pmatrix} g(0) & g(1) & g(2) & \dots & g(n-1) \\ g(n-1) & g(0) & g(1) & \dots & g(n-2) \\ g(n-2) & g(n-1) & \dots & \dots & g(n-3) \\ \vdots & \vdots & \dots & \dots & \vdots \\ g(1) & g(2) & \dots & g(n-1) & g(0) \end{pmatrix} \text{ and } \hat{f} = \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{pmatrix}$$

notice that every row in  $G$  is a cyclic permutation of the previous row

■ **Example 4.1 — using  $G$  and  $\hat{f}$ .** denote  $h = f * g$  and let  $f, g$  be defined in the coordinate range  $[0, 5]$  as followed:  $f = (0, 0, 0, 1, 0, 0)$  and  $g = (0, 0, 0, 1, -1, 0)$ , for specific  $y = 6$

$$h(6) = \sum_{s=0}^5 f(s)g(6-s) = f(0)g(6) + \dots + f(3)g(3) + \dots + f(5)g(1) = f(3)g(3) = -1$$

for every value of  $h$ , we can use  $G$  and  $\hat{f}$ :

$$h = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

■ **Example 4.2 — derivative.** consider the filter  $g = (-1, 1)$ , and some general  $f(x)$  notice that  $f * g = f(x+1) - f(x)$ .

how does this relates to derivatives? from definition

$$\frac{\partial f(x)}{\partial x} = \lim_{\epsilon \rightarrow \infty} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

this can be approximated when  $f$  is discrete (for example, if  $f$  is some image, we can consider a single pixel as our minimum length, which has size 1). That is

$$\frac{\partial f(x)}{\partial x} \approx f(x+1) - f(x)$$

before we move on, let us introduce the 2D convolution, as it is relevant for many convolutional NN architectures

**Definition 4.1.1 — discrete 2D convolution.** given 2D functions  $f(x,y)$  and  $g(x,y)$  and two integers  $a,b \in \mathbb{N}$ , we define the 2D discrete convolution as

$$f * g = \sum_{k=-a}^a \sum_{l=-b}^b g(k,l) f(x-k, y-l)$$

intuitively, we can think of  $g$  as a kernel that we "run-through" our function  $f$  and for every  $k,l$ , the new value we assign to  $f * g$  is the sum of all multiplications of points where  $g$  lays on top of  $f$ .

## 4.2 Convolutional NN

### 4.2.1 the outline

A convolution layer is a layer in our architecture that operates convolution instead of just summing up weights (like we saw in the MLP architecture), that is - the values of the  $i^{th}$  layer  $L_i$  is given by convolving the previous layer  $L_{i-1}$  with some chosen kernel function  $g$ , i.e  $L_i = L_{i-1} * g$  (up to bias addition and non linearity). Our weights would be the values of the kernel, and these would be (along with the biases) the parameters which we aim to learn in the iterative learning process. Generally speaking, the transition between layers in such architecture would be

- apply  $N$  dimensional convolution - same as applying dot product to the input with the kernel
- add bias
- apply some nonlinear activation
- Optional: apply pooling - sample every  $j^{th}$  pixel, for some  $j$  (see below)

Note that compared to multi layer fully connected NN, applying convolution greatly reduces the parameters of the network, as the matrix  $G$  possesses "only"  $n$  different values, compared to fully connected layer that applies a matrix with  $n^2$  parameters. Having said that, in convolutional layers it is not clear a-priori what a neuron really are. Most literature defines the neurons as the rows of  $G$ , which are neurons with shared weights (as  $g_1, \dots, g_n$  are the same for every neuron, with some permutation).

It is common to think of the convolution layers as layers that extract features that are increasingly concrete. For example, the first layers would be responsible for only edge detection, and deeper layers may focus on features that are less vague, such as specific features of a face if we try to detect faces.

### 4.2.2 pooling layer and strides

A pooling layer reduces the size of the image, which may be relevant for example if our input is too large to be fed through our network efficiently. Say we wish to scale down our input by a factor of 2 - few variations exist

- max pooling - chose a  $2 \times 2$  section in the input (for example grid of  $2 \times 2$  pixels) and choose only the pixel that has maximum grey-scale. One downside to this is that we lose the information of which pixel was indeed the one with the max grey-scale.
- average pooling - take the average of your input points in a given window

Stride, on the other-hand is the process of sub-sampling our input. This is mostly implemented inside the convolution process - instead of shifting our kernel one value at a time, we shift it two or more values

### 4.2.3 the architecture

for a more rigorous description, let's consider the case of 2D convolutions applied on images. An image is a tensor that has dimensions  $w \times h \times c$ , where in most cases  $c$  represents Red, Green, Blue, therefore it satisfies  $c = 3$ . A convolution layer receives the image as input, and generates an output that is called an activation map, which is a tensor with dimensions  $w' \times h' \times c'$ , which is the product of convolving with a kernel with dimensions  $w_g \times h_g \times c \times c'$ . What this actually means is that we map every cube with dimensions  $h_g \times w_g \times c$  to a single pixel in our output, and we perform this  $c'$  times, for  $c'$  kernels. figure 4.1 illustrates a concrete example

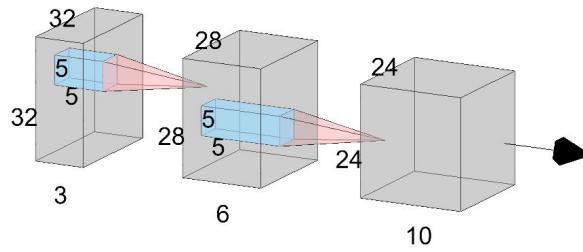


Figure 4.1: left to right -  $layer_1$  ( $32 \times 32 \times 3$ ) convolved with  $kernel_1$  ( $5 \times 5 \times 3 \times 6$ ) outputs  $layer_2$  ( $28 \times 28 \times 6$ ) convolved with  $kernel_2$  ( $5 \times 5 \times 6 \times 10$ ) outputs  $layer_3$  ( $24 \times 24 \times 10$ ). notice there is no padding, so the width and height decreases, and between each layer we may perform some non linearity  $\sigma$

note that (when no padding is being made) the dimensions of the  $i^{th}$  tensor are fully determined by the dimensions of the kernel and the previous tensor  $i - 1$ :

$$w_i = w_{i-1} - w_{g,i-1} + 1, h_i = h_{i-1} - h_{g,i-1} + 1 \text{ and } c_i = c_{g,i-1}$$

this is a mere result of how convolution works.

## 4.3 Other Invariances

Setting aside time and space, it is easy to come up with more cases for which our model should be invariant to. For example, two images of a cat - one that is dark and one that is bright, should always be classified the same way. This means that I want my system to be invariant to changes in lighting. A good network might classify such cases properly without apriori intervention, but we can make it's life easier if we apply some smart preprocessing to our data, say normalize each pixel beforehand. As this may indeed work, it is more common to normalize one of the layers and not the images (or add a new layer that simply normalized its input). This process/layer is known as Local Response Normalization Layer, and is formally defined as the following mapping:

$$R_i(x) \rightarrow \frac{R_i(x)}{(c + \alpha \sum_j R_j(x))^\beta}$$

where  $c, \alpha, \beta$  are some predefined parameters, and  $R_i(x)$  represents some response of sample  $i$

another example is when we wish that our results would satisfy some relation - say we output  $(x, y)$ , but the only relevance is that  $(x, y)$  would be positioned on a circle defined by  $r = \sqrt{x^2 + y^2}$ . What we can do is map the predicted position  $(x, y)$  to  $\frac{(x,y)}{\sqrt{x^2+y^2}}$ , which formally means (in terms of

invariance) that all dots  $(x, y)$  on the same ray should land on the same point where the ray intersects the circle, and for that reason we add the normalization.

Lastly, there's also the case of projecting our data to values bigger than 0 that sums to one, this could be done using the mapping of our output vector  $x = (x_1, \dots, x_n)^T$  to  $\text{softmax}(x)$



## 5. Recurrent Neural Networks (RNN)

As of now, we only considered inputs that were fixed and predefined, that had similar dimension to the matrices that made up our architecture (either in a fully connected MLP or in Conv NN). Let us relax that assumption, and discuss new types of inputs.

### 5.1 Sequence signals

we consider Sequence signal as any signal that has the following properties

- the prediction at some time  $t$  depends on history (say, time  $t - \tau$ )
- the input may have varying length

for example, Google Translate receives its input as a sequence signal, as the translation of some sentence relays on the words that were previously inserted, and the sentence provided could be of any length.

### 5.2 Recurrent NN

the main idea would be that we can receive an input, perform some algorithm to it and then output it to a future version of our model, that would also be able to receive future input, as well as the prediction of the previous model. If so, for every time stamp  $t$ , we

- receive an input  $x_t$
- produce a prediction  $y_t$
- produce a memory hidden state  $h_t$  that can be transferred to future states

generally speaking, we can think of this model as some generic predictor  $h$  that is constantly receiving inputs and constantly predicts outputs, or (unfolding) as a sequence of predictors, as the below figure describes. This model is known Elman network notice that every input  $x_i$  and every output  $o_i$  are of constant size, but the number of such inputs and outputs is unknown, and isn't necessarily constant.

usually we will look only at the output at the very end, and back to our Google Translate example, this is intuitively clear as a portion of a sentence may not have any meaning up until it is finished.

we can formally write  $h_t$  as

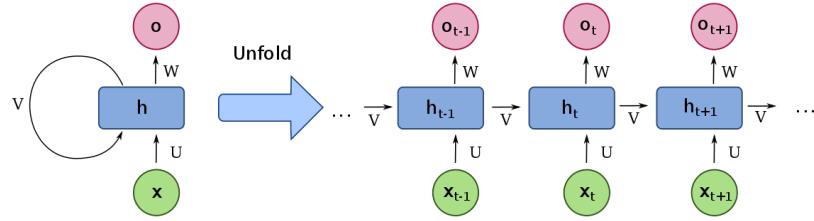


Figure 5.1: Elman-network diagram.  $U, V, W$  are some matrix that we multiply  $x_i, o_i$  and  $h_i$  with (aka the network parameters).

$$h_t = \sigma_h(U_h x_t + V_h h_{t-1} + b_h)$$

where  $W_h$  and  $U_h$  are some matrices that we learn in the iterative learning process (they are not different per  $t$ , as we do not know how big  $t$  is), and  $b_h$  is the bias. our prediction  $o_t$  would be

$$o_t = \sigma_y(W_y h_t + b_y)$$

In general notation we could also write  $h_t, y_t = N_\theta(h_{t-1}, x_t)$

### 5.2.1 RNN architectures

One key feature of Elman's network model is that we have the freedom to play with the number of inputs/outputs. In most cases we distinguish between the following designs

- one-to-one: the input and the output are of fixes size, this is similar to models we already saw before.
- one-to-many: fixed size input, and arbitrary size output. for example we input an image and output its textual representation
- many-to-one: exactly the opposite. for example - trying to predict stock price given many future values
- many-to-many: for example, translating Hebrew to English.

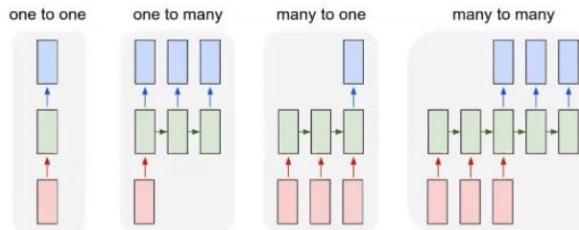


Figure 5.2: Various RNN architectures.

### 5.3 Back Propagation Through Time (BPTT)

In regular back-propagation, The loss was only a function of the weights and biases, and therefore the derivatives of the loss were with respect to the weights and biases. On the other hand, in RNN

there are more inputs to the loss, so we have to be careful when deriving with respect to the weights. Lets denote the model prediction and hidden state as

$$h_t, o_t = N(\theta, h_{t-1}, x_t)$$

that is, the network outputs both  $h_t$  and  $y_t$ . we also denote  $N_h = h$ ,  $N_o = o$  (as  $N$  outputs a tuple). from here, we may choose some lose function  $L$  (that could, for example take the squared differences between  $N$  and some labels  $y$ ), but regardless of how we chose  $L$ , there would be a tern in which we specifically derive the network's prediction, so let us focus on that term

$$\begin{aligned} \frac{\partial h_t}{\partial \theta} &= \frac{\partial}{\partial \theta} N_h(\theta, h_{t-1}, x_t) = \frac{\partial N_h}{\partial \theta} \frac{d\theta}{d\theta} + \frac{\partial N_h}{\partial h_{t-1}} \frac{\partial h_{t-1}}{\partial \theta} + \frac{\partial N_h}{\partial x_t} \frac{\partial x_t}{\partial \theta} = \\ &= \frac{\partial}{\partial \theta} N_h(\theta, h_{t-1}, x_t) + \frac{\partial}{\partial h_{t-1}} N_h(\theta, h_{t-1}, x_t) \frac{\partial h_{t-1}}{\partial \theta} \end{aligned}$$

where  $\frac{\partial x_t}{\partial \theta} = 0$  because the input is not a function of the network parameters. Now comes the tricky part, as we recursively expand the values of  $\frac{\partial h_t}{\partial \theta}$  for every  $t$  to achieve the final results

$$\frac{\partial h_t}{\partial \theta} = \frac{\partial}{\partial \theta} N_h(\theta, h_{t-1}, x_t) + \sum_{i=1}^{t-1} (\prod_{j=i+1}^t \frac{\partial}{\partial h_{j-1}} N_h(\theta, h_{j-1}, x_j)) \frac{\partial}{\partial \theta} N_h(\theta, h_{i-1}, x_i)$$

which can be written in shorter notation as

$$a_t = b_t + \sum_{i=1}^{t-1} (\prod_{j=i+1}^t c_j) b_i$$

where

$$a_t = \frac{\partial h_t}{\partial \theta}, b_t = \frac{\partial}{\partial \theta} N_h(\theta, h_{t-1}, x_t) \text{ and } c_t = \frac{\partial}{\partial h_{j-1}} N_h(\theta, h_{j-1}, x_j)$$

this gives us a way to derive the loss - suppose  $L = \frac{1}{T} \sum_{t=1}^T l(y_t, o_t)$ , for some loss function  $l$  (say,  $l = l_2$ ) and  $T$  samples. The gradient is given by

$$\frac{\partial L}{\partial \theta} = \frac{1}{T} \sum_{t=1}^T \frac{\partial l(y_t, o_t)}{\partial \theta} = \frac{1}{T} \sum_{t=1}^T \frac{\partial l(y_t, o_t)}{\partial o_t} \frac{\partial o_t}{\partial h_t} \frac{\partial h_t}{\partial \theta}$$

note that  $l$  is some loss function over  $o_t$ , so we can easily derive it with respect to  $o_t$ . Furthermore,  $o_t = \sigma_y(W_y h_t + b_y)$  so we can derive it with respect to  $h$  as well. the third factor is given from the calculation above, and this concluded what we wished to achieve.



Though the derivation of  $\frac{\partial h_t}{\partial \theta}$  from above can be computed fully, for large  $T$  this gets extremely long. There are some strategies for dealing with such problem, but we will not discuss them in the course

## 5.4 Vanishing/Exploding gradient problem

when we look at the term  $\prod_{j=i+1}^t c_j$ , it is reasonable to think that if  $c_j$  is relatively big, the product would converge exponentially. This will obviously cause a problem, as our gradient iteration will not bring us closer a good solution (in fact, it will probably won't provide any solution). On the contrary, if the term  $c_j$  is sufficiently small, we will face the opposite problem of vanishing gradient, and the network will work hard to converge. the problem described above isn't solely related to RNNs. For example, a very deep NN could encounter vanishing gradients. To minimize the effect, skip connection was introduced. This means, in general, that some input  $x$  (of some layer) would not be multiplied (or convolved) with the layers that follows him , and instead would be transferred directly deeper down the network blocks.

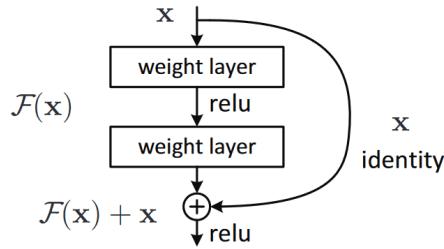


Figure 5.3: skip connection diagram

#### 5.4.1 First solution: Long-Short-Term-Memory (LSTM)

LSTM is an architecture that was introduced to try and reproduce the concept of skip connections in RNNs. the model itself is comprised of a concatenation of identical "cells", and each cell has a few components that are in charge of various applications

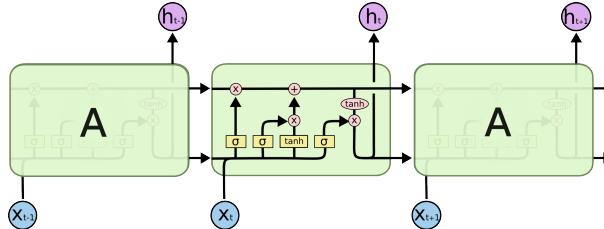
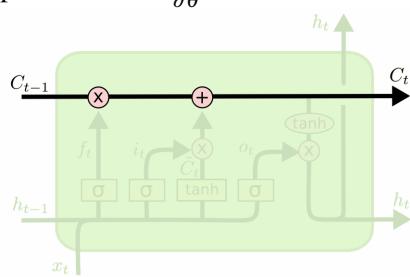


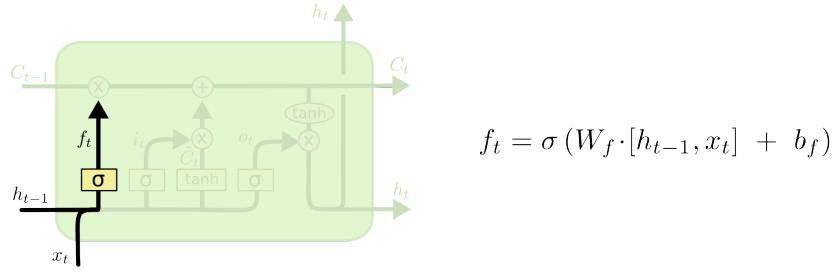
Figure 5.4: LSTM is a concatenation of cells

let us break the cell to its main components:

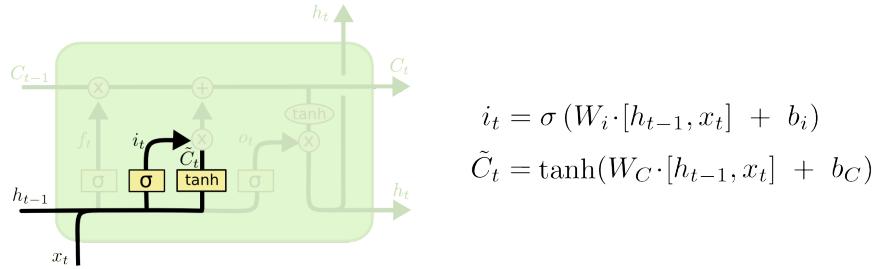
1. **Cell State:** represented with  $c_t$ , is the analog of the skip connection. The cell state is to be thought of as the "memory" that it passed along from previous cells. With that intuition in mind, notice that  $c_t$  can be multiplied by a number  $\in (0, 1)$  (output of some sigmoid), and this will allow us to control how much from past iterations we wish to "remember" (0 - forget the history and 1 let it pass fully). Finally, our  $c_t$  is added with "+" to our processing of  $h_{t-1}$  and  $x_t$ , which is important as we remember that when deriving with respect to our network parameters  $\theta$ ,  $\frac{\partial c_t}{\partial \theta} = 0$ , and therefore  $c_t$  will not contribute to vanishing gradients.



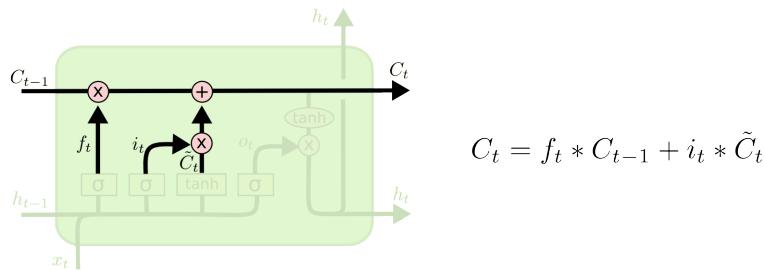
2. **Forget State:** both the current input  $x_t$  and the output from the previous cell  $h_{t-1}$  are inserted as input to a layer that outputs some value  $f_t = \sigma(W_f \cdot [h_{t-1}, x_t] + b_f) \in (0, 1)$  ( $[ \cdot, \cdot ]$  means concatenation).This value, as described before, is what chooses whether to "forget" or "remember"  $c_t$ .



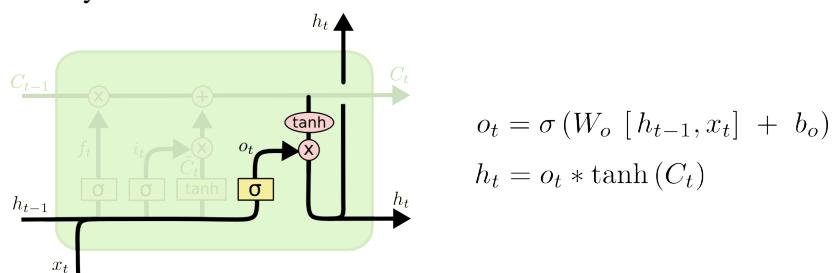
3. Update (Input) State: the current input and previous output can update the previous memory by adding to it new memory  $\tilde{C} = \tanh(W_C \cdot [h_{t-1}, x_t] + b_C)$ . Before  $\tilde{C}$  is added to the previous  $c$ , we also calculate  $i_t = \sigma(W_i \cdot [h_{t-1}, x_t] + b_i)$  which will provide a percentage for how much of  $\tilde{C}$  we wish to transfer. We use tanh as it allows us an update that is both addition and subtraction (as  $\tanh(x) \in [-1, 1]$ )



4. Forgetting and Updating Cell State: This state is responsible of updating the memory  $c$  given the outputs of the Update state and Forget state, that is it sets  $c_t = f_t \times c_{t-1} + i_t \times \tilde{C}_t$ , where  $\times$  is element-wise multiplication.



5. Output State: The last state is responsible for generating the output of the current cell,  $o_t$  and the hidden state  $h_t$ . specifically we have  $o_t = \sigma(W_o \cdot [h_{t-1}, x_t] + b_o)$  and  $h_t = o_t \times \tanh(c_t)$ , which indicates that the hidden state is some factor of the output with the previous layers memory.




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Provided an LSTM architecture, the problem of vanishing gradients greatly reduces, as the cell state can pass either a long or short memories without changing them, which is what skip connection is

all about. This allows us to build a deep model with elongated time dependencies. Do notice though that there are still assignments to the parameters of the LSTM that will result in vanishing gradient, and therefore the method isn't bullet proof.

The above could be concisely summarized to the following set of instructions, where notice we've added matrices  $U$ , which won't change the architecture (you could either use  $W \cdot [h_{t-1}, x_t]$  or  $Wh_{t-1} + Ux_t$ ):

**forget gate** - controls what is kept vs forgotten from previous cell states:

$$f_t = \sigma(W_f h_{t-1} + U_f x_t + b_f)$$

**input gate** - controls what parts of the new cell content are written to cell:

$$i_t = \sigma(W_i h_{t-1} + U_i x_t + b_i)$$

**output gate** - controls what parts of cells are output to hidden state:

$$o_t = \sigma(W_o h_{t-1} + U_o x_t + b_o)$$

**new cell content** - is the new content to be written to the cell:

$$\tilde{c}_t = \tanh(W_c h_{t-1} + U_c x_t + b_c)$$

**cell state** - forgets some content from last cell state and writes some new cell content

$$c_t = f_t \times c_{t-1} + i_t \times \tilde{c}_t$$

**hidden state** - outputs some content from the cell

$$h_t = o_t \times \tanh(c_t)$$

#### 5.4.2 Second solution: Gated Recurrent Unit (GRU)

The GRU is a recurrent unit that was introduced by Cho et al (2014), that has similar performance as the basic LSTM, though is more compact and has less inner parameters. Let us briefly go over its main components:

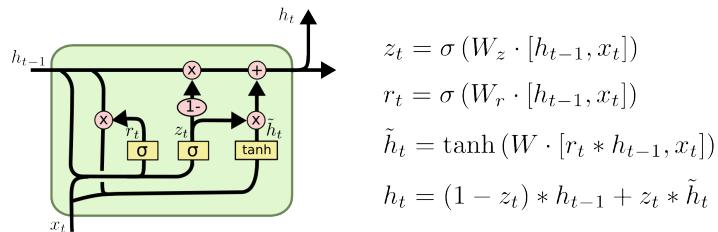


Figure 5.5: GRU cell

The first thing to notice is we do not use  $c_t$  any more - the hidden state  $h_t$  will hold the information that was previously attributed to  $c_t$ . More formally,  $h_t$  can be either updated (using  $z_t$ ) or rebooted (using  $r_t$ ) in the following manner: if we choose to reset,  $r_t$  will be set to  $(0, 0, \dots, 0)$  and the updated hidden state  $h_t$  will only be affected by the input of the current time stamp,  $x_t$ . Eventually the hidden state will be a factor of both the updated state  $\tilde{h}_t$  and the previous state  $h_{t-1}$ , as a sum with factoring by  $z_t$  - if  $z_t$  is closer to 1,  $h_t$  will be less similar to the previous  $h_{t-1}$ , and vice versa (see figure). Also notice that the update is performed using tanh, as to incorporate both addition and subtraction.



## 6. Attention Layers

As we dive deeper to the realm of deep neural networks, we are facing the realization that getting acquainted with the inductive bias of our network, that is, finding how differently our network tends to learn the data based on its specific architecture, is indeed a key component to the success of the network. A great example to that is the usage of CNNs instead of MLPs, as we had some prior knowledge about our data - we assumed it was LTI, and this allowed us to specifically build our model to use that prior knowledge. We can take this general concept to the next level, as in broader terms, we would also like our network to distinguish what is essential and what is bland, that is, equip our network with some way to divide its "attention" between samples that have significant meaning, and samples that do not.

In this chapter we will see how the intuition above was translated to what is called an **Attention Layer**

### 6.1 The problem with encoder-decoder models

encoder-decoder are a general term to models that have (no surprise here) two phases

- encoding - we receive an input (say, some sentence), and encode it to hidden states. This phase is finished when the input was fully processed (for example, when we have reached the end of the sentence)
- decoding - we decode the data possessed in the hidden layers to some meaningful output (for example, some translation of text)

The problem is that our input may very well be extremely long, with some complex dependencies inside of it, but still, at some point  $t$  in time we expect our encoder to give us some output vector  $o_t$  of constant length that should contain all that information. The problem here is evident, and indeed it was shown that large and complex inputs weren't successfully decoded to a meaning-full output (even when using GRU)

another example of the complexity of the problem arises when discussing the following learning task: given some image input, provide a textual representation to it. Why is this so difficult? consider an image of a girl throwing Frisbee. Even if the image is very clear, it may consist of various other

parts - for example, there ought to be some background (trees maybe?), etc.. Long story short, as we feed our input through some fancy deep vision CNN concatenated with an LSTM cell, we still output some vector that should in theory represent the entire image (and not only the girl throwing Frisbee), What we would have wanted is some way to tell the network which parts of the image are more relevant than others, aka, to which pixels should the network pay more attention

## 6.2 The solution - Attention Layer

Attention Layer is essentially a mapping/table, that given keys  $\{k_i\}$ , values  $\{v_i\}$  and a query  $q$

- compute the proximity  $d_i$  between  $q$  and every key  $k_i$
- apply softmax to define a probability  $\alpha_i$  that signifies the relevance of each value
- produce some soft prediction, mean value with respect to the distribution

several variants exists, and the simplest one is

$$d_i = \frac{\langle k_i, q \rangle}{\sqrt{N}}, \alpha_i = \text{softmax}(d_i) \text{ and } v_{out} = \sum_i \alpha_i v_i$$

where  $N$  is the size of the table

or shortly in matrix form, where  $q$  is a query vector,  $K$  is a matrix where each column is a vector of keys and  $V$  is a matrix where each column is a set of values:

$$v_{out} = V^T \text{softmax}(Kq) / \sqrt{N}$$

another option is to concatenate  $k_i$  and  $q$ , and add some learnable weights (a vector  $w$ , and a matrix  $W$ ). In that case  $d_i = w^T \tanh(W[k_i, q])$ , where we use tanh to get both positive and negative values. lastly, there's also the variant of

$$d_i = \frac{\langle W_K k_i, W_Q q \rangle}{\sqrt{N}}, \alpha_i = \text{softmax}(d_i) \text{ and } v_{out} = \sum_i \alpha_i W_V v_i$$

which is a common variant, that allows  $k$  and  $q$  to have different dimensions.

lets see how this layer can be added to the task of image captioning to provide significantly improved results:

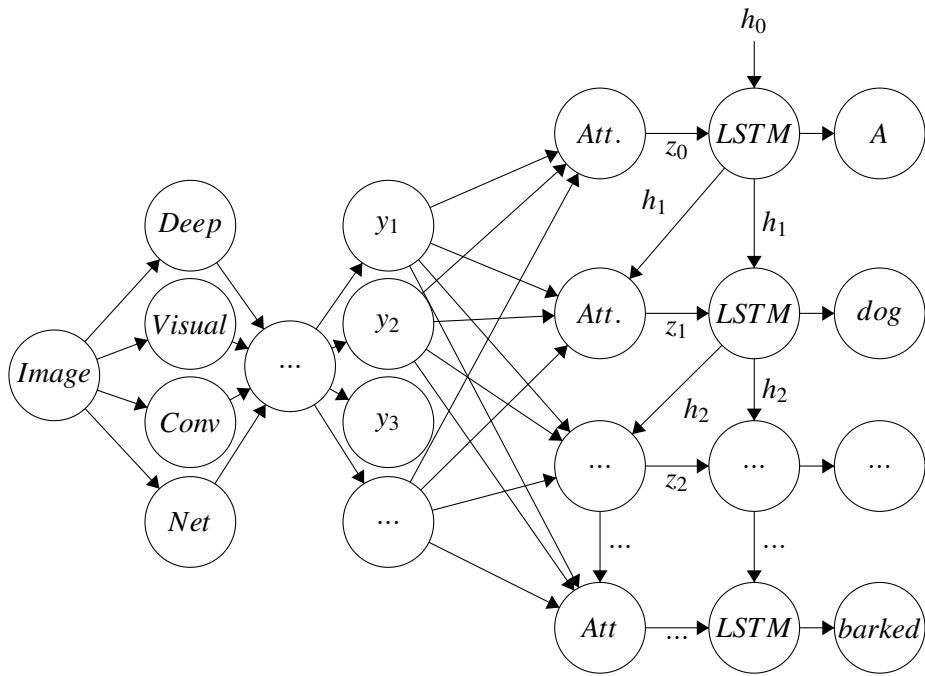
### 6.2.1 Image Captioning

based on the paper Show, Attend and Tell

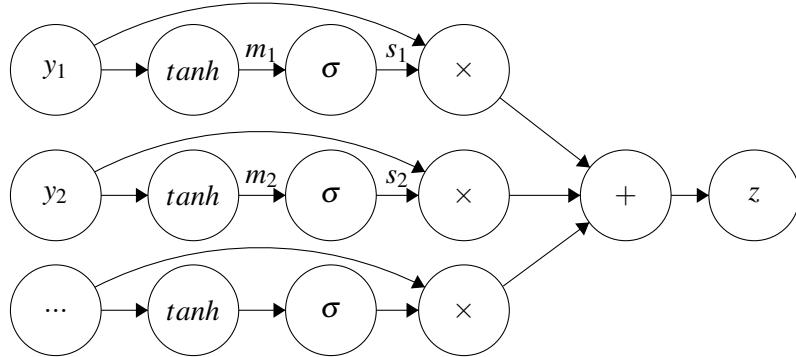
As a reminder, the task in hand is to provide meaningful caption to a given image. We faced the problem of encoder-decoder, in which some of the problems were that the encoding is being made on the entire image, without taking into consideration different regions that may be more relevant. In this approach, we will define attention layers that will do just that. Consider the following architecture (See figure)

- the input is fed to a Deep Vision CNN
- we zoom in on some (non-output) layer that outputs a vector  $y$
- every element  $y_i$  is to be considered both as key and as value, and is connected to Attention models that outputs some  $z_j$
- every  $z_j$  is fed as sequential input to an LSTM cell
- the LSTM cell outputs a hidden state  $h_i$  and some output (for example, a word), where  $h_i$  is inserted back to the Attention layer as a query  $q$ ,

this can be visualized as



where the Attention layer is as followed ( $\sigma \equiv softmax$ ):



Where in fact, the input to tanh is both  $y_i$  and the output of the  $LSTM$  cells, which is the hidden state (omitted from the diagram to avoid cluttering). Notice that every attention layer is provided with all  $\{y_i\}$ . under these notations, we have  $m_i = \tanh(y_i W_{y_i} + c W_c)$ , where  $c$  is really the hidden states memory  $h$ . we applied tanh so that very high/small values would have small differences. This is not obligatory, and we could also use inner product, as we have described in the previous section. these  $m$ 's are then fed to a softmax function  $s_i = e^{m_i} / \sum e^{m_i}$ , and finally, the output  $z$  is the expectation of  $y$  under the probabilities provided as  $s_i$  (was denoted  $\alpha_i$  before), i.e  $z = \sum s_i y_i$ . It is important to understand that the output of the CNN is provided without the notion of a future sentence that is to be constructed - it only gives us the information for the various key features of an image. the attention layer takes that data and combined with the LSTM cell provides a mapping between the features in the image and their appropriate positioning in a sentence

- R taking  $v_{out} = \sum \alpha_i v_i$  is known as soft Attention. on the other hand, hard Attention samples one of the  $v_i$  according to  $\alpha_i$ . This may be relevant when taking a single value makes more sense than averaging

**6.3 Neural Machine Translation**