Geophysical Fluid Dynamics

PH20101 Fluid Mechanics and Elasticity

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1 Introduction

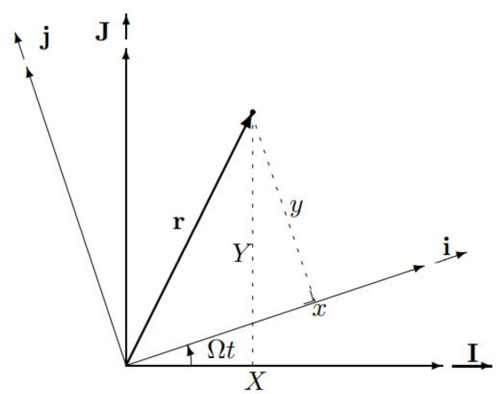
Geophysical fluid dynamics is concerned with all geophysical things, it deals with fluid phenomena such as Earth's interior,volcanoes,lava flows,ocean circulation and planetary atmospheres.One of the main features of geophysical fluids is the rotation of fluid due to planetary rotation.In this we will discuss the general equations used in geophysical fluid phenomenon and modelling.We discuss coriolis force and accelaration in 3D rotating frame like earth ,the equations used,some approximations used like Boussinesq approximation and the boundary conditions.

2 The Coriolis Force

The equations governing geophysical fluid process can be either described in an inertial frame of reference. But it makes more sense to describe them in the frame of reference attached to earth as the boundaries such as the ocean bed are at rest with respect to earth.To get a sense of the effect of rotation let's start by analysing a 2 dimensional rotating frame.

X and Y form the inertial frame of reference while x and y for the rotating frame of reference with same origin and rotating with an angular speed of Ω. $(\mathbf{I}, \mathbf{J})$ and $(\mathbf{i}, \mathbf{j})$ are the corresponding unit vectors. The position vector $\mathbf{r}$ is given by.

$$\mathbf{r} = X\mathbf{I} + Y\mathbf{J} = x\mathbf{i} + y\mathbf{j}$$



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$$\mathbf{i} = \mathbf{I}\cos\Omega t + \mathbf{J}\sin\Omega t$$
$$\mathbf{j} = -\mathbf{I}\sin\Omega t + \mathbf{J}\cos\Omega t$$
$$\mathbf{I} = \mathbf{i}\cos\Omega t - \mathbf{j}\sin\Omega t$$
$$\mathbf{J} = \mathbf{i}\sin\Omega t + \mathbf{j}\cos\Omega t$$

$$x = X\cos\Omega t + Y\sin\Omega t$$
$$y = -X\sin\Omega t + Y\cos\Omega t$$

On differentiating with respect to time

$$\frac{dx}{dt} = \frac{dX}{dt}\cos\Omega t + \frac{dY}{dt}\sin\Omega t + \Omega(-X\sin\Omega t + Y\cos\Omega t) = \frac{dX}{dt}\cos\Omega t + \frac{dY}{dt}\sin\Omega t + \Omega y$$

$$\frac{dy}{dt} = -\frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t - \Omega(X\cos\Omega t + Y\sin\Omega t) = -\frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t - \Omega x$$

$$\mathbf{u} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = u\mathbf{i} + v\mathbf{j}$$

$$\mathbf{U} = \frac{dX}{dt}\mathbf{I} + \frac{dY}{dt}\mathbf{J} =$$

$$\mathbf{U} = \left(\frac{dX}{dt}\cos\Omega t + \frac{dY}{dt}\sin\Omega t\right)\mathbf{i} + \left(-\frac{dX}{dt}\sin\Omega t + \frac{dY}{dt}\cos\Omega t\right) = U\mathbf{i} + V\mathbf{j}$$

Using the above equations we can easily say:

$$U = u - \Omega y \text{ and } V = v + \Omega x$$

Analogous calculations for acceleration $\mathbf{a} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{A} = A\mathbf{I} + B\mathbf{J}$ yields:

$$A = a - 2\Omega v - \Omega^2 x \text{ and } B = b + 2\Omega u - \Omega^2 y$$

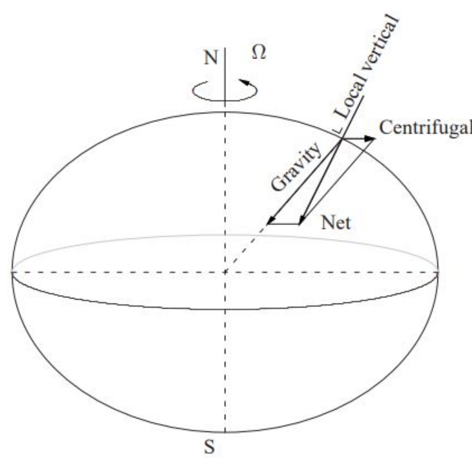
Rewriting these results in vector form (taking $\mathbf{\Omega} = \Omega\mathbf{k}$)

$$\mathbf{U} = \mathbf{u} + \mathbf{\Omega} \times \mathbf{r}$$

$$\mathbf{A} = \mathbf{a} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

2.1 The centrifugal force and its unimportance

The centrifugal force depends only on the rotation rate and the point's distance from the centre.Objects don't fly out into space because the gravitational force keeps everything together.When there is no rotation,gravitational force keeps everything together and forms a spherical body, but the centrifugal force due to rotation causes the distortion in the spherical shape and the planet assumes slightly a flattened shape.



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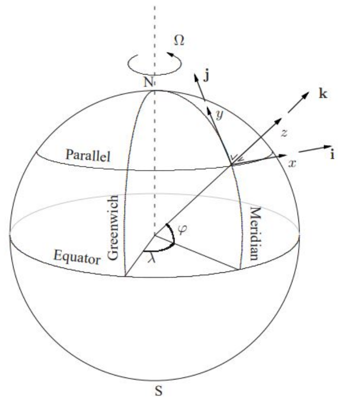
Gravity is directed towards centre while centrifugal force is outward,the resulting force is in intermediate direction,and this direction is perpendicular to the tangent at the surface of the flattened planet.In case of earth the distortion is very slight because gravity by far exceeds the centrifugal force.

2.2 Acceleration on a 3D rotating planet

Despite the previous discussion, the Earth can be assumed to be a perfect sphere for all practical purposes. The Cartesian coordinates on this sphere are traditionally represented as follows:



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Hence,

$$\boldsymbol{\Omega} = \Omega\cos\varphi\mathbf{j} + \Omega\sin\varphi\mathbf{k}$$

Subtracting the centrifugal component from absolute acceleration yields

$$\frac{d\mathbf{u}}{dt} + 2\boldsymbol{\Omega}\times\mathbf{u}$$

whose three components are

$$\frac{du}{dt} + 2\Omega\cos\varphi w - 2\Omega\sin\varphi v$$

$$\frac{dv}{dt} + 2\Omega\sin\varphi u$$

$$\frac{dw}{dt} - 2\Omega\cos\varphi u$$

Define

$$f = 2\Omega\sin\varphi$$

$$f_* = 2\Omega\cos\varphi$$

The coefficient f is called the Coriolis coefficient and f_* is called the reciprocal Coriolis coefficient. In the absence of centrifugal terms we get

$$\tfrac{du}{dt} - fv = 0 \text{ and } \tfrac{dv}{dt} + fu = 0$$

The general solution to this system is

$$u = A\sin(ft+\phi) \text{ and } v = A\cos(ft+\phi)$$

Hence, inertial oscillations on the surface of a planet have period $\frac{2\pi}{f} = \frac{\pi}{\Omega\sin\varphi}$.

3 Governing equations

The following equations describe the geophysical flow of a fluid :

The Continuity Equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

Navier Stokes Equations

$$\frac{Du}{Dt} + f_* w - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{Dw}{Dt} - f_* u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$, i.e., the material derivative.

Equation of State
In general the density of water in oceans depends on pressure, temperature and salinity. Continuing our assumption of incomprehensibility of fluids from the Navier Stokes equations, we consider ρ to be independent of pressure. Moreover, as a first order approximation, we assume

$$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)]$$

where T_0 and S_0 are the reference values about which ρ is expanded.

Conservation of Energy
From the First Law of Thermodynamics, we have

$$\frac{DU}{Dt} = Q - W$$

where U is the internal energy, Q is the rate of heat gain , and W is the rate of work done by pressure on the surrounding fluid. By definition,

$$\frac{DU}{Dt} = C_v\frac{DT}{Dt}$$

Fourier's law of heat conduction gives

$$Q = \frac{k_T}{\rho}\nabla^2 T$$

and rate of Work can be considered to be (pressure)×(rate of change of volume per unit mass, per unit time)

$$\Rightarrow W = p\frac{d\xi}{dt}$$

where $\xi = \frac{1}{\rho}$. Replacing these values in the energy equation-

$$C_v\frac{DT}{Dt} = \frac{k_T}{\rho}\nabla^2 T - p\frac{d\xi}{dt} = \frac{k_T}{\rho}\nabla^2 T + \frac{p}{\rho^2}\frac{d\rho}{dt}$$

Simplifying further using the continuity equation-

$$\rho C_v\frac{DT}{Dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = k_T\nabla^2 T$$

This is the equation that governs the variation in Temperature.

Salt Budget The total salt in a oceanic fluid packet is conserved. Hence,

$$\frac{dS}{dt} = k_S\nabla^2 S$$

3.1 Boussinesq approximation

For the purpose of geophysical fluid we use some simplifications without much loss of accuracy and it's called the Boussinesq approximation.

The density of the fluid varies around a mean value in most geophysical system but the extent to which it varies is very very less ,for example the density of water is 1000 in SI units and the variation is less than 5.The variation of air in atmosphere is great,it's maximum at ground level and becomes nearly 0 at great heights,however this happens mainly due to hydrostatic pressure effects,in troposphere the variations is less than 5

$$\rho = \rho_0 + \rho'(x, y, z, t),\ |\rho| << \rho_0$$

substituting this in the equation of continuity we get

$$\rho_0(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \rho'(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + (\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \rho'}{\partial z}) = 0$$

since ρ' is much lesser than ρ_0 so we retain only the first term ,so we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

similarly we can ignore containing ρ' whenever it's in the shadow of ρ_0 so the x and y momentum eqn becomes

$$\frac{Du}{Dt} + f_* w - fv = \frac{-1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + fu = \frac{-1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

but it's slightly different for z momentum equation.

$$\frac{\partial w}{\partial t} - f_* u = \frac{-1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu\nabla^2 w$$

4 Geophysical flows

4.1 Reynolds averaged equation

We decompose each variable into mean and a fluctuation component

$$u = \langle u \rangle + u'$$

where $\langle \cdot \rangle$ denotes the temporal average over rapid turbulent fluctuations in the fluid.
In general we define $\langle u' \rangle = 0$. Averages of quadratic expressions are written as

$$\langle uv \rangle = \langle \langle u \rangle \langle v \rangle \rangle + \langle \langle u \rangle v' \rangle + \langle u' \langle v \rangle \rangle + \langle u'v' \rangle$$

$$\Rightarrow \langle uv \rangle = \langle u \rangle \langle v \rangle + \langle u'v' \rangle$$

The objective of Reynolds's approximation is to express the governing equations in terms of mean quantities. Hence, the x-component of Navier Stokes equation:

$$\frac{\partial \langle u \rangle}{\partial t} + \frac{\partial(\langle u \rangle \langle u \rangle)}{\partial x}\frac{\partial(\langle u \rangle \langle v \rangle)}{\partial y} + \frac{\partial(\langle u \rangle \langle w \rangle)}{\partial z} + f_* \langle w \rangle - f\langle v \rangle = -\frac{1}{\rho_0}\frac{\partial \langle p \rangle}{\partial x} + \mu\nabla^2\langle u \rangle - \frac{\partial \langle u'u' \rangle}{\partial x} - \frac{\partial \langle u'v' \rangle}{\partial y} - \frac{\partial \langle u'w' \rangle}{\partial z}$$

We can combine the last three additional terms with the viscosity term in the following manner:

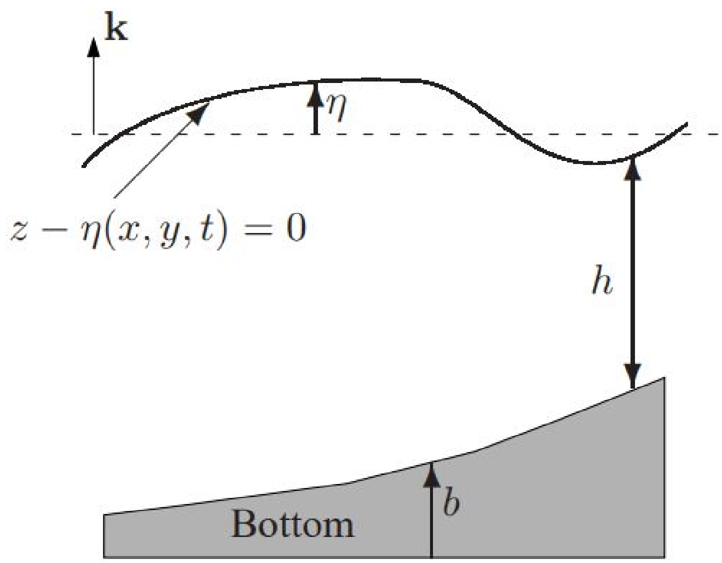
$$\frac{\partial}{\partial x}\left(\nu\frac{\partial \langle u \rangle}{\partial x} - \langle u'u' \rangle\right),\ \frac{\partial}{\partial y}\left(\nu\frac{\partial \langle u \rangle}{\partial y} - \langle u'v' \rangle\right),\ \frac{\partial}{\partial x}\left(\nu\frac{\partial \langle u \rangle}{\partial z} - \langle u'w' \rangle\right)$$

From this, we can infer that the averages of the quadratic velocity fluctuations add to the viscous stress in the fluid. Thus, these averages are also known as Reynolds Stresses.
Similar combinations can be made for the other two components as well.

4.2 Boundary conditions

The partial differential equations governing a geophysical flow forms a closed set of equations, the number of unknowns is equal to the number of equations.But the solution of them is uniquely defined only when information like initial condition and boundary conditions are defined.The boundary conditions in geophysical flows are defined by the geography of the boundary.
One of the most important condition independent of any physical property is that air and water flows doesn't penetrate solid boundary,this is called the no penetration condition.Mathematically the velocity must be tangential to the boundary.(normal vector to the boundary and the velocity vector are perpendicular to each other).At a free surface also it is similar to the above situation but the boundary will be moving with the fluid.
Another important boundary condition is implicating forces,pressure called the dynamical conditions are necessary, example is the requiring continuity of pressure at the sea air interface.

$$p_atm = p_sea$$



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If the sea surface elevation is η the continuity of the pressure is at the actual surface z=η

$$p_sea(z = 0) = p_atm + \rho_0 g\eta$$

Another boundary condition depends on wether the fluid is viscous or not.If it is viscous then the fluid particle next to fixed boundary must have 0 velocity.Boundary layer is the distance over which the velocity falls to 0.This short distance restricts the influence of friction to a narrow band of fluid.If the boundary layer if neglibible then we can ignore the frictional effects in the momentum equations.Slip between the boundary and the fluid is allowed in this casem,but if we consider viscosity then 0 velocity must be imposed at a fixed boundary.If there is a moving boundary between 2 fluids continuity of both velocity and tangential stress are required.