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2.1 (a) given $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

we have:

$\det(\lambda I - A) = 0$

$\Rightarrow \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\right) = 0$

$\Rightarrow \det\left(\begin{bmatrix} \lambda - 4 & -2 \\ -1 & \lambda - 3 \end{bmatrix}\right) = 0$

$\Rightarrow (\lambda - 4)(\lambda - 3) - 2 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4\lambda + 12 - 2 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 10 = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm 3}{2}$$

So the corresponding eigenvalues are $\lambda = 5, 2$

at $\lambda = 5$

$Av = \lambda v$

$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\Rightarrow -v_1 + 2v_2 = 0$

$\Rightarrow v_1 - 2v_2 = 0$

so $v_1 = 2$ and $v_2 = 1$ is an eigenvector.

ie $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

at $\lambda = 2$

$Av = \lambda v$

$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\Rightarrow 4v_1 + 2v_2 = 2v_1$

$\Rightarrow v_1 + 3v_2 = 2v_2$

at $v_1 = -1$ and $v_2 = +1$, the eigenvector will be $\begin{bmatrix} -1 \\ +1 \end{bmatrix}$

2.1 (b) we have

$\Rightarrow Av = \lambda v$

$\Rightarrow A^{-1}Av = A^{-1}\lambda v$

$\Rightarrow v = A^{-1}\lambda v$

$\Rightarrow A^{-1}\lambda v = v$

$\Rightarrow A^{-1}v = \frac{1}{\lambda}v$

2.1 (c) we have

$\Rightarrow ABv = \lambda v$

$\Rightarrow BABv = B\lambda v$

$\Rightarrow BABv = \lambda Bv$

Let $x = Bv$

$\Rightarrow BAx = \lambda x$

hence BA has the same eigenvalue λ as AB.

2.2 (a) we have;

$\Rightarrow f(x) = w^T x$

$\Rightarrow f(x) = \sum_{i=1}^{n} w_i x_i$

$$\Rightarrow \nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = w$$

2.2 (b) we have;

$\Rightarrow f(x) = x^T A x$

$\Rightarrow f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ji} x_j x_i$

partial derivative of $f(x)$ w.r.t x_k

at $k = i \neq j$: $\frac{\partial f}{\partial x_k} = \sum_{j \neq k} a_{jk} x_j$

at $k = j \neq i$: $\frac{\partial f}{\partial x_k} = \sum_{i \neq k} a_{ki} x_i$

at $k = i = j$: $\frac{\partial f}{\partial x_k} = 2a_{kk} x_k$

combining them we get.

$$\Rightarrow \frac{\partial f}{\partial x_k} = \sum_{j=1}^{n} a_{jk} x_j + \sum_{i=1}^{n} a_{ki} x_i$$

$$\Rightarrow \frac{\partial f}{\partial x_k} = A_{k_{col}}^T x + A_{k_{row}} x$$

$$\Rightarrow \frac{\partial f}{\partial x_k} = A_{k_{row}} x + A_{k_{col}}^T x$$

where

→ $A_{k_{col}}$ represents the k^{th} column of A matrix.

→ $A_{k_{row}}$ represents the k^{th} row of A matrix.

thus, stacking up components from all the derivatives we will have

$\Rightarrow \nabla_x f(x) = Ax + A^T x$

2.2 (c) we have;

$$\Rightarrow f(x) = \|Bx\|_2^2 = \left[\sqrt{\left(\sum_j (B_{1j} x_j)\right)^2 + \left(\sum_j (B_{2j} x_j)\right)^2 + \dots \left(\sum_j (B_{nj} x_j)\right)^2}\,\right]^2$$

$$\Rightarrow f(x) = \sum_i^n \left(\sum_j^n B_{ij} x_j\right)^2$$

differentiating $f(x)$ w.r.t k^{th} component of x i.e x_k

$$\Rightarrow \frac{\partial f}{\partial x_k} = \sum_i^n 2\sum_j B_{ij} x_j B_{ik}$$

$$\Rightarrow \frac{\partial f}{\partial x_k} = 2\sum_i^n\sum_j^n B_{ik} B_{ij} x_j$$

combining derivatives w.r.t all the parameter of x i.e from x_1 to x_n we get.

$\Rightarrow \nabla_x f = 2B^T B x$

2.2 (d) we have;

$\Rightarrow f(x) = \|Bx - c\|_2^2$

$$\Rightarrow f(x) = \sum_i^n \left(\sum_j^n B_{ij} x_j - c_i\right)^2$$

differentiating w.r.t the k^{th} component of x i.e x_k.

$$\Rightarrow \frac{\partial f(x)}{\partial x_k} = 2\sum_i^n \left(\sum_j^n B_{ij} x_j - c_i\right) B_{ik}$$

$$\Rightarrow \frac{\partial f(x)}{\partial x_k} = 2\sum_i^n B_{ik}\left(\sum_j^n B_{ij} x_j - c_i\right)$$

$$\Rightarrow \frac{\partial f(x)}{\partial x_k} = 2\sum_i^n \left(\sum_j^n B_{ik} B_{ij} x_j - B_{ik} c_i\right)$$

now combining derivative w.r.t all the parameters of x i.e x_1 to x_n we get

$\Rightarrow \nabla_x f(x) = 2B^T(Bx - c)$