Word-representable graphs. The basics

Sergey Kitaev

University of Strathclyde

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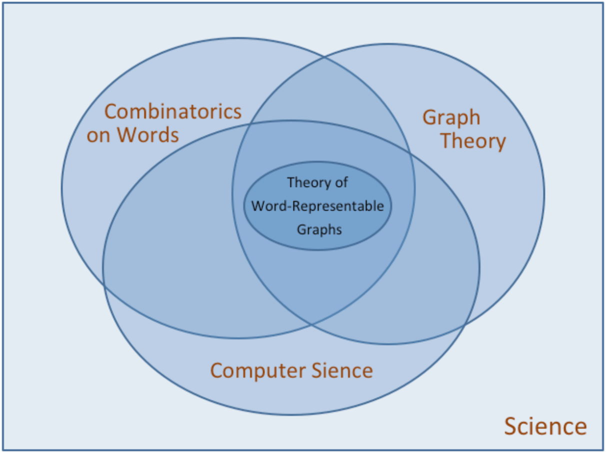
What is this about?



Motivation

- Study of the Perkins semigroup (original motivation) — Algebra

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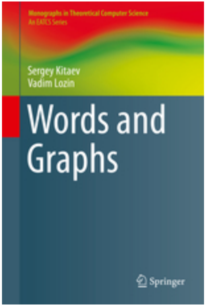
- Just fun — Human Science

Literature



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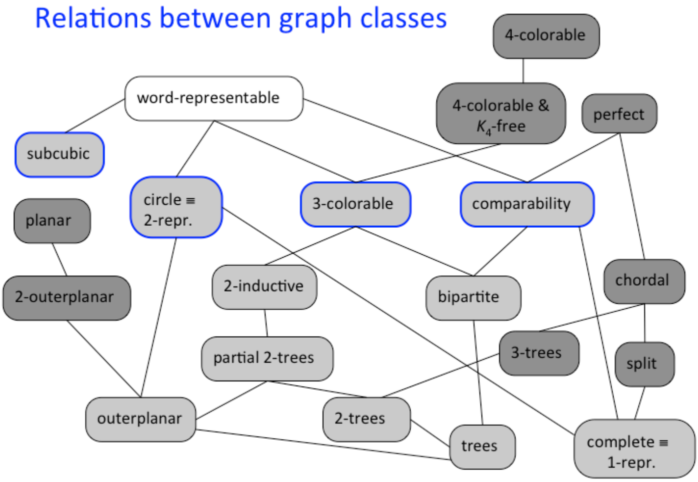
Relations between graph classes



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Any consecutive letters in a word generate a factor of the word. All different factors of the word 113 are 1, 3, 11, 13 and 113.

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Note that removing all letters but 5 and 6 we obtain 56 showing that the letters 5 and 6 alternate (by definition).

Basic definitions

All graphs considered by us are simple (no loops, no multiple edges).

Word-representable graph

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y, $x \neq y$, alternate in w **if and only if** $xy \in E$. (w **must** contain **each** letter in V)

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Remark

We deal with **unlabelled graphs**. However, to apply the definition, we need to label graphs. Any labelling of a graph is **equivalent** to any other labelling because letters in w can always be renamed.

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Remark

The class of word-representable graphs is hereditary. That is, removing a vertex v in a word-representable graph G results in a word-representable graph G'. Indeed, if w represents G then w with v removed represents G'.

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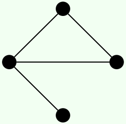
Example

The graph  is word-representable.

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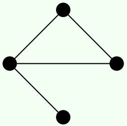
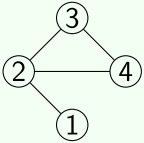
The graph  is word-representable.

Indeed, 

can be represented by 1213423.

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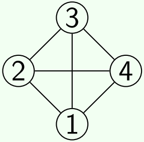
Example: representing complete graphs and empty graphs



can be represented by 1234 or 12341234.
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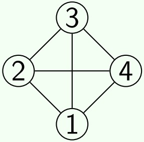

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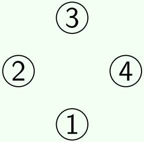

can be represented by 12344321 or 11223344.

k-representability and graph's representation number

Uniform word

k-uniform word = **each** letter occurs k times
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k-representability implies $(k+1)$-representability.

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- **5**34253412132154 - a uniform word-representant for the same graph.

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k-representability and graph's representation number

Graph's representation number

Graph's representation number is the **least** k such that the graph is k-representable. By a theorem above, this notion is well-defined for word-representable graphs. For non-word-representable graphs, we let $k = \infty$.

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Also, let $\mathcal{R}_k = \{G : \mathcal{R}(G) = k\}$.

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Observation

$\mathcal{R}_1 = \{G : G \text{ is a complete graph}\}$.

Graphs with representation number 2

Empty graphs

If G is an empty graph on **at least two** vertices then $\mathcal{R}(G) = 2$.

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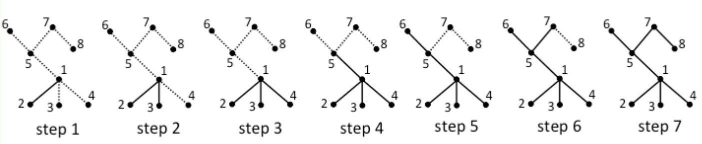
Trees

Trees on **at least three** vertices belong to $\mathcal{R}_2$. The idea of a simple inductive proof is shown for the tree in "step 7" below.



Step 1	$w_1 = 1212$
Step 2	$w_2 = 123132$
Step 3	$w_3 = 12341432$
Step 4	$w_4 = 5152341432$
Step 5	$w_5 = 656152341432$
Step 6	$w_6 = 65617572341432$
Step 7 (the tree itself)	$w_7 = 6561758782341432$

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Lemma

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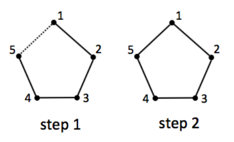
Cycle graphs on **at least four** vertices belong to $\mathcal{R}_2$. E.g. see C_5:



■ As step 1, remove the edge 15 and represent the resulting tree as 1213243545.

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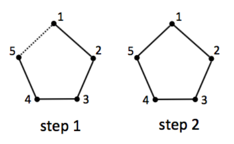

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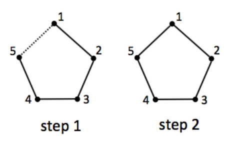


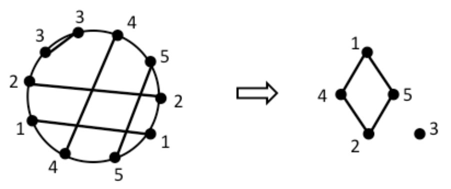
■ As step 1, remove the edge 15 and represent the resulting tree as 1213243545.

■ Make one letter cyclic shift (moving the last letter): 5121324354.

■ Swap the first two letters to obtain a word-representant for C_5: 1521324354.

Characterization of graphs with representation number 2

Circle graphs



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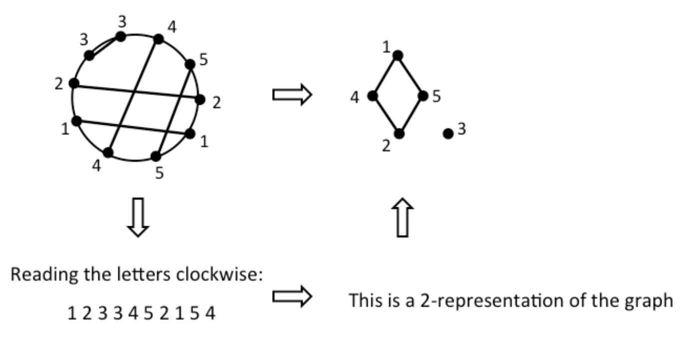

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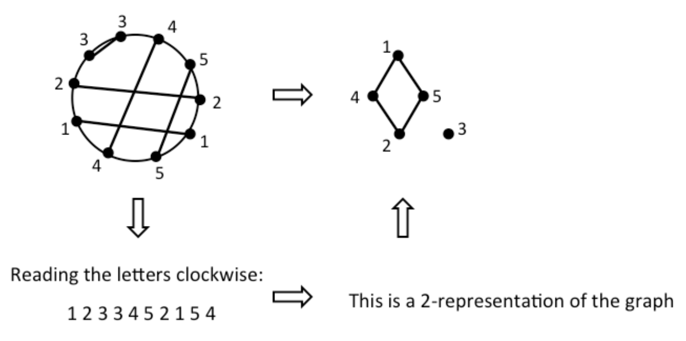
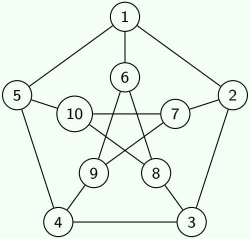


Theorem (Halldórsson, SK, Pyatkin; 2011)

For a graph G **different from** *a complete graph, $\mathcal{R}(G) = 2$* **iff** *G is a circle graph.*

Graphs with representation number 3

Petersen graph



Two **non-equivalent** 3-representations (by Konovalov and Linton):

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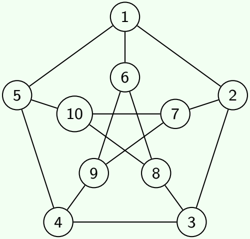

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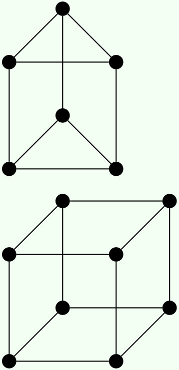
Theorem (Halldórsson, SK, Pyatkin; 2010)

Petersen graph is **not** *2-representable.*

Graphs with representation number 3

Prisms



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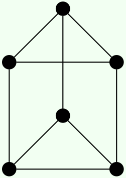

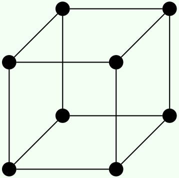

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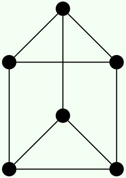

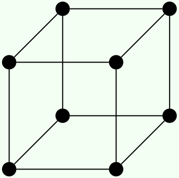

Theorem (SK, Pyatkin; 2008)

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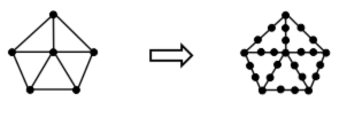
Theorem (SK; 2013)

None *of the prisms is 2-representable.*

Graphs with representation number 3

Subdivisions of graphs



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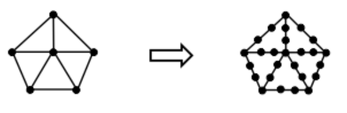

Theorem (SK, Pyatkin; 2008)

3-subdivision of **any** *graph is 3-representable. In particular, for* **every** *graph G there exists a 3-representable graph H that contains G as a minor.*

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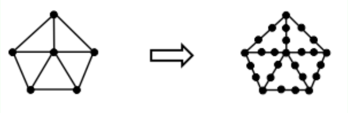

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Remark

In fact, **any** subdivision of a graph is 3-representable as long as **at least two new vertices** are added on each edge.

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- Are all graphs word-representable?

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- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph's representation number for a given graph? Essentially, what is the minimal length of a word-representant?
- How hard is it to decide whether a graph is word-representable or not? (complexity)

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- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph's representation number for a given graph? Essentially, what is the minimal length of a word-representant?
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- Which graphs are word-representable in your favourite class of graphs?

Comparability graphs

Transitive orientation

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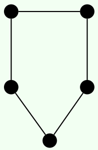

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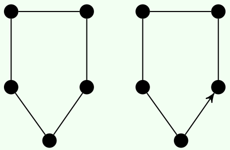

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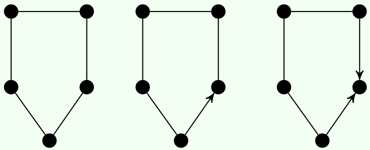

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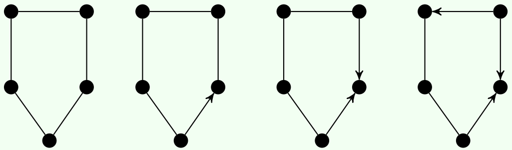

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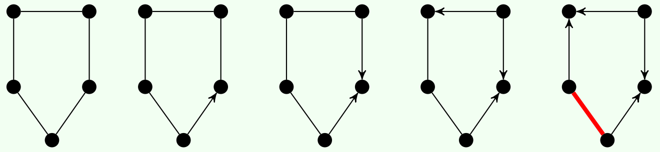

Permutationally representable graphs

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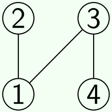

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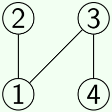

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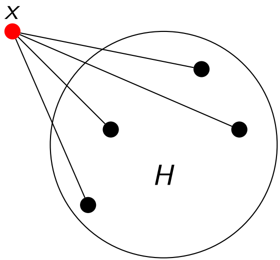
Theorem (SK, Seif; 2008)

A graph is permutationally representable **iff** *it is a comparability graph.*

Significance of permutational representability

The graph G below is obtained from a graph H by adding an all-adjacent vertex (apex):

$G =$ 

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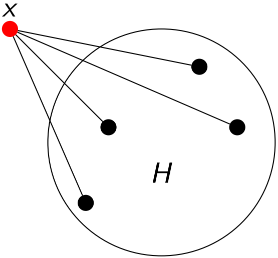
$G =$ 

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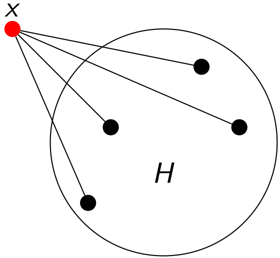
$G =$ 

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Theorem (SK, Pyatkin; 2008)

G is word-representable $\Rightarrow$ the neighbourhood of each vertex is **permutationally representable** *(is a* **comparability graph***).*

Converse to the last theorem is not true

Theorem (Halldórsson, SK, Pyatkin; 2010)

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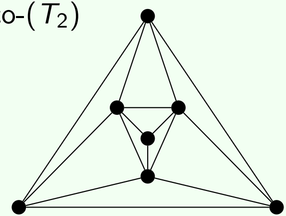
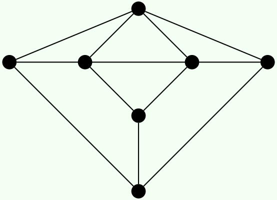
Minimal counterexamples

co-(T_2)





Maximum clique problem on word-representable graphs

Maximum clique

A clique in an undirected graph is a subset of pairwise adjacent vertices. A maximum clique is a clique of the **maximum size**.

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Given a graph G, the Maximum Clique problem is to find a **maximum clique** in G.

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Remark

The Maximum Clique problem is NP-complete.

Maximum clique problem on word-representable graphs

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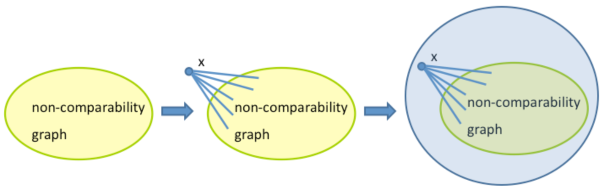
■ Each neighbourhood of a word-representable graph G is a comparability graph.
■ The Maximum Clique problem is known to be solvable on comparability graphs in polynomial time.
■ Thus the problem is solvable on G in polynomial time, since any maximum clique belongs to the neighbourhood of a vertex including the vertex itself.

□

Non-word-representable graphs

A general construction via adding an apex



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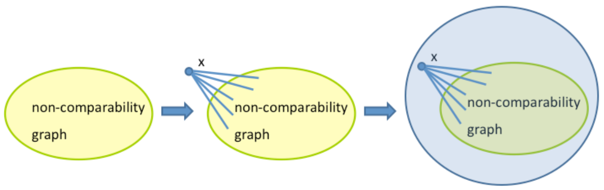

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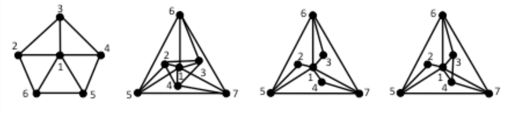

The wheel graph W_5 (to the left) is the **smallest** non-word-represnetable graph. It is the only such graph on 6 vertices.

Odd wheels

Observation

The cycle graphs C_{2k+1} for $k \geq 2$ are **non-comparability graphs** $\Rightarrow$ the odd wheels W_{2k+1} for $k \geq 2$ are **non-word-representable**.

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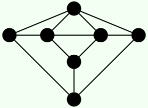
Observation

The wheel graph W_5 is **non-word representable** $\Rightarrow$ **almost all** graphs are **non-word-representable** (since almost all graphs contain W_5 as an induced subgraph).

Non-word-representable graphs

Non-word-representable graphs of maximum degree 4

The **minimal non-comparability** graph is on 5 vertices, and thus the construction of non-word-representable graphs above gives a graph with a **vertex of degree at least 5**. Collins, SK and Lozin showed non-word-representability of 

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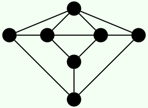


Triangle-free non-word-representable graphs

Adding an apex to a **non-empty** graph gives a graph containing a triangle. Are there any triangle-free non-word-representable graphs?

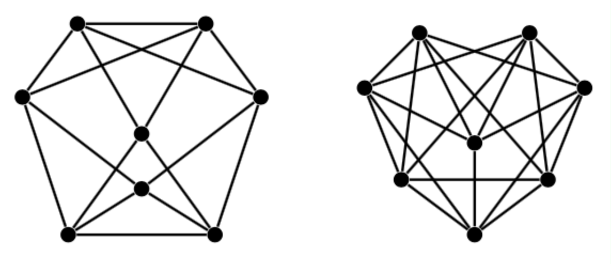
Theorem (Halldórsson, SK, Pyatkin; 2011)

There exist triangle-free non-word-representable graphs.

Non-word-representable graphs

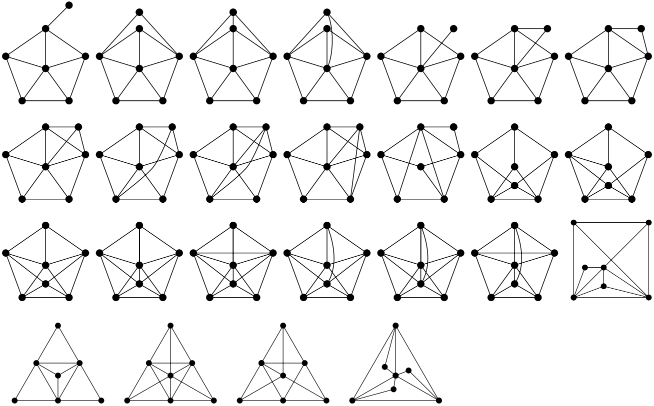
Regular non-word-representable graphs

A regular graph is a graph having degree of each vertex the **same**. It was found out by Herman Chen that the **smallest regular** non-word-representable graphs are on 8 vertices.



All 25 non-word-representable graphs on 7 vertices

The following picture was created by Herman Chen.



Ozgur Akgun, Ian Gent, Chris Jefferson found the number of non-word-representable graphs on up to 10 nodes: 1, 25, 929, 68545, 4880093 (ca 42% of all connected graphs)

Asymptotic enumeration of word-representable graphs

Theorem (Collins, SK, Lozin; 2017)

The number of n-vertex word-representable graphs is $2^{\frac{n^2}{3}+o(n^2)}$.

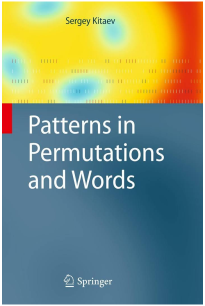
Proof.

Proof idea: Apply to the case of word-representable graphs Alekseev-Bollobás-Thomason Theorem related to asymptotic growth of **every** hereditary class. Details are skipped due time constraints, but they can be found here:

Collins, Kitaev, Lozin. New results on word-representable graphs. *Discr. Appl. Math.* 216 (2017) 136–141.

□

Word-representants avoiding patterns



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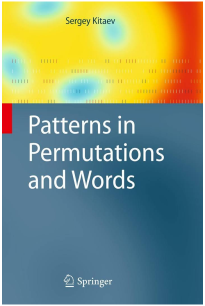

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Merging two areas of research

In the context of word-representable graphs, which graphs can be represented if we require that word-representants **must avoid** a given **pattern** or a **set of patterns**.

Word-representants avoiding patterns

A trivial example

Describe graphs representable by words avoiding the pattern 21.

Solution: Clearly, any 21-avoiding word is of the form

$$w = 11\cdots122\cdots2\cdots nn\cdots n.$$

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Papers in this direction

- Gao, Kitaev, Zhang. On 132-representable graphs. arXiv:1602.08965 (2016)
- Mandelshtam. On graphs representable by pattern-avoiding words. arXiv:1608.07614. (2016)

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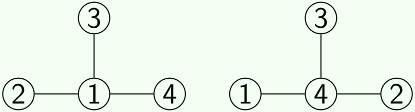

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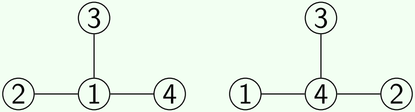

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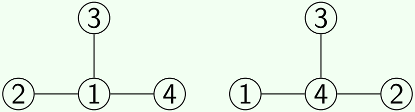

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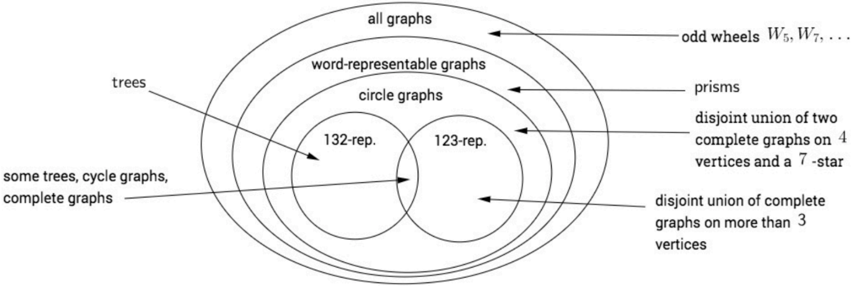

Word-representants avoiding patterns

Examples of simple, but useful general type results:

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Let G be a word-representable graph, which can be represented by a word avoiding a pattern τ of length $k+1$. Let x be a vertex in G such that its degree $d(x) \geq k$. Then, **any** *word w representing G that avoids τ must contain no more than k copies of x.*

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Corollary (Mandelshtam, 2016)

Let w be a word-representant for a graph which avoids a pattern of length $k+1$. If some vertex y adjacent to x has degree at least k, then x occurs **at most** $k+1$ *times in w.*

Semi-transitive orientations as the main tool in the theory of word-representable graphs discovered so far

Sergey Kitaev

University of Strathclyde

April 21, 2017

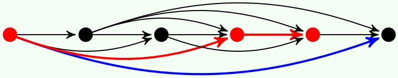
Semi-transitive orientations

Shortcut

A shortcut is an oriented graph that

- is **acyclic** (that it, there are **no directed cycles**);
- has **at least 4 vertices**;
- has **exactly one** source (no edges coming in), **exactly one** sink (no edges coming out), and a **directed path** from the source to the sink that goes through **every** vertex in the graph;
- has an edge connecting the **source** to the **sink**;
- is **not** transitive (that it, there exist vertices u, v and z such that $u \to v$ and $v \to z$ are edges, but there is **no** edge $u \to z$).

Example of a shortcut



The part of the graph in red shows **non-transitivity**. There are **two other violations** of transitivity.

Semi-transitive orientations

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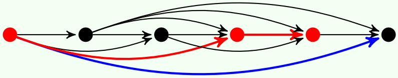


The part of the graph in red shows **non-transitivity**. There are **two other violations** of transitivity.

The blue edge, from the **source** to the **sink**, justifies the name "shortcut" for this type of graphs. Indeed, **instead** of going through the **longest directed path** from the **source** to the **sink**, we can shortcut it by going directly through the single edge. The blue edge is called shortcutting edge.

Semi-transitive orientations

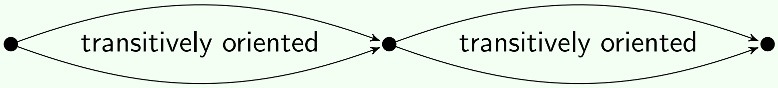
Semi-transitive orientation

An orientation of a graph is semi-transitive if it is

- **acyclic**, and
- **shortcut-free**.

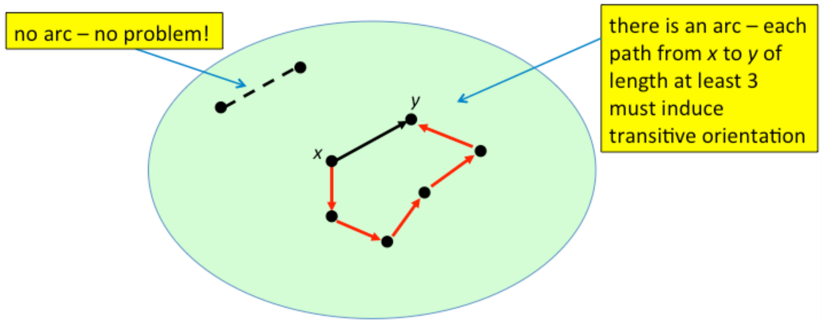
Remark

Any transitive orientation is necessary **semi-transitive**. The converse is **not** true, e.g. the schematic semi-transitively oriented graph below is **not** transitively oriented:



Thus semi-transitive orientations **generalize** transitive orientations.

Checking if a given acyclic orientation is semi-transitive



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Finding a semi-transitive orientation

- Pick **any** edge and orient it **arbitrarily**.

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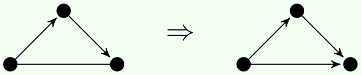
- After that repeat the following procedure: pick an edge connected to an already oriented edge and **branch** the process by orienting it in one way and the other way assuming such an orientation does **not** introduce a cycle or a shortcut. E.g. **no** branching is required for the following situation:



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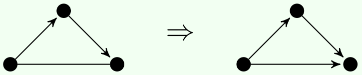
Finding a semi-transitive orientation

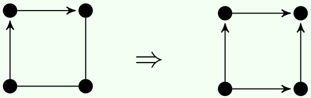
- Pick **any** edge and orient it **arbitrarily**.
- After that repeat the following procedure: pick an edge connected to an already oriented edge and **branch** the process by orienting it in one way and the other way assuming such an orientation does **not** introduce a cycle or a shortcut. E.g. **no** branching is required for the following situation:



- The process can normally be shorten by e.g. completing orientation of **quadrilaterals** as shown on **next slide**, which is unique to **avoid** cycles and shortcuts.

Semi-transitive orientations

Finding a semi-transitive orientation



Semi-transitive orientations

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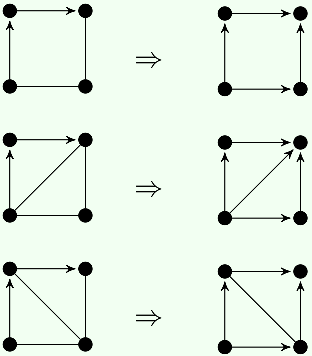

The **diagonal** in the last case **may require** branching.

A key result in the theory of word-representable graphs

Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

A graph G is word-representable **iff** *G admits a semi-transitive orientation.*

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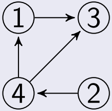
"$\Rightarrow$" **Proof idea:** Given a word, say, $w = 2421341$, orient the graph represented by w by letting $x \to y$ be an edge if the **leftmost** x is to the left of the **leftmost** y in w, to obtain a **semi-transitive orientation**:



□

The shortest length of a word-representant

An upper bound on the length of a word-representant

Any **complete graph** is 1-representable. The algorithm turning semi-transitive orientations into word-representants gave:

Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

Each **non-complete** *word-representable graph G is $2(n - \kappa(G))$-representable, where $\kappa(G)$ is the size of the maximum clique in G.*

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The recognition problem of word-representability is in **NP**. Indeed, any word-representant is of length at most $O(n^2)$, and we need $O(n^2)$ passes through such a word to check alternation properties of **all** pairs of letters.

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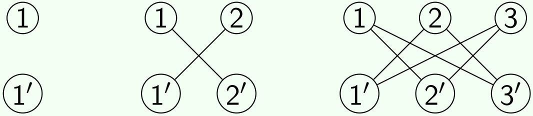
Graphs requiring long word-representants

Crown graph (Cocktail party graph)

Crown graph $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.



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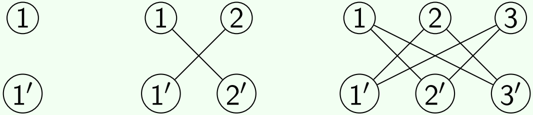

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Word-representability of crown graphs

$H_{n,n}$ is a **comparability graph** $\Rightarrow$ it is **permutationally representable**. In fact, $H_{n,n}$ requires n permutations to be represented. Can $H_{n,n}$ be represented in a shorter way if not to require permutational representability? E.g. $H_{3,3}$ is **2-representable**, while $H_{4,4}$ is 3-dimensional cube (a prism) and is **3-representable**.

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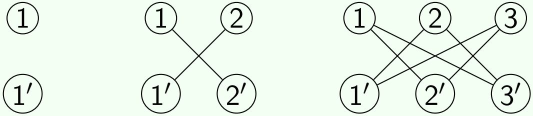

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Theorem (Glen, Kitaev, Pyatkin; 2016)

If $n \geq 5$ then the **representation number** *of $H_{n,n}$ is $\lceil n/2 \rceil$ (that is, one needs $\lceil n/2 \rceil$ copies of each letter to represent $H_{n,n}$ but* **not fewer***).*

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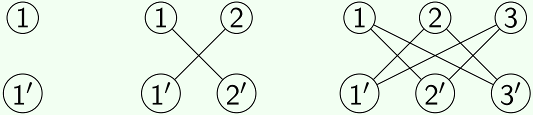


$H_{1,1}$ $H_{2,2}$ $H_{3,3}$

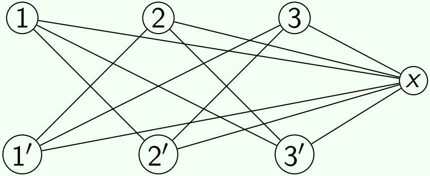
Open problem

Is it true that out of all bipartite graphs, **crown graphs** require **longest** word-representants?

Graphs requiring long word-representants

The "worst" known word-representable graph

The graph G_n is obtained from a **crown graph** $H_{n,n}$ by adding an **apex** (an **all-adjacent vertex**). The **representation number** of G_n is $\lfloor n/2 \rfloor$, which is the **highest known** representation number.

$G_3 =$ 

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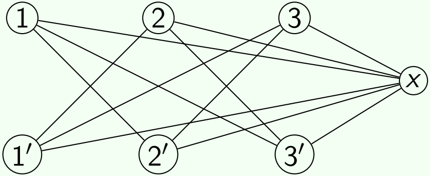
$G_3 =$ 

Open problem

Are there any graphs whose representation requires **more** than $\lfloor n/2 \rfloor$ copies of each letter? Recall that **any** word-representable graph can be represented by $2n$ copies of each letter (a bit fewer depending on the size of the maximum clique).

3-colorable graphs

Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

Any 3-colorable graph is word-representable.

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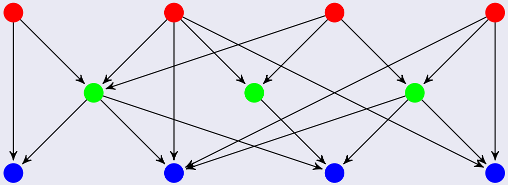
Proof.

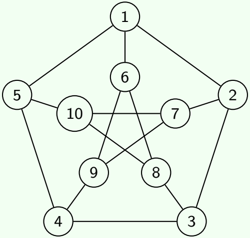
Coloring a 3-**colorable graph** in three colors Red, Green and Blue, and orienting the edges as Red $\rightarrow$ Green $\rightarrow$ Blue, we obtain a **semi-transitive orientation**. Indeed, obviously there are **no cycles**, and because the longest directed path involves only three vertices, there are **no shortcuts**.



□

Some corollaries to the last theorem

Petersen graph



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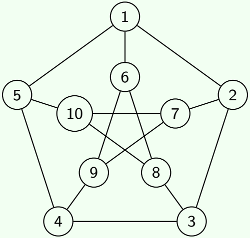

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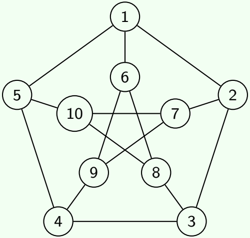

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Theorem (Halldórsson, Kitaev, Pyatkin; 2011)

Triangle-free planar graphs are word-representable.

Proof.

By Grötzch's theorem, **every** triangle-free planar graph is 3-colorable. □

Some corollaries to the last theorem

Optimization problems

The following optimization problems are **NP-hard** on 3-colorable graphs $\Rightarrow$ they are **NP-hard** on word-representable graphs:

- Dominating Set,
- Vertex Coloring,
- Clique Covering, and
- Maximum Independent Set.

Two complexity results

Theorem (Limouzy; 2014)

It is an **NP-complete problem** *to recognize whether a given graph is word-representable.*

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The proof of Limouzy's result appears in the book "Words and Graphs" and it is based on the observation that the class of **triangle-free word-representable graphs** is exactly the class of cover graphs of posets, recognising which is **NP-complete**.

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Theorem (Halldórsson, Kitaev, Pyatkin; 2011)

Deciding whether a given graph is k-**representable**, *for any fixed* k, $3 \leq k \leq \lceil n/2 \rceil$, *is* **NP-complete**.

Graph operations preserving word-representability

- Replacing a vertex v by a module H (**clique** or **any comparability graph**); Neighbors of v become neighbors of all vertices in H. **[Proof is straightforward via word-representants]**

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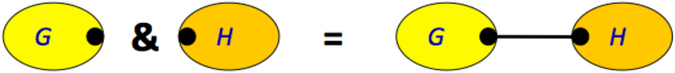
- Joining two word-representable graphs by an edge:



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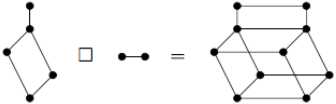
Graph operations preserving word-representability

- Cartesian product of two graphs (shown by Bruce Sagan):



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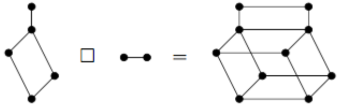
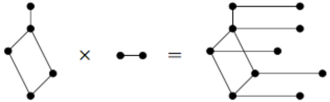


- Rooted product of graphs:



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Graph operations **not** preserving word-representability

- Taking the complement. The complement to the cycle graph C_5 **and** an isolated vertex is the **non-word-representable** wheel graph W_5.

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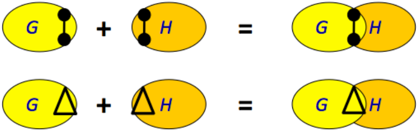
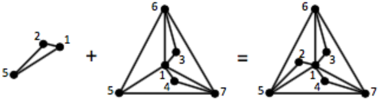
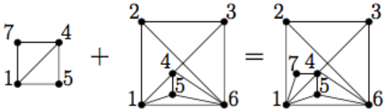
- Gluing two graphs at an **edge** or a **triangle**







Graph operations **not** preserving word-representability

- Taking the line graph operation. **[Example is on next slide.]**

Theorem (SK, Salimov, Severs, Úlfarsson; 2011)

For **any** *wheel graph W_n and $n \geq 4$, the line graph $L(W_n)$ is* **not** *word-representable.*

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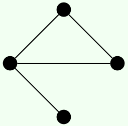
Open problem

Is the line graph of a non-word-representable graph **always** non-word-representable? (This is the case in all known cases.)

Taking the line graph operation

Example of taking the line graph operation



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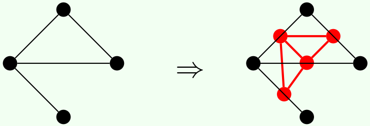

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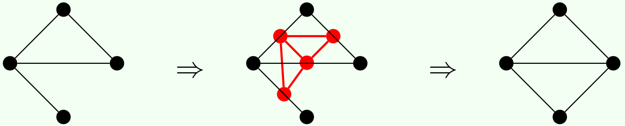

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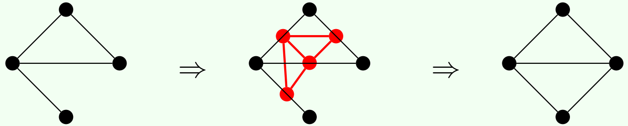
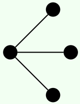
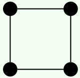
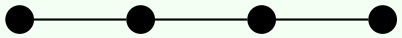


The claw graph; a cycle graph; a path graph

$K_{1,3} =$  $C_4 =$  $P_4 =$ 

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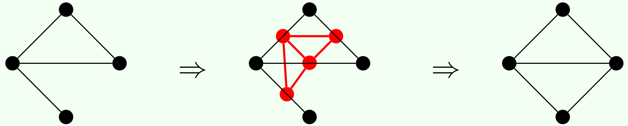

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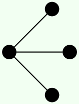
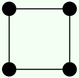
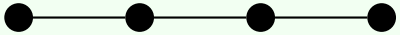
$K_{1,3} =$ 

$C_4 =$ 

$P_4 =$ 

Theorem (SK, Salimov, Severs, Úlfarsson; 2011)

If a connected graph G is **not** *a path graph, a cycle graph or the claw graph $K_{1,3}$, then the line graph $L^n(G)$ is* **not** *word-representable for $n \geq 4$.*

Word-representability of planar graphs

- Not all planar graphs are word-representable (e.g. **odd** wheel graphs on at least 5 vertices are **non-word-representable**).

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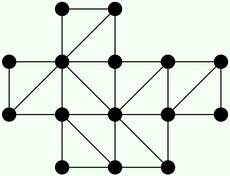
- Towards solving the open problem various, triangulations of planar graphs were considered to be discussed next. **Key tools** here are **3-colorability** and **semi-transitive orientations**.

Word-representability of polyomino triangulations

Convex polyomino triangulation

Convex = no "holes" (missing squares) in a column or a row.



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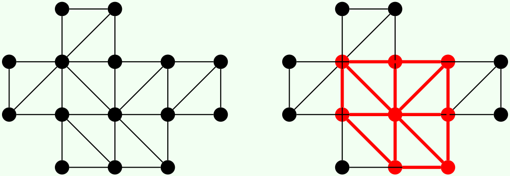

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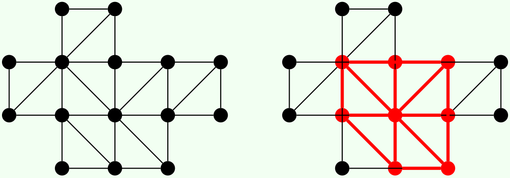
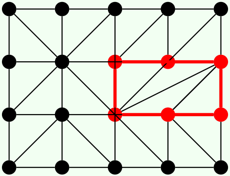


Theorem (Akrobotu, SK, Masárova; 2015)

A triangulation of a **convex** *polyomino is word-representable* **iff** *it is 3-colorable. There are not 3-colorable word-representable* **non-convex** *polyomino triangulations.*

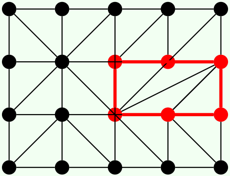
Word-representability of polyomino triangulations

Rectangular polyomino triangulation with a single domino tile



Word-representability of polyomino triangulations

Rectangular polyomino triangulation with a single domino tile



Theorem (Glen, SK; 2015)

A triangulation of a rectangular polyomino with a single domino tile is word-representable **iff** *it is 3-colorable.*

Word-representability of near-triangulations

Near-triangulation

A near-triangulation is a planar graph in which **each** inner bounded face is a **triangle** (where the outer face may possibly not be a triangle).

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The following theorem is a far-reaching generalization of the results from the last two slides:

Theorem (Glen; 2016)

A K_4-free near-triangulation is 3-colorable **iff** *it is word-representable.*

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The following theorem is a far-reaching generalization of the results from the last two slides:

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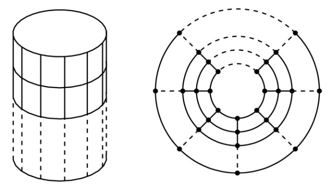
Open problem

Characterize word-representable near-triangulations (containing K_4).

Triangulations of grid-covered cylinder graphs

Grid-covered cylinder graph



Triangulations of grid-covered cylinder graphs

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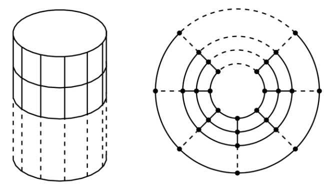
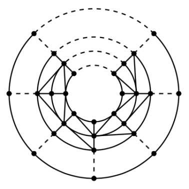


Triangulation of a grid-covered cylinder graph



Triangulations of grid-covered cylinder graphs

Theorem (Chen, SK, Sun; 2016)

A triangulation of a grid-covered cylinder graph with **more than three** *sectors is word-representable* **iff** *it contains no W_5 or W_7 as an induced subgraph.*

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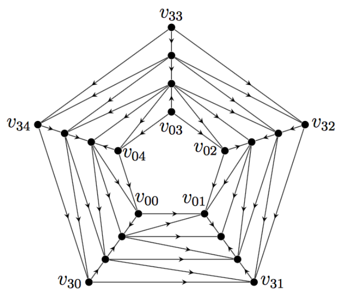
Semi-transitive orientation involved in the proof



Triangulations of grid-covered cylinder graphs

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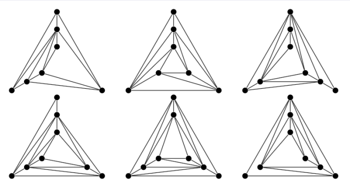
A triangulation of a grid-covered cylinder graph with **three** *sectors is word-representable* **iff** *it contains as an induced subgraph* **none** *of*



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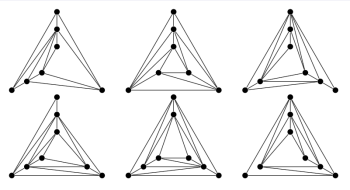

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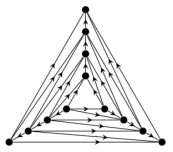
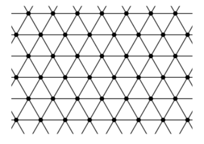


Subdivisions of triangular grid graphs

The infinite graph T^∞



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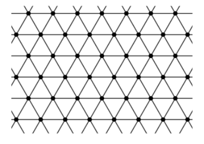
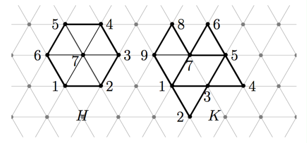


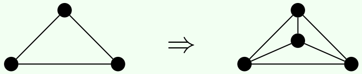
Triangular grid graph

A triangular grid graph is a subgraph of T^∞, which is formed by all edges bounding **finitely** many cells. Note that a triangular grid graph does **not** have to be an **induced subgraph** of T^∞.



Subdivisions of triangular grid graphs

Subdivision of a cell



Subdivisions of triangular grid graphs

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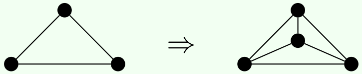
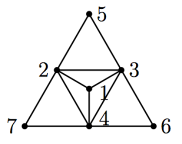


Interior and exterior cells

An edge of a triangular grid graph G shared with a cell in T^{∞} that does not belong to G is a boundary edge. A cell in G that is incident to at **least one** boundary edge is a boundary cell. A **non-boundary** cell in G is an interior cell.

Subdivisions of triangular grid graphs

Minimal non-word-representable subdivision of a triangular grid graph



Subdivisions of triangular grid graphs

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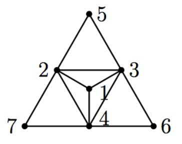
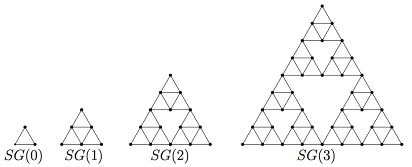


Theorem (Chen, SK, Sun; 2016)

A subdivision of a triangular grid graph G is word-representable **iff** *it has* **no** *induced subgraph isomorphic to* **the graph above***, that is, if G has* **no** *subdivided interior cell.*

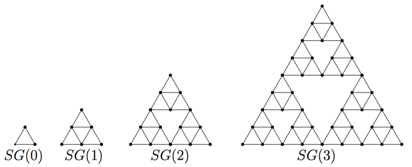
Subdivisions of triangular grid graphs

2-dimensional Sierpiński gasket graph $SG(n)$

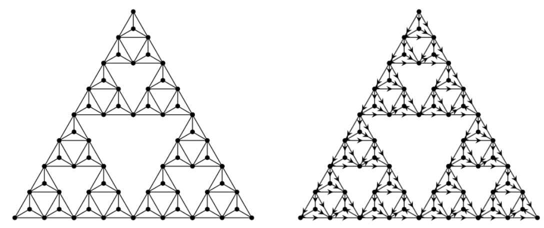


Subdivisions of triangular grid graphs

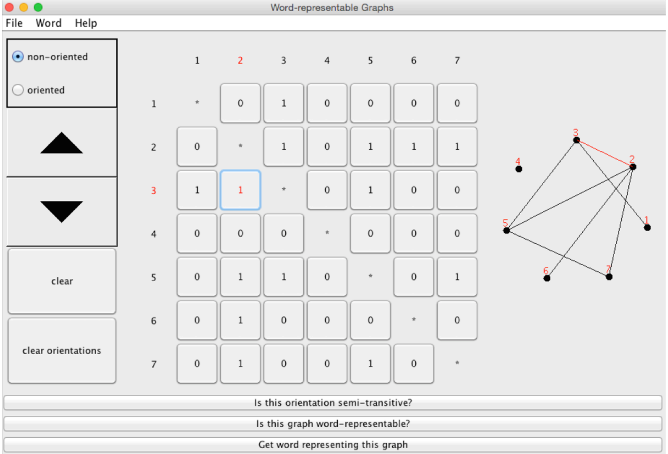
2-dimensional Sierpiński gasket graph $SG(n)$



A semi-transitive orientation of $SG(3)$



Software by Marc Glen to study word-representable graphs



Available at

https://personal.cis.strath.ac.uk/sergey.kitaev/word-representable-graphs.html

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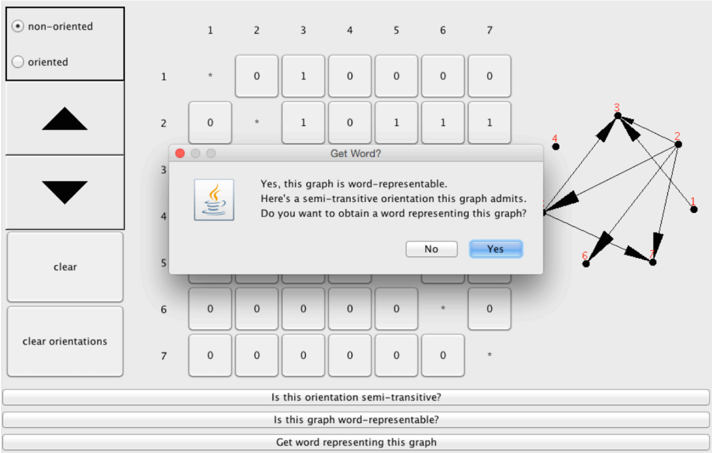

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Open problems

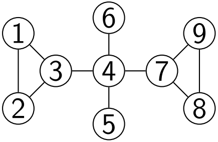
The software should be of **great help** in tackling the problems below.

- Which graphs in your favourite class of graphs are word-representable?
- Characterize (non-)word-representable planar graphs.
- Characterize word-representable near-triangulations (**containing** K_4).
- Describe graphs representable by words avoiding a **pattern** τ of length ≥ 4.
- Is it true that out of all bipartite graphs, **crown graphs** require **longest** word-representants?
- Are there any graphs whose representation requires **more** than $\lfloor n/2 \rfloor$ copies of each letter?
- Is the line graph of a non-word-representable graph **always** non-word-representable?
- Characterize word-representable graphs in terms of forbidden subgraphs.
- Translate a known to you problem from **graphs** to **words** representing these graphs, and find an **efficient algorithm** to solve the obtained **problem**, and thus the **original problem**.

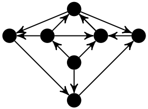
  [The last two problems are of fundamental importance!]

Exercises for this afternoon

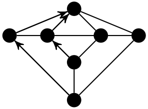
1. Represent the following graph using **two** copies of **each** letter:



2. The graph below contains **lots of shortcuts**. How many can you see?



3. Use a branching process to show that the partial orientation below **cannot** be extended to a **semi-transitive orientation**:



Acknowledgment

Thank you for your attention!