THE GENERAL THEORY OF NOTATIONAL RELATIVITY

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P R E F A C E

The following pages present, in outline, a new method in mathematical logic.

In a volume entitled Analytic Knowledge, which the writer hopes to publish in the near future, this method--which may be characterized as a sort of Prolegomenon to Every Future Postulate Set--is developed in detail, and is then applied to the solution of a number of fundamental problems in logic, mathematics, and Mengenlehre.

Some of the ideas which led, through a period of more than twelve years, to the gradual formulation of this method, were presented, from time to time, in papers read before the Americal Mathematical Society. We refer in particular to the following abstracts:

(1) "Total Determinations of Deductive Systems with special reference to the Algebra of Logic", Bulletin of the American Mathematical Society, March, 1910, p. 285;

(2) "Superpostulates: Introduction to the Science of Deductive Systems", ib., November, 1913, p. 76;

(3) "Deductive Systems and Postulate Theory; I. Finite Case", ib., February, 1915, p. 220;

(4) "The Elimination of Modular Existence Postulates", ib., March, 1916, p. 269;

(5) "Mutually Prime Postulates", ib., March, 1916, p. 287.

(6) "Principia Analytica", Philosophical Review, March, 1919, pp. 187-8. This paper was read before the American Philosophical Association. We quote the first paragraph of the abstract:

"Deductive systems, it is well known, may be determined by means of postulate sets in various ways. Euclidean geometry, for example, is determined by the widely different postulate sets of Hilbert, of Veblen, and of Huntington. These distinct determinations are all 'equivalent'--any two of the postulate sets are uniquely intertranslatable. May there not be, then, a set of 'superpostulates', of which Hilbert's, Veblen's, Huntington's, and other postulate sets are special cases? There is. And, as a matter of fact, the 'invariant' of these postulate sets turns out to be of an extraordinarily simple character".

Cambridge, Mass.

April, 1921.

SYSTEMS

Ordinary euclidean three dimensional geometry may be 'given' in various ways. In particular, it may be 'given'--by the use of "logical principles"--in terms of:

1. The two undefined euclidean-geometric notions, class of solid spheres, and sphere-inclusion; and

2. A small number of unproved euclidean-geometric propositions, each statable entirely in terms of "logic" and the two undefined notions in 1.

By means of "logic", all euclidean-geometric notions--other than those in 1.--can be defined, and all the euclidean-geometric propositions--other than those in 2.--can be proved. [Huntington]

We shall call the two undefined notions, [class of solid spheres, sphere-inclusion], the Huntingtonian language, and the set of unproved propositions in 2., the Huntingtonian assumptions. By the system of euclidean geometry in terms of the Huntingtonian language, we shall mean the set of Huntingtonian assumptions together with the set of all those and only those propositions "logically" deducible from these assumptions.

SYSTEM FUNCTIONS

Russell has introduced the important generalization of the concept of proposition to that of propositional function. We find it useful to coin analogous terms for certain analogous generalizations.

When, in the current literature, the relation "sphere-inclusion" is replaced by R, we have no right to call R a relation. R is, and should be called, a relational function. Similarly, when the class of "solid spheres" is replaced by K, we must call K a class function. When the class of "solid spheres" is replaced by K, and the relation "sphere-inclusion" is replaced by R, the Huntingtonian language becomes the Huntingtonian language function [K, R]. The name "language function" is conveniently replaced by the name base. [K, R] is therefore the Huntingtonian base. The propositions that we have called the Huntingtonian assumptions become now the propositional functions that we may call the Huntingtonian assumptional functions. However, the name "assumptional functions" is conveniently replaced by the name postulates. The set of propositions that we have called "the system of euclidean geometry in terms of the Huntingtonian language" becomes the set of propositional functions that we shall call "the

Huntingtonian system function in terms of the Huntingtonian base [K, R]". [Footnote. Cf. Keyser, "doctrinal functions"]

DOUBLE TERMINOLOGY

If we are to avoid fallacious reasoning, we must always distinguish between the two sets of concepts involved in system and in system function. The sets of concepts are:

class	::	class function
relation	::	relational function
language	::	language function
	::	[=base]
assumption	::	assumptional function
	::	[=postulate]
system	::	system function

Striking examples of the confusion resulting from a disregard of this distinction are found in many expositions of Einstein's "generalized theory of relativity". Results that are valid only for "the Einsteinian system function in terms of the Einsteinian base" are supposed to hold also for the various systems derivable from interpretations of the Einsteinian base; or, results that are valid for a "particular Einsteinian system in terms of a particular Einsteinian language" are supposed to hold necessarily for "any Einsteinian system in terms of any Einsteinian language".

In the following pages we shall be concerned exclusively with system functions.

GRAFS

Although, for purposes of analytic proof, grafic methods are either irrelevant or inconclusive or both, yet, for purposes of exposition, they may well be indispensable.

In the following discussion we shall employ various grafic devices, and shall develop, on the grafic side, a sort of "deductive analytic geometry". For the simplest case of system functions, viz.,for those given in terms of a base involving just one K and just one R, the K being finite, and the R dyadic, we shall use Schroeder's grafs. For all the other cases we shall introduce new types of diagram.

BASES

We shall always symbolize a base involving a K of cardinality n and an R of degree m, i. e., an n-element K and an m-adic R, by

$$K^n \; R_m$$

or, for typografic convenience, by Kn Rm.

I S O T R O P Y

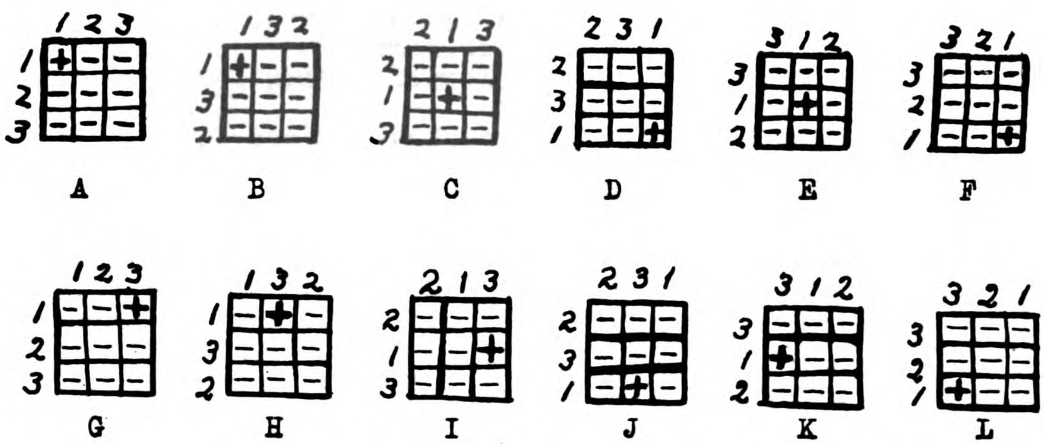
DYADIC ISOTROPY

Dyadic Graf

Let the following be a postulate set for a certain system function on base K3 R2:

0. K has just three distinct elements.
1. There is a unique K element, "1", such that R11 holds.
2. For any distinct K elements a, b, Rab fails.

This postulate set may be represented diagrammatically



as in Fig. A, where the K elements are labelled 1, 2, 3. In this figure the element a is found on the left, and the element b at the top. When Rab holds, the square labelled (a,b) is marked "+"; when Rab fails, the square is marked "-"

Factorial Family

So far as postulational properties of a system function are concerned, it is immaterial whether or not such properties are ever represented by any grafic scheme. And, if they are so represented, it is equally immaterial, from the standpoint of the postulational properties, whether the K elements are taken with one permutation or another. [Footnote. We use the term "permutation" rather than "order", because, in the case of a non-finite K, the elements may be taken with different permutations within one order type. Thus, 1, 2, 3, 4, 5, 6, ..., and 2, 1, 4, 3, 6, 5, ..., are different permutations of a countable set of elements within the one order type ω.]

Hence, postulationally, it does not matter whether the three elements of our system function on K3 R2 are taken with permutation 1, 2, 3, as in Fig. A, or with any other of the possible 3! permutations, Figs. A-F.

This leads to the introduction of the first of our new auxiliary concepts. We shall call the set of 3! grafs, A-F, a factorial family of grafs. More generally: the set of all those and only those n! grafs obtained from a given graf by arranging the K elements with all the n! permutations constitutes a factorial family of grafs.

Thus, when a postulate set is represented grafically by a single graf, rather than by the whole factorial family of grafs, the particular permutation of K elements chosen for the graf is extra-postulational, and therefore postulationally irrelevant.

Ordinarily, the n! grafs of a factorial family are neither all identical nor all distinct. This is the case with factorial family A-F. It may happen, however, that all the n! grafs of a factorial family are distinct. Thus, if one of the grafs for a certain postulate set on K3 R2 is Fig. G, the factorial family, Figs. G-L, consists of n! grafs which are all distinct.

Isotropy

Finally, it may happen that all the n! grafs of a factorial family are identical.

Let us consider, for example, the following postulate set for a system function on base K3 Z2 :

0. K has just three distinct elements.
1. For any K element a,............. Zaa holds.
2. For any distinct K elements a, b, Zab fails.

The class function K may be interpreted as any class we please, and Zab may be interpreted as "a is identical with b". This postulate set may be represented by the graf of Fig. 1, where the K elements are labelled 1, 2, 3. The factorial family consists of the six grafs, Figs. 1-6, which are all identical.

In the present system function, whatever postulational property holds for any K element holds also for any other;

and whatever postulational property fails for any K element fails also for any other. That is to say, no two of the elements are KZ-distinguishable, and therefore no one of the elements is KZ-identifiable.

The grafic equivalent of this state of affairs is the identity of all the grafs of the factorial family.

A system function which has the property just mentioned will be said to have the property of isotropy, and we shall say that the system function is isotropic. An isotropic system function is therefore one that is determined by a postulate set that makes identically the same statement about any two of the elements.

For the base of an isotropic system function we shall always use KZ, the letter Z, as the initial of "zero", being intended to suggest that the system function furnishes "zero" information about postulational distinctions of elements. We shall symbolize an isotropic system function by O (zero), and shall indicate an isotropic system function of, say, three elements by writing the zero as a subscript to the number of elements involved, thus: 3_0 .

How many isotropic system functions are possible with a three element K and a dyadic Z? To obtain a general method for answering this type of question we introduce a new concept.

Validands

The symbol "+", or "-", put into a graf, means that, for certain combinations of K elements, the given relational function respectively holds, or fails, and thus "+" and "-" represent, respectively, positive and negative relational-functional validation. We shall call each of these symbols a validatum. Hence, "+" is a positive validatum, and "-" a negative validatum.

Just as we generalized the notion of class to that of class function, relation to relational function, language to language function, and system to systemfunction, so we find it useful to generalize the concept of validatum to that of validatum function. By a validatum function we shall mean a two-valued function whose values are validata. We shall symbolize validatum functions by Greek letters. A validatum function, α, represents, therefore, ambiguously, either the positive validatum, "+", or the negative validatum, "-".

Having defined the new concept, we shall henceforth replace the name validatum function by the more convenient name validand.

If we change Fig. 1 by putting validands $\alpha_{11}, \ldots, \alpha_{33}$, respectively in place of the nine validata, the result is Fig. 7. This figure, which is a graf function, we shall call a hypergraf. When all the validands are replaced by suitable (positive or negative) validata, the hypergraf becomes a graf.

We may now write out all the 3! hypergrafs obtainable from Fig. 7 by arranging the K elements with all the 3! permutations, Figs. 7-12. This set of 3! hypergrafs we shall call a factorial family of hypergrafs.

An examination of these hypergrafs shows that a necessary and sufficient condition for their identity is that the nine validands reduce to two, as follows:

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{aa}\ ; \quad \alpha_{12} = \alpha_{21} = \alpha_{13} = \alpha_{31} = \alpha_{23} = \alpha_{32} = \alpha_{ab}\ .$$

When this condition is satisfied, each of the hypergrafs, Figs. 7-12, reduces to Fig. 13. When all the n! hypergrafs of a factorial family are identical, we shall call the hypergrafs isotropic. Hence, Fig. 13 represents an isotropic hypergraf.

Dyadic Isotropy Superpostulates for K3

The hypergraf of Fig. 13 is determined analytically by the following conditions:

0. K has just three distinct elements.

1. For any K element a, ... aa α_{aa}

2. For any distinct K elements a, b, ab α_{ab}

The forms aa and ab we shall call atomic dyads. An atomic dyad together with a validand--e. g., aa together with α_{aa}, or ab together with α_{ab}--we shall call an atomic superpostulate. When the validand is replaced by a (positive or negative) validatum, the atomic superpostulate becomes an atomic postulate. Thus aaα_{aa} may become either aa+ or aa-; abα_{ab} may become, independently, ab+ or ab-.

Hence, Fig. 13 is determined completely by the K3-postulate together with the two atomic superpostulates 1 and 2. To distinguish this type of atomic superpostulate from other types to be considered later, we shall call 1 and 2 a set of isotropy atomic superpostulates.

The validands α_{aa} and α_{ab} may assume "+" values or "-" values independently, and therefore the hypergraf of Fig. 13 determines four grafs, and represents four system functions. Since the hypergraf is isotropic, each of the grafs is isotropic. If we call the relational functions Z', Z'', Z''', Z'''', the system functions represented by Fig. 13 are the following:

(1) $\alpha_{aa}=+\ ;\ \alpha_{ab}=+$.

Base=K3 Z'2. In this three element dyadic isotropic system function, Z' holds universally. Hence Z'ab may be interpreted as "a is either identical with or different from b".

(2) α_{aa} =- ; α_{ab} =- .

Base=K3 Z''2. In this system function Z'' fails universally. Hence, Z''ab is interpretable as "a is neither identical with nor different from b".

(3) α_{aa} =+ ; α_{ab} =- .

Base=K3 Z'''2. In this system fuhction Z'''ab is interpretable as "a is identical with b". Fig. 1.

(4) α_{aa} =- ; α_{ab} =+ .

Base=K3 Z''''2. Z''''ab is interpretable as "a is diferent from b".

Each of these four system functions is determined by a postulate set on a distinct base; i. e., these system functions are non-cobasal. Therefore, the graf of any one of these system functions cannot be obtained from that of any other by a mere permutation of elements; i. e., the four grafs are non-cofactorial. It is not difficult to see that cofactoriality of grafs and cobasality of system functions imply each other.

We have now answered the question as to how many isotropic system functions are possible with a three element K and a dyadic Z. There are just four such system functions. These are interpretable in terms of four basic logical relations that we may call truism, falsism, identity, and difference.

Dyadic Isotropy Superpostulates for K4

If we increase the number of K elements from three to four, we obtain the same results. See Figs. 14 and 15.

Dyadic Isotropy Superpostulates for Kn

More generally: Any finite dyadic n-element hypergraf, Fig. 16, when reduced to a finite dyadic n-element isotropic hypergraf on Kn Z2, Fig. 17, involves just two validands, α_{aa} and α_{ab} , and is determined analytically by the dyadic isotropy superpostulate set:

0. K has just n (finite) distinct elements.
1. For any K element a, ... aaα_{aa}
2. For any distinct K elements a, b, abα_{ab}

For any finite n, this superpostulate set determines just four distinct dyadic system functions, which are the only dyadic n-element isotropic system functions, viz., those which are interpretable with the dyadic relations of truism, falsism, identity, and difference.

<u>Isotropy: Extensional Definition</u>

So far, we have defined isotropy as a certain <u>property</u> of a certain system function; <u>i. e.</u>, we have defined isotropy <u>intensionally</u>. We may also define it <u>extensionally</u>, thus: An isotropy is the set of all those and only those system functions determined by a hypergraf whose factorial family consists of hypergrafs which are all identical.

TRIADIC ISOTROPY

<u>Triadic Graf</u>

From finite isotropic <u>dyadic</u> system functions, grafs, and hypergrafs, we proceed to the triadic case.

Let the following be a postulate set for a system function on base K3 Z3:

0. K has just three distinct elements.
1. For any K element a, Zaaa holds.
2. For any K element a, b, c, not all identical, Zabc fails.

K is interpretable as any class we choose, and Zabc may be interpreted as "a, b, c, are all identical".

This postulate set may be represented by a triadic graf in various ways. The form of triadic graf that we find most useful for our present inquiry is that of Fig. 18. This figure represents a "cubic" graf divided into "layers" or "slices", so that layer (1.v2v3) is at the bottom, layer (2.v2v3) is on top of (1.v2v3), and layer (3.v2v3) is on top of (2.v2v3). In each of these layers, v2 is found on the left, and v3 at the top; and each layer involves just one value of v1. This form of graf facilitates the extension to tetradic and higher degrees, and also foreshadows a generalization to be discussed later.

The 3! grafs of the factorial family of this triadic graf are easily found to be all identical. For example, the permutation of elements 2 and 3, throughout v1, v2, and v3, leads to Fig. 19. (Here it should be noted that the second and third layers must be interchanged) The triadic graf of Fig. 18, and the system function it represents, are therefore isotropic.

<u>Triadic Hypergraf</u>

From the graf of Fig. 18 we proceed to the corresponding general three element triadic hypergraf, Fig. 20, with its 3^3 independent validands. As in the dyadic case, the necessary and sufficient condition for the identity of the 3! triadic hypergrafs of the factorial family is obtained easily, and is found to be the following, where the 27 unrestricted validands are reduced to five:

$$\alpha_{111} = \alpha_{222} = \alpha_{333} = \alpha_{aa}$$
$$\alpha_{112} = \alpha_{113} = \alpha_{221} = \alpha_{223} = \alpha_{331} = \alpha_{332} = \alpha_{ab}$$
$$\alpha_{121} = \alpha_{131} = \alpha_{212} = \alpha_{232} = \alpha_{313} = \alpha_{323} = \alpha_{ba}$$
$$\alpha_{122} = \alpha_{133} = \alpha_{211} = \alpha_{233} = \alpha_{311} = \alpha_{322} = \alpha_{bb}$$
$$\alpha_{123} = \alpha_{132} = \alpha_{213} = \alpha_{231} = \alpha_{312} = \alpha_{321} = \alpha_{bc}$$

When this condition is satisfied, each of the hypergrafs of the factorial family of Fig. 20 reduces to Fig. 21. Hence, Fig. 21 represents a three element triadic <u>isotropic</u> hypergraf.

<u>Triadic Isotropy Superpostulates for K3</u>

The hypergraf of Fig. 21 is determined analytically by the following triadic atomic superpostulate set:

0. K has just three distinct elements.
1. For any K element a, ... a.aa α_{aa}
2. For any distinct K elements a, b, ... a.ab α_{ab}
3. " " " " " a, b, ... a.ba α_{ba}
4. " " " " " a, b, ... a.bb α_{bb}
5. " " " " " a, b, c,. a.bc α_{bc}

This superpostulate set involves five <u>atomic triads</u>, viz., aaa, aab, aba, abb, abc, and each of these atomic triads involves just one validand.

Since each of these validands may assume the value "+" or the value "-" independently of any other, this hypergraf determines 2.2.2.2.2, or 32, non-cofactorial grafs, and therefore 32 non-cobasal system functions.

<u>Triadic Isotropy Superpostulates for Kn</u>

If we increase the number of K elements from three to four, the result is the same. See Figs. 22 and 23.

More generally: Any finite triadic n-element hypergraf, Fig. 24, when reduced to a finite triadic n-element <u>isotropic</u> hypergraf on Kn Z3, Fig. 25, involves the same five validands, and is determined analytically by the same superpostulate set as for K3, except that the number of K elements is changed from three to any finite number, n.

TETRADIC ISOTROPY

<u>Tetradic Graf</u>

Let the following be a postulate set for a system function on base K4 Z4:

0. K has just four distinct elements.
1. For any K element a, Zaaaa holds.
2. For any K elements a, b, c, d, not all identical, Zabcd fails.

As in previous cases, K may be taken as any class we please, and Zabcd may be interpreted as "a, b, c, and d are all identical". This postulate set may be represented by a tetradic graf in various ways. For our present purpose we shall use the form of tetradic graf given by Fig. 26, which is an extension of the form we employed for the triadic case, Fig. 18. In Fig. 26, the "four dimensional" graf is divided into cubic layers, (1.v2v3v4), (2.v2v3v4), (3.v2v3v4), (4.v2v3v4), and these, in turn, are divided into dyadic layers, or "slices", (11.v3v4), ..., (44.v3v4). In each of the 16 dyadic layers v3 is found on the left, and v4 at the top; and each of these layers involves just one value of v1, and just one value of v2.

The 4! grafs of the factorial family of this graf will be found to be all identical, and therefore this tetradic graf, and the system function it represents, are isotropic.

<u>Tetradic Hypergraf</u>

As in preceding cases, we generalize from the graf to the corresponding hypergraf. If we were to replace, in Fig. 26, each "+" and each "-" by an independent validand, we should obtain the general four element tetradic hypergraf, with its 4.4.4.4, or 256, unrestricted validands. The necessary and sufficient condition for the identity of all the 4! hypergrafs of the factorial family of this general hypergraf will be found to reduce the 256 validands to fifteen, Fig. 27. (In order to gain certain notational advantages, we represent some of the 15 validands by an α with a double subscript, and the others by a β with a double subscript)

<u>Tetradic Isotropy Superpostulates for K4</u>

Analytically, the tetradic isotropic hypergraf of Fig. 27 is determined by the following atomic isotropy superpostulate set:

0. K has just four distinct elements.

For any distinct K elements a, b, c, d:

1.	aa.aa α_{aa}	4.	aa.bb α_{bb}	7.	ab.ab β_{ab}
2.	aa.ab α_{ab}	5.	aa.bc α_{bc}	8.	ab.ac β_{ac}
3.	aa.ba α_{ba}	6.	ab.aa β_{aa}	9.	ab.ba β_{ba}

10. ab.bb β_{bb} 13. ab.cb β_{cb}

11. ab.bc β_{bc} 14. ab.cc β_{cc}

12. ab.ca β_{ca} 15. ab.cd β_{cd}

This superpostulate set involves 15 atomic tetrads, aaaa, aaab, ..., abcc, abcd, and each of these tetrads involves just one validand.

Since each of the 15 validands may be assigned a "+" value or a "-" value independently of any other, the number of non-cofactorial grafs, and therefore of non-cobasal system functions, determined by this tetradic hypergraf is 2^{15}.

Tetradic Isotropy Superpostulates for Kn

Any finite tetradic n-element hypergraf, Fig. 28, when reduced to a fintte tetradic n-element isotropic hypergraf on Kn Z4, Fig. 29, involves the same 15 validands, and is determined analytically by the same superpostulate set as for K4, except that the number of K elements is changed from four to any finite number, n.

PENTADIC ISOTROPY

Pentadic Hypergraf

Let the following be a postulate set for a system function on K5 Z5:

0. K has just five distinct elements.
1. For any K element a, Zaaaaa holds.
2. For any K elements a, b, c, d, e, not all identical, Zabcde fails.

As in former cases, K may be taken as any class whatsoever, and Zabcde may be interpreted as "a, b, c, d, e, are all identical". This pentadic postulate set may be represented by a pentadic graf in various ways, --in particular, by an extension of the form of graf we have already employed for the triadic and the tetradic cases. This mode of grafing the given pentadic system function results in a set of 5.5.5, or 125, dyadic slices or layers. Of these 125 layers, the five which belong to type (iii.v4v5), i=1, 2, 3, 4, 5, have exactly one "+" each, viz.,

(111.11)+, (222.22)+, . . ., (555.55)+ .

All the ~~othe~~ other dyadic layers have "-" throughout.

The 5! grafs of the factorial family of this pentadic graf are all identical, and therefore this graf, and the system function it represents, are isotropic.

If, in this pentadic graf, we were to replace each "+" and each "-" by an independent validand, we should obtain the general five element pentadic hypergraf, with its 5.5.5.5.5, or 3125, unrestricted validands. The necessary and sufficient condition for the identity of all the 5! hypergrafs of the factorial family of this general hypergraf will be found to reduce the 3125 validands to 52, Fig. 30. (For notational simplicity, we represent these 52 validands by the letters $\alpha, \beta, \gamma, \delta, \varepsilon$ with double subscripts)

Pentadic Isotropy Superpostulates for K5

Analytically, the five element pentadic isotropic hypergraf, whose typical layers are given in Fig. 30, is determined by the following atomic isotropy superpostulate set:

0. K has just five distinct elements.

For any distinct K elements a, b, c, d, e:

1.	aaa.aa α_{aa}	6.	aab.aa β_{aa}	16.	aba.aa γ_{aa}
2.	aaa.ab α_{ab}				
3.	aaa.ba α_{ba}		....		....
4.	aaa.bb α_{bb}				
5.	aaa.bc α_{bc}	15.	aab.cd β_{cd}	25.	aba.cd γ_{cd}
26.	abb.aa δ_{aa}	36.	abc.aa ε_{aa}		
....		....			
35.	abb.cd δ_{cd}	52.	abc.de ε_{de}		

This superpostulate set involves 52 atomic pentads, aaaaa, aaaab, ... , abcdc, abcdd, abcde, and each of these pentads involves just one validand.

Since these 52 validands are independent of each other, the number of non-cofactorial grafs, and therefore of non-cobasal system functions, determined by this pentadic hypergraf, is 2^{52}.

Pentadic Isotropy Superpostulates for Kn

Any finite pentadic n-element hypergraf, when reduced to a finite pentadic n-element isotropic hypergraf on Kn Z5, Fig. 31, involves the same 52 validands, and is determined analytically by the same superpostulate as for K5, except that the number of K elements is changed from five to any finite number, n.

M-ADIC ISOTROPY

If we proceed from finite pentadic isotropic hypergrafs to the hexadic and higher degrees, we obtain more complicated but analogous results. We find that:

Any finite m-adic n-element isotropic hypergraf may be represented by a set of $n^{(m-2)}$ dyadic layers, and may be determined analytically by a definite atomic isotropy superpostulate set, according to an obvious generalization of the method we have employed for m=2, 3, 4, 5.

S T R A T I G R A F Y

DYADIC FIRST STRATIGRAFY

First Stratigrafic Family of K3 Z3

From the consideration of the theory of isotropic, or, as we shall now call them, all-isotropic, system functions,-- *i. e.*, of system functions which have *all* their elements isotropic--we proceed to the study of the theory of system functions which have *not necessarily all* their elements isotropic. In order to do this, we introduce a new fundamental concept.

Consider the three element triadic isotropic hypergraf of Fig. 21. This hypergraf, if represented in three dimensions, would be a cube. Let us imagine it so represented, and let us also imagine this cube cut by planes parallel to the v2v3-axes, and therefore perpendicular to the v1-axis. The result will be the set of three *dyadic* layers, labelled (1.v2v3), (2.v2v3), (3.v2v3). Hence, Fig. 21 may be regarded from two distinct points of view. In the first place it may be regarded as one of many possible ways of representing a *single triadic* hypergraf, as we did in our discussion of isotropy; in the second place it may be viewed as representing a *set of three dyadic* hypergrafs, viz., (1.v2v3), (2.v2v3), (3.v2v3).

We shall call this set of three dyadic hypergrafs, Fig. 21, the *first stratigrafic family* of the original *single triadic isotropic* hypergraf.

Let us study some of the consequences of regarding Fig. 21 from the second standpoint.

First Stratigrafy of K3 Z3

The dyadic hypergrafs which we have just defined as the first stratigrafic family of the given triadic hypergraf have the important property of constituting a *factorial family* of hypergrafs. For example, if we take the first of these dyadic hypergrafs, (1.v2v3), and write out the 3! hypergrafs obtainable from all the permutations of the K elements, we find that the interchange of elements 2 and 3 leaves the hypergraf invariant, whereas any change in the position of element 1 changes the hypergraf, so that the factorial family of (1.v2v3) consists of the three dyadic hypergrafs of Fig. 21. We obtain exactly the same result if we start with (2.v2v3) or with (3.v2v3).

If we take the general dyadic three element hypergraf, Fig. 7, with its nine independent validands, and require that at least *two* of the elements, say 2 and 3, be isotropic, --i. e., that the permutation of 2 and 3 shall leave the hypergraf unchanged--we find that the necessary and sufficient condition for this is that the nine validands be reduced to five, exactly as in (1.v2v3), Fig. 21.

Since we have as yet imposed no other condition upon the five validands, the isotropy-value of element 1, *i. e.*, its isotropy or non-isotropy, is still entirely undetermined. Hence, any one of the dyadic hypergrafs of Fig. 21 represents all those and only those three element dyadic system functions which have *at least two* elements isotropic. This fact we indicate by saying that any one of the dyadic hypergrafs of Fig. 21 has *stratigrafy* $1_{(\ )}\, 2_0$. Since the number of elements whose isotropy value is yet undetermined is one, we call this the *first* stratigrafy of the original triadic isotropic hypergraf.

First Stratigrafic Family of Kn Z3

In like manner, Fig. 25 may be viewed as representing *either* a *single* n-element *triadic* isotropic hypergraf, as we did earlier, *or* a *set of n dyadic* hypergrafs, viz., those labelled (1.v2v3), ..., (n.v2v3). This set of n dyadic hypergrafs is, by definition, the *first stratigrafic family* of the original single n-element triadic isotropic hypergraf.

This first stratigrafic family constitutes a factorial family of hypergrafs. If we take, for example, the dyadic hypergraf (1.v2v3), and write out the n! hypergrafs obtainable from all the permutations of the K elements, we find that the permutation of any two of the elements 2, 3,..., n, leaves the hypergraf invariant, whereas any change in the position of element 1 changes the hypergraf, so that the factorial family of (1.v2v3) consists of the n hypergrafs of Fig. 25. We obtain the same result if we start with (2.v2v3), or (3.v2v3), ..., or (n.v2v3).

First Stratigrafy of Kn Z3

If we were to take the general n-element dyadic hypergraf, Fig. 16, with its n.n independent validands, and were to require thatat least n-1 of the n elements be isotropic, we should find that the necessary and sufficient condition for this is that the n^2 validands be reduced to five, exactly as in (1.v2v3), Fig. 25. Hence, any one of the dyadic hypergrafs, (i.v2v3), i=1, 2, .., n, of Fig. 25 represents all those and only those n-element dyadic system functions which have at least n-1 elements isotropic. In other words, any one of these dyadic hypergrafs has stratigrafy $1_{(\ )}(n-1)_0$ or, the first stratigrafy of the given n-element triadic isotropic hypergraf.

Dyadic First-Stratigrafy Superpostulates for Kn

From these diagrammatic results we proceed to the analytic version.

When Fig. 25 was considered as a single n-element triadic hypergraf, it was determined by the isotropy superpostulate set, 1-5, given on page 11. Hence, layer (1.v2v3), Fig. 25, can be determined by the same superpostulate set, provided only we replace a by 1 in each of the five atomic superpostulates. But layer (2.v2v3), Fig. 25, belonging to the same factorial family as (1.v2v3), is determined by identically the same superpostulate set, if we merely replace 1 by 2. Likewise, any layer (i.v2v3), i=1, 2, .., n, Fig. 25, is determined by this superpostulate set, if we replace 1 by i.

This is only a special case of the relation which holds between any superpostulate set and its hypergrafic representation; for, any two hypergrafs of a factorial family differ only by a permutation of elements, and not by any formal superpostulational property. Hence, if we choose to represent a superpostulate set by a single hypergraf rather than by the whole factorial family, the specific permutation of K elements employed introduces an analytically irrelevant factor. Furthermore, since the given triadic hypergraf is isotropic, there is no formally statable difference between any two of the elements. Hence, analytically, there is no difference whatsoever between writing 111, etc., and writing 222, etc., or iii, etc. In other words, from the analytic point of view, the superpostulate set in question determines any one of the n dyadic hypergrafs chosen arbitrarily, but--

once chosen--regarded as fixed. That is to say, the analytic change from abc, where a is a variable, like b and c, is not to 1bc, where 1 is a "constant", or 2bc, where 2 is a "constant", but to $p_1$bc, where p_1 is a parameter.

Hence, we may say, indifferently, that the complete set of n dyadic hypergrafs, Fig. 25, or any arbitrarily chosen one of these hypergrafs, is determined by the following conditions:

0. K has just n (finite) distinct elements.
1. For the stratigrafic K element p_1, $p_1 . p_1 p_1 \alpha_{aa}$
   " " " " " p_1 and any K element b, distinct from p_1,
2. $p_1 . p_1 b \ \alpha_{ab}$
3. $p_1 . b \ p_1 \alpha_{ba}$
4. $p_1 . b \ b \ \alpha_{bb}$
5. For the stratigrafic K element p_1 and any K elements b, c, distinct from each other and from p_1,
   $p_1 . bc \ \alpha_{bc}$

These conditions we shall call the dyadic first-stratigrafy superpostulate set. This set determines analytically all those and only those n-element dyadic system functions which have at least n-1 elements isotropic; i. e., which belong to the dyadic stratigrafy $1_{(\ )} (n-1)_0$.

DYADIC SECOND STRATIGRAFY

First Stratigrafic Family of K4 Z4

Consider the four element tetradic isotropic hypergraf of Fig. 27. This figure, if represented in four dimensions, would be a regular four dimensional solid. Let us imagine it so represented, and let us also imagine this regular solid cut by planes parallel to the v2, v3, v4 -axes, and therefore perpendicular to the v1-axis. The result will be the set of four triadic layers, labelled (1.v2v3v4), (2.v2v3v4), (3.v2v3v4), (4.v2v3v4), Fig. 27. Hence, Fig. 27 may be regarded, first, as representing a single tetradic isotropic hypergraf; secondly, as representing the set of four triadic hypergrafs, (1.v2v3v4),..., (4.v2v3v4).

This set of four triadic hypergrafs we define as the first stratigrafic family of the original tetradic isotropic hypergraf.

Second Stratigrafic Family of K4 Z4

We may now take the first stratigrafic family of each of these triadic hypergrafs. The triadic hypergraf (1.v2v3v4) has for its first stratigrafic family the set of four dyadic hypergrafs, labelled (11.v3v4), .., (14.v3v4), Fig. 27. Similarly for the other triadic hypergrafs. Hence, Fig. 27 may be viewed not only as a single tetradic hypergraf, or as a set of four triadic hypergrafs, but also as a set of 16 dyadic hypergrafs. These belong to two types, viz., (ii.v3v4) and (ij.v3v4), i, j=1, 2, 3, 4; $i \neq j$.

This set of 16 dyadic hypergrafs we define as the second stratigrafic family of the original tetradic isotropic hypergraf.

First Stratigrafy of K4 Z4

If we take the general four element triadic hypergraf of Fig. 22, with its 64 independent validands, and require that at least three of the elements--say 2, 3, and 4--be isotropic, we find that the necessary and sufficient condition for this is that the 64 validands be reduced to 15, exactly as in (1.v2v3v4), Fig. 27. If, instead of 2, 3, 4, the isotropic elements are to be 1, 3, 4, then the result is (2.v2v3v4), Fig. 27. Similarly for (3.v2v3v4), and (4.v2v3v3). This is due to the fact that the four triadic hypergrafs, Fig. 27, constitute a factorial family of hypergrafs.

Since we have as yet imposed no other condition upon the fifteen validands of (1.v2v3v4), the isotropy-value of element 1 is still entirely undetermined. Similarly, for element 2 in (2.v2v3v4), and for element i in (i.v2v3v4). Hence, any one of these four triadic hypergrafs represents all those and only those four element triadic system functions which have at least three of their elements isotropic. In other words, any one of these triadic hypergrafs has triadic stratigrafy $1_{(\;)}\,3_{0}$. Since the number of elements whose isotropy value is yet undetermined is one, we call this the first stratigrafy of the original tetradic isotropic hypergraf.

Second Stratigrafy of K4 Z4

If we take the general four element dyadic hypergraf, Fig. 14, with its 16 independent validands, and require that at least two of the elements, say 3 and 4, be isotropic, we find that the necessary and sufficient condition for this is that the 16 validands be reduced to ten, exactly as in layer (12.v3v4), Fig. 27. If, instead of 3 and 4, the isotropic elements are to be 2 and 3, then the result is layer (14.v3v4), Fig. 27. Similarly for (ij.v3v4), i,j=1, 2, 3, 4; i≠j. This is due to the fact that the (4.3=) 12 dyadic hypergrafs of type (ij.v3v4) constitute a factorial family of hypergrafs.

Since we have as yet imposed no other condition upon the ten validands of this factorial family, the isotropy-values of elements 1 and 2, in (12.v3v4), are still entirely undetermined. Hence, this factorial family represents all those and only those four element dyadic system functions which have at least two elements isotropic. This fact we indicate by saying that any one of the dyadic hypergrafs of type (ij.v3v4) has dyadic stratigrafy $2_{(\ )}\ 2_0$. Since the num- of elements whose isotropy-value is still undetermined is two, we call this the second stratigrafy of the original tetradic isotropic hypergraf.

First Stratigrafic Family of Kn Z4

The generalization from K4 to Kn is obvious. We may regard Fig. 29 either as representing a single n-element tetradic isotropic hypergraf, or as representing the set of n triadic hypergrafs of type (i.v2v3v4), i. e., the set (1.v2v3v4), ..., (n.v2v3v4). This set is, by definition, the first stratigrafic family of the given tetradic hypergraf.

Second Stratigrafic Family of Kn Z4

In like manner, we may regard Fig. 29 also as representing the set of n.n dyadic hypergrafs of types (ii.v3v4) and (ij.v3v4). This set of dyadic hypergrafs is, by definition, the second stratigrafic family of the given tetradic hypergraf

First Stratigrafy of Kn Z4

If we were to take the general n-element triadic hypergraf, Fig. 24, with its n.n.n independent validands, and were to require that at least n-1 of the elements be isotropic, we should find that the necessary and sufficient condition for this is that the n.n.n validands be reduced to 15, exactly as in (1.v2v3v4), Fig. 29. But the n triadic layers, (i.v2v3v4), Fig. 29, are cofactorial, i. e., form one factorial family. Hence, any one of these cofactorial triadic hypergrafs represents all those and only those n-element triadic system functions which have at least n-1 elements isotropic. That is to say, any one of these triadic hypergrafs has triadic stratigrafy $1_{(\ )}\ (n-1)_0$, or, the first stratigrafy of the given n-element tetradic isotropic hypergraf.

Triadic First-Stratigrafy Superpostulates for Kn

Fig. 29, considered as a single tetradic isotropic hypergraf, is determined by the isotropy superpostulate set, 1-15, given on pages 12, 13 . Hence, layer (1.v2v3v4), Fig. 29, is determined by the same superpostulate set, provided only we replace the variable a by the "constant"1, throughout the fifteen atomic superpostulates. But layer (2.v2v3v4), being cofactorial with (1.v2v3v4), is determined by identically the same set, if we merely replace 1 by 2. Similarly for any layer, (i.v2v3v4). This arises from the fact that although the element 1 seems to be a constant, it is actually a parameter. We must therefore replace 1 by p_1. When the variable, a, is replaced throughout by the parameter p_1, we obtain the triadic first-stratigrafy superpostulate set. This set determines the n cofactorial triadic hypergrafs (i.v2v3v4), Fig. 29, or, if we wish to put it so, it determines any arbitrarily chosen one of these hypergrafs.

Second Stratigrafy of Kn Z4

If we were to take the general n-element dyadic hypergraf, Fig. 16, with its n.n independent validands, and were to require that at least n-2 of the elements be isotropic, we should find that the necessary and sufficient for this is that the n.n validands be reduced to ten, exactly as in (12.v3v4), Fig. 29. But the set of n(n-1) dyadic layers of type (ij.v3v4), i,j=1, 2, 3, .., n; i≠j, Fig. 29, forms one factorial family. Hence, any one of these dyadic hypergrafs represents all those and only those n-element dyadic system functions which have at least n-2 elements isotropic.

Since we have as yet imposed no other conditions upon the ten validands, the isotropy-value of the remaining two elements is still entirely undetermined. In other words, any one of these dyadic hypergrafs has dyadic stratigrafy $2_{(\ )}(n-2)_0$. Since the number of elements whose isotropy value is still undetermined is two, we call this set of n(n-1) layers the second stratigrafy of the original tetradic isotropic hypergraf.

Thus, while the second stratigrafic family includes all the n.n dyadic layers, the set of layers which have the second stratigrafy includes only the n(n-1) layers of type (ij.v3v4).

Dyadic Second-Stratigrafy Superpostulates for Kn

These n(n-1) dyadic layers, Fig. 29, are determined analytically by the superpostulate set of pages 12, 13, after we omit the five atomic superpostulates of type (ii.v3v4), and after we replace the variable a by the parameter p_1, and the variable b by the parameter p_2 . Thus, the entire set of cofactorial dyadic layers of type (ij.v3v4), Fig. 29, -- or, if we wish to put it so, any arbitrarily chosen one of these layers--is determined by the following dyadic second-stratigrafy superpostulate set:

0. K has just n (finite) elements.

For the distinct stratigrafic elements p_1, p_2, and for any K elements c, d, distinct from each other and from p_1, p_2 :

6. $p_1p_2 \cdot p_1p_1$ β_{aa}
7. $p_1p_2 \cdot p_1p_2$ β_{ab}
8. $p_1p_2 \cdot p_1c$ β_{ac}

....

14. $p_1p_2 \cdot cc$ β_{cc}
15. $p_1p_2 \cdot cd$ β_{cd}

This atomic superpostulate set determines analytically the set of all those and only those n-element dyadic system functions which belong to the dyadic second stratigrafy $2_{(\ )}\ (n-2)_0$

DYADIC THIRD STRATIGRAFY

Stratigrafic Families of Kn Z5

Any finite n-element pentadic isotropic hypergraf (see typical dyadic layers, Fig. 31) has:

	a first	stratigrafic family,	of n	tetradic hypergrafs
	a second	" "	of n.n	triadic "
and	a third	" "	of n.n.n	dyadic "

Stratigrafies of Kn Z5

Any finite n-element pentadic isotropic hypergraf has:

a first stratigrafy, $1_{(\ )}\ (n-1)_0$, of n cofactorial tetradic hypergrafs of type (i.v2v3v4v5);
a second stratigrafy, $2_{(\ )}\ (n-2)_0$, of n(n-1) cofactorial triadic hypergrafs of type (ij.v3v4v5);

and a third stratigrafy, $3(\ )\ (n-3)_0$, of $n(n-1)(n-2)$ cofactorial dyadic hypergrafs of type (ijk.v4v5).

M-ADIC S-TH STRATIGRAFY

If we proceed from pentadic to hexadic and highèr degrees we obtain analogous results. These may be summarized as follows:

Any finite n-element m-adic isotropic hypergraf has m-2 <u>stratigrafic families</u>, viz.:

a first strat. family			of n	$(m-1)$-adic	hypergrafs	;
a second	"	"	of $n.n$	$(m-2)$-adic	"	;
a third	"	"	of n^3	$(m-3)$-adic	"	;
		. . .	. . .			
an s-th	"	"	of n^s	$(m-s)$-adic	"	;
		. . .	. . .			
an $(m-2)$-th	"	"	of n^{m-2}	dyadic	"	.

Also, any finite n-element m-adic isotropic hypergraf has m-2 <u>stratigrafies</u>, viz.:

a first	statigrafy	of n	$(m-1)$-adic
a second	"	of $n(n-1)$	$(m-2)$-adic
a third	"	of $n(n-1)(n-2)$	$(m-3)$-adic
a fourth	"	of $n(n-1)(n-2)(n-3)$	$(m-4)$-adic
		. . .	
an s-th	"	of $n(n-1)(n-2)\ldots(n-s+1)$	$(m-s)$-adic
		. . .	
an $(m-2)$-th	"	of $n(n-1)\ldots(n-m+3)$	dyadic

cofactorial hypergrafs.

The dyadic (m-2)-th atomic stratigrafy superpostulate set consists of $(m-1)^2 + 1$ atomic dyadic superpostulates.

P E R M U T I V I T Y

CLASS 1 PERMUTIVITY

In the preceding sections we studied

(1) the general theory of all-isotropic system functions--
i. e., of system functions which have all their
elements isotropic; and

(2) the general theory of stratigrafic system functions--
i. e., of system functions which have at least
n-s elements isotropic, the isotropy values of
the stratigrafic elements being entirely undetermined.

We proceed now to the consideration of stratigrafic system functions which have the isotropy values of the stratigrafic elements completely determined. In order to do this, we introduce a new fundamental concept.

Let us consider Fig. 25 stratigrafically, i. e., as representing the dyadic first stratigrafy of Kn Z3. Then for each of the n dyadic layers of Fig. 25 it is determined that n-1 of the K elements shall be isotropic. But the isotropy or non-isotropy of the n-th, that is, of the stratigrafic, element is still undetermined. We may add, therefore, either a certain condition on the five validands that shall make the n-th element isotropic with the others; or, the contradictory of that condition.

If we do the latter--i. e., if we add the contradictory of the isotropizing condition, we find that any change whatsoever in the position of the stratigrafic element--irrespective of any permutations among the other n-1 elements--changes the hypergraf. Thus the isotropy value of the stratigrafic element is now completely determined.

Permutivity of an element

We shall call the complete isotropy value of a stratigrafic element the permutivity of that element. The permutivity we have just found ~~found~~ for the stratigrafic element in question belongs to a type that is of extreme importance. It is the type that is the precise opposite of isotropy. For, whereas any permutation of a set of isotropic elements leaves the hypergraf invariant, any change whatsoever in the position of an element having the permutivity just defined, irrespective of what may happen to the other elements, changes the hypergraf.

We find it convenient to symbolize this type of permutivity by writing 1 as a suffix to the number of elements involved. Since, in the present case, there is a single element that has this permutivity, we indicate this fact by the notation 1_1 (the first "1" denotes the number of elements, the suffix "1" denotes the type of permutivity). This corresponds to the notation 2_0 , 3_0 ,..., for isotropy.

Permutivity of a System Function

Just as it leads to no confusion to speak of isotropy of elements, and also of system functions, of grafs, and of hypergrafs, and to speak of stratigrafy of elements, and also of system functions, of grafs, and of hypergrafs, so it will lead to no confusion to speak of permutivity of elements, and also of system functions, of grafs, and of hypergrafs. By the permutivity of a system function we shall mean the permutivities of all the stratigrafic elements together with the isotropy of all the non-stratigrafic elements. Similarly for grafs and for hypergrafs.

Thus the n elements under consideration, Fig. 25, consist of a single element which has permutivity 1, and of n-1 elements which are isotropic. This fact we indicate by saying that any one of the dyadic layers of Fig. 25 has dyadic permutivity $1_1+(n-1)_0$

Since this permutivity is obtained from the first dyadic stratigrafy, we shall call it the dyadic permutivity of class one.

In like manner we obtain the triadic, tetradic,..., (m-1)-adic permutivity, $1_1+(n-1)_0$, of class one.

CLASS 2 PERMUTIVITIES

Permutivity $2_0+(n-2)_0$

Let us consider, in Fig. 29--viewed stratigrafically-- the n(n-1) dyadic layers of type (ij.v3v4), i≠j, which represent the dyadic second stratigrafy of Kn Z4 [Foothote. Although we have defined stratigrafy as a property of certain layers or hypergrafs, i. e., we have stratigrafy intensionally, it is possible also to define it extensionally, as the set of all those and only those hypergrafs which have a certain superpostulationally defined property. And often it is more convenient to use the ex- rather the in- tensional definition. Similar remarks apply also to the two possible ways of defining permutivity]. For each of these n(n-1) dyadic layers, it is determined that n-2 of the K elements shall be isotropic. But the isotropy values--or, as we shall now call them, the permutivities--of the two stratigrafic elements are still undetermined.

We may add, therefore, any of the following additional conditions on the ten validands:

(1) Two isotropizing conditions, that will make both stratigrafic elements isotropic with the other n-2 isotropic elements.

(2) One isotropizing, and one non-isotropizing, condition, that will make one of the stratigrafic elements isotropic, and the other non-isotropic, with the n-2 isotropic elements.

(3) Two non-isotropizing conditions, that will make neither of the stratigrafic elements isotropic with the other elements.

Condition (3) does not yet determine the permutivities of the two stratigrafic elements completely. For, although neither of these elements can be isotropic with the n-2 isotropic elements, it may or may not be the case that these two elements are isotropic with each other. Hence, in addition to (3), we may still add either

(4) A condition for the isotropy of the two stratigrafic elements; or

(5) The contradictory of condition (4).

Conditions (3) and (4) reduce layer (12.v3v4), Fig. 29, to Fig. 32, where the ten validands are specialized to six. In Fig. 32 there are, thus, two isotropies, viz., the original isotropy of n-2 elements, and the isotropy of the two stratigrafic elements. Henceforth, we shall refer to the original isotropy of n-s elements as the main isotropy, and to isotropies formed by stratigrafic elements as stratigrafic isotropies. Conditions (3) and (4) add, therefore, to the main (n-2)-element isotropy a stratigrafic two-element isotropy. The given system function has, then, the dyadic permutivity $2_0+(n-2)_0$.

Since this permutivity is obtained from the second dyadic stratigrafy, we call it a dyadic permutivity of class two.

Permutivity $2_1+(n-2)_0$

If to condition (3) we add the contradictory of (4), we impose certain restrictions upon the ten validands of layer (12.v3v4), Fig. 29, but we do not reduce the number of these validands. The two stratigrafic elements are now prevented from joining the main (n-2)-element isotropy, and also from being isotropic with each other. It will be found that any change whatsoever in the position of either of the stratigrafic elements--irrespective of any permutations among the other n-1 elements--changes the hypergraf. Hence, each of these two elements has permutivity 1_1. Their combined permutivity is therefore 1_1+1_1, which we abbreviate to 2_1

Thus the permutivity of the given system function consists of a main isotropy of n-2 elements, and of permutivity 1 for each of the two stratigrafic elements. In other words, the complete dyadic class two permutivity is $2_1+(n-2)_0$.

For $n-2 > 2$, the permutivities $2_0+(n-2)_0$ and $2_1+(n-2)_0$ are the only ones determined by the second stratigrafy $2_{(\ )}(n-2)_0$.

In like manner we obtain, for $n-2 > 2$, the triadic, tetradic,...., (m-2)-adic permutivities, $2_0+(n-2)_0$ and $2_1+(n-2)_0$, of class two.

For the special case, n-2=2, permutivity $2_1+(n-2)_0$ reduces to 2_1+2_0, and requires no comment. But permutivity $2_0+(n-2)_0$, which becomes 2_0+2_0, introduces an ambiguity. This results from the fact that there are now two two-element isotropies, the main and the stratigrafic. And these two pairs, 2_0 and 2_0, may or may not be isotropic with each other. To indicate this ambiguity, we replace 2_0+2_0 by the notation $(2_0+2_0)_{(\ )}$; Fig. 34.

When the two pairs are isotropic with each other, we write $(2_0+2_0)_0$. Here the zero subscripts inside the parentheses refer to the single elements, and the zero outside refers to the pairs. See Fig. 35. When the two pairs are non-isotropic with each other, we shall write $(2_0+2_0)_1$. Here the subscript 1 refers to the pairs. See Fig. 34 (where additional conditions are supposed to have been put upon the six validands).

CLASS 3 PERMUTIVITIES

Let us consider, in [partial] Fig. 31--viewed stratigrafically--the n(n-1)(n-2) dyadic layers of type (ijk.v4v5), i, j, k, all distinct, which represent the dyadic third stratigrafy of Kn Z5. For each of these dyadic layers it is determined that n-3 of the K elements shall be isotropic. But the permutivities of the three stratigrafic elements are still undetermined.

For $n-3 > 3$, there are just four such permutivities, as follows:

<u>Permutivity</u> $3_0+(n-3)_0$

Enough conditions may be imposed upon the 17 validands of layer (123.v4v5), Fig. 31, to produce a complete permutivity consisting of the main (n-3)-element isotropy and a three-element stratigrafic isotropy. See Fig. 33.

<u>Permutivity</u> $(1_1+2_0)+(n-3)_0$

Or, the conditions may impose permutivity 1 upon a single stratigrafic element, and stratigrafic isotropy upon the two others. This leads to permutivity $(1_1+2_0)+(n-3)_0$. See Fig. 36.

<u>Permutivity</u> $3_1+(n-3)_0$

Also, the conditions may require permutivity 1 for <u>each</u> of the three stratigrafic elements. In this case, although several restrictions are imposed upon the validands of layer (123.v4v5), Fig. 31, the <u>number</u> of these validands is not reduced.

<u>Permutivity</u> $3_\kappa+(n-3)_0$ <u>Cyclopermutivity</u>

Finally, we may impose conditions that will give the three stratigrafic elements a new type of permutivity.

It may happen, for instance, Fig. 37, that the permutation that replaces element 1 by 2, 2 by 3, and 3 by 1, leaves the hypergraf unchanged. This introduces a new type of isotropy value. We shall call this <u>cyclopermutivity</u>, and symbolize it by writing κ to the number of elements involved. Thus we indicate a cyclopermutivity of three elements by 3_κ. The smallest number of elements that can be cyclopermutive is three, just as the smallest number of elements that can be isotropic is two. On the other hand, a single element can have the permutivity of type 1.

<u>Maxpermutivity</u>

In a set of n elements which are isotropic, any change in the position of any subset of elements leaves the hypergraf unchanged. On the contrary, in a set of n elements each of permutivity 1, any change in the position of any element, regardless of what happens to the other elements, changes the hypergraf. Hence, in an n-element isotropy all the n! hypergrafs of the factorial family are identical; in an n-element set each of permutivity 1, all the n! hypergrafs of the factorial family are distinct. Thus isotropy involves a minimum number of distinct hypergrafs in the factorial family; permutivity 1 involves a maximum number. For this reason we shall call permutivity 1--for a single element, or for a set of elements--the maximum permutivity, or, briefly, maxpermutivity.

Any permutivity which makes the n! hypergrafs of a factorial family neither all identical nor all distinct--i. e., which is "between" an isotropy and a maxpermutivity--we shall call a mediopermutivity. An example of a mediopermutivity is cyclopermutivity.

Summary of Class 3 Permutivities

For $n-3 > 3$, the dyadic third stratigrafy, $3_{(\ )}\ (n-3)_0$, determines
the four class-3 dyadic permutivities:
$3_0+(n-3)_0$, $(1_1+2_0)+(n-3)_0$, $3_1+(n-3)_0$, $3_K+(n-3)_0$.

For the special case $n-3=3$, the ambiguous $(3_0+3_0)_{(\ )}$, Fig. 38, leads to the two permutivities

$(3_0+3_0)_0$, Fig. 39, and $(3_0+3_0)_1$, Fig. 38 (where certain

additional restrictions are supposed to be put upon the six validands).

In the same way we obtain, for $n-3 > 3$, the triadic, tetradic, ..., (m-3)-adic permutivities of class three.

CLASS 4 PERMUTIVITIES

Fig. 40 is a sample of the n(n-1)(n-2)(n-3)dyadic layers of type $(i_1 i_2 i_3 i_4 . v_5 v_6)$, $[i_1, i_2, i_3, i_4$, all distinct] which represent the dyadic fourth stratigrafy of Kn 26. For each of these dyadic layers it is determined that n-4 of the K elements shall be isotropic. But the permutivities of the four stratigrafic elements are still undetermined.

For $n-4 > 4$, there are just nine such permutivities, as follows: Permutivity $4_0+(n-4)_0$, Fig. 41;

" $(1_1+3_0)+(n-4)_0$, Fig. 42;

Permutivity $(2_1+2_0)+(n-4)_0$, Fig. 43;

" $(2_0+2_0)_1+(n-4)_0$, Fig. 44;

" $(2_0+2_0)_0+(n-4)_0$, Fig. 45;

" $4_K+(n-4)_0$, Fig. 46;

" $1_1+3_K+(n-4)_0$, Fig. 47;

" $4_1+(n-4)_0$, Fig. 40.

These eight permutivities require no explanation.

Multiplicational Permutivities

The dyadic fourth stratigrafy, $4_{(\ )}(n-4)_0$, determines, in addition to the above eight permutivities, also a ninth. We may impose upon the 26 validands of Fig. 40 enough conditions to reduce Fig. 40 to Fig. 48. In this case, the permutation of the elements 1 and 2 alone changes the hypergraf, and the permutation of the elements 3 and 4 alone also changes the hypergraf. But the simultaneous permutation of 1 and 2, and also of 3 and 4, leaves the hypergraf unchanged. This simultaneous permutation of sets of elements is an example of what we shall call a multiplicational permutivity. In this case, the "multiplication" consists of two "factors", of two elements each. We shall symbolize this permutivity by 2.2, or, briefly, by 2^2 . Just as isotropy requires a minimum of two elements, and cyclopermutivity a minimum of three elements, so a multiplicational permutivity requires a minimum of four elements.

The ninth permutivity determined by $4_{(\ )}(n-4)_0$, is therefore

Permutivity $2^2+(n-4)_0$, Fig. 48.

For the special case n-4=4, the ambiguous $(4_0+4_0)_{(\ )}$ leads to the two permutations, $(4_0+4_0)_0$ and $(4_0+4_0)_1$.

In a similar way we obtain the triadic, tetradic, ... (m-4)-adic permutivities of class four.

CLASS S PERMUTIVITIES

By precisely analogous methods we find the sets of permutivities determined by the fifth, the sixth, ..., the s-th stratigrafies, by determining, for each of these stratigrafies, the necessary dyadic, triadic, ..., (m-s)-adic conditions to be imposed upon the respective sets of validands.

<u>Class 5 Permutivities</u>

For example, among the permutivities determined by the fifth stratigrafy, $5_{(\ )}(n-5)_0$, are the following [We omit, in each case, ~~the~~ the main isotropy, "$+(n-5)_0$"].

5_0, (1_1+4_0), (2_0+3_0), (2_1+3_0), $1_1+(2_0+2_0)_0$, $1_1+(2_0+2_0)_1$, (3_k+2_0), (3_1+2_0), 5_k, (1_1+4_k), (2_1+3_k), (1_1+2^2), 5_1.

<u>Class 6 Permutivities</u>

Similarly, among the permutivities determined by the sixth stratigrafy, $6_{(\ )}(n-6)_0$, are following [We omit, in each case, the main isotropy, "$+(n-6)_0$"].

6_0, (1_1+5_0), (2_0+4_0), (2_1+4_0), $(3_0+3_0)_0$, $(3_0+3_0)_1$, $(1_1+2_0+3_0)$, (3_k+3_0), (3_1+3_0), $2_1+(2_0+2_0)_0$, $2_1+(2_0+2_0)_1$, (4_k+2_0), $(1_1+3_k+2_0)$, (2^2+2_0), (4_1+2_0), 6_k, (1_1+5_k), (2_1+4_k), (3_1+3_k), (2_1+2^2), 6_1, $(2_0+2_0+2_0)_0$, $(2_0+2_0+2_0)_1$, $(2_0+2_0+2_0)_k$, $(3_k+3_k)_0$, $(3_k+3_k)_1$, 2^3 [=2.2.2, a "multiplicational" permutivity].

PERMUTIVITY SUPERPOSTULATE SETS

When we add, to a given stratigrafy superpostulate set, enough conditions upon the validands to determine the permutivity of all the stratigrafic elements, we obtain complete ("categorical") superpostulate sets. These we shall call <u>permutivity</u> superpostulate sets.

Thus we proceed, by a definite method, <u>from</u> complete isotropy superpostulate sets, <u>through</u> incomplete stratigrafy superpostulate sets, <u>to</u> complete permutivity superpostulate sets.

A P P L I C A T I O N S

In the preceding sections we presented a somewhat detailed exposition of our method, this method involving the use of the new concepts of isotropy, stratigrafy, and permutivity. We shall now mention very briefly, and without proof, some of the more important applications of this method.

<u>Atomic Postulate Sets</u>

If, in any permutivity superpostulate set, we replace all the validands by an appropriate set of validata, by permissible "+'s" and "-'s", we obtain a complete set of atomic postulates.

<u>Destratigrafized Atomic Postulate Sets</u>

In this set of atomic postulates, we may suppress the mention of stratigrafic elements by "camouflaging" these elements as "constants". [This accounts for the existence of the "principle of duality" in such subjects as Boolean algebra and projective geometry. Where several stratigrafic elements are suppressed, we have the "principle of multiplicity".]

<u>Destratigrafized Deatomized Postulate Sets</u>

Finally, we may combine atomic postulates in various ways into single non-atomic postulates. We then obtain the ordinary postulate sets of current mathematical logic.

<u>Analytic Katabasis</u>

We may thus proceed, by definite steps, from isotropy to stratigrafy, then to permutivity, then to atomic postulates and finally to "ordinary postulates".

<u>Analytic Anabasis</u>

We may also reverse the process, begin with ordinary postulates, and end with isotropy superpostulates.

<u>Maximal Independence</u>

A set of p postulates such that no p-1 of the postulates imply the p-th is called an independent set. Thus, if A and B are any two postulates of an independent set, A does not imply the whole of B. A may imply, however, a part--perhaps a very large part--of B. Likewise, the subset consisting of all the postulates except A may imply a very large part of A. For this reason we call such independence, <u>minimal</u> independence. By <u>maximal</u> independence, on the other hand, we shall mean a set of postulates such no postulate implies <u>any</u>

part of any other postulate. It is not difficult to see that any atomic postulate set, as we have defined it, is maximally independent [See the writer's abstract entitled "Mutually Prime Postulates", Bulletin of the Americal Mathematical Society, March, 1916, p. 287].

Consistency

A single atomic postulate involves just one validatum, i. e., either "+" alone, or "-" alone. It is therefore bound to be self-consistent. Any two atomic postulates have "spheres of influence" which are always mutually exclusive. There is consequently no possibility for one atomic postulate to interfere with any other. They are therefore bound to be interconsistent.

Completeness

Atomic superpostulate sets, and thus atomic postulate sets, are constructed according to a definite formula, and so we can always ascertain whether or not we have included all the atomic postulates generated by the formula. Hence, we know whether or not our postulate set is complete ("categorical").

Postulational Technique

By the use of atomic postulates we are enabled to solve the three important problems of postulational technique, viz., consistency, independence, and completeness, by a non-interpretational method. And we show that this holds for non-finite as well as for finite cases.

Order Types

A theory of ordinal isotropy, stratigrafy, and permutivity can be developed, to a great extent parallel to the relational isotropy, stratigrafy, and permutivity expounded above. By means of this ordinal theory, all of our results may be extended to the case of transfinite order types.

Cardinoid Numbers

The concept of permutivity enables us to generalize the notion of cardinal number to that of cardinoid number (which we do not define here). The theory of cardinoid number is used especially for those system functions where the assumption of some equivalent of Zermelo's axiom, or of the multiplicative axiom, seems otherwise unavoidable.

Equivalence of System Functions

Any two system functions are shown to be equivalent, i.e uniquely intertranslatable, if they are cocardinal. This

holds: (1) for finite system functions;
(2) for (co-ordinal or non-co-ordinal) countable system functions;
(3) for lineable system functions [We propose to call a class "which has the power of the continuum" a lineable class. And we shall speak of a lineably infinite class, and of lineability].
(4) for any two system functions, each of which is known to have a cardinal, the two having the same cardinal.

Cocardinality, however, is not a necessary condition for equivalence. Any two cocardinoid system functions are also equivalent.

Systems

So far we have been concerned with system functions rather than with systems. There is a theory of interpretational isotropy, stratigrafy, and permutivity, which leads to the general theory of interpretational relativity, just as relational isotropy, stratigrafy, and permutivity leads to the general theory of notational relativity.

Special Example: Einstein's Relativity

Einstein's "generalized theory of relativity" may be called:

(1) from the notational point of view, the general theory of tensor relativity;
(2) from the interpretational point of view, the the general theory of motional relativity.

(1) is a special case of what we have called the general theory of notational relativity; (2) is a special case of what we have called the general theory of interpretational relativity.

Foundations of Deductive Logic

Instead of basing deductive logic on such relations and operations as propositional negation, implication, disjunction, or incompatability, and on corresponding "ordinary" postulates (e. g., those of Principia Mathematica), it is possible to base deductive logic on the more fundamental notions of propositional isotropy, stratigrafy, and permutivity. This superpostulational foundation for logic has important philosophical bearings.

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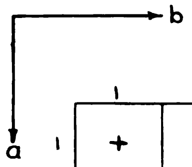


	1	2	3
1	+	−	−
2	−	+	−
3	−	−	+

Fig. 1

	1	3	2
1	+	−	−
3	−	+	−
2	−	−	+

Fig 2

	2	1	3
2	+	−	−
1	−	+	−
3	−	−	+

Fig. 3

	2	3	1
2	+	−	−
3	−	+	−
1	−	−	+

Fig. 4

	3	1	2
3	+	−	−
1	−	+	−
2	−	−	+

Fig. 5

	3	2	1
3	+	−	−
2	−	+	−
1	−	−	+

Fig. 6

	1	2	3
1	α_{11}	α_{12}	α_{13}
2	α_{21}	α_{22}	α_{23}
3	α_{31}	α_{32}	α_{33}

Fig. 7

	1	3	2
1	α_{11}	α_{13}	α_{12}
3	α_{31}	α_{33}	α_{32}
2	α_{21}	α_{23}	α_{22}

Fig 8

	2	1	3
2	α_{22}	α_{21}	α_{23}
1	α_{12}	α_{11}	α_{13}
3	α_{32}	α_{31}	α_{33}

Fig 9

	2	3	1
2	α_{22}	α_{23}	α_{21}
3	α_{32}	α_{33}	α_{31}
1	α_{12}	α_{13}	α_{11}

Fig. 10

	3	1	2
3	α_{33}	α_{31}	α_{32}
1	α_{13}	α_{11}	α_{12}
2	α_{23}	α_{21}	α_{22}

Fig 11

	3	2	1
3	α_{33}	α_{32}	α_{31}
2	α_{23}	α_{22}	α_{21}
1	α_{13}	α_{12}	α_{11}

Fig 12

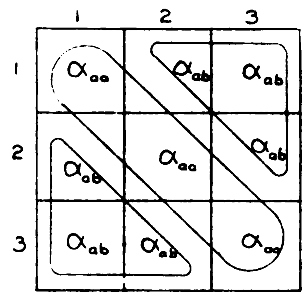


Fig 13

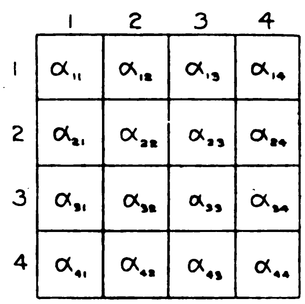


Fig. 14

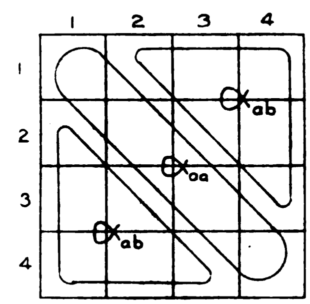


Fig 15

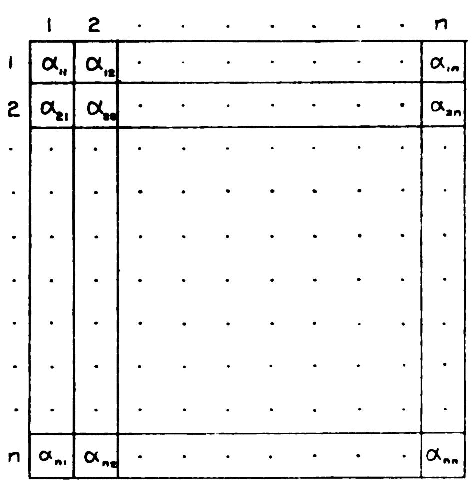


Fig 16

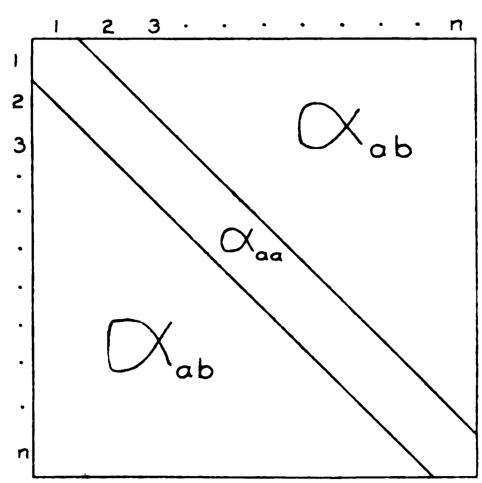


Fig 17

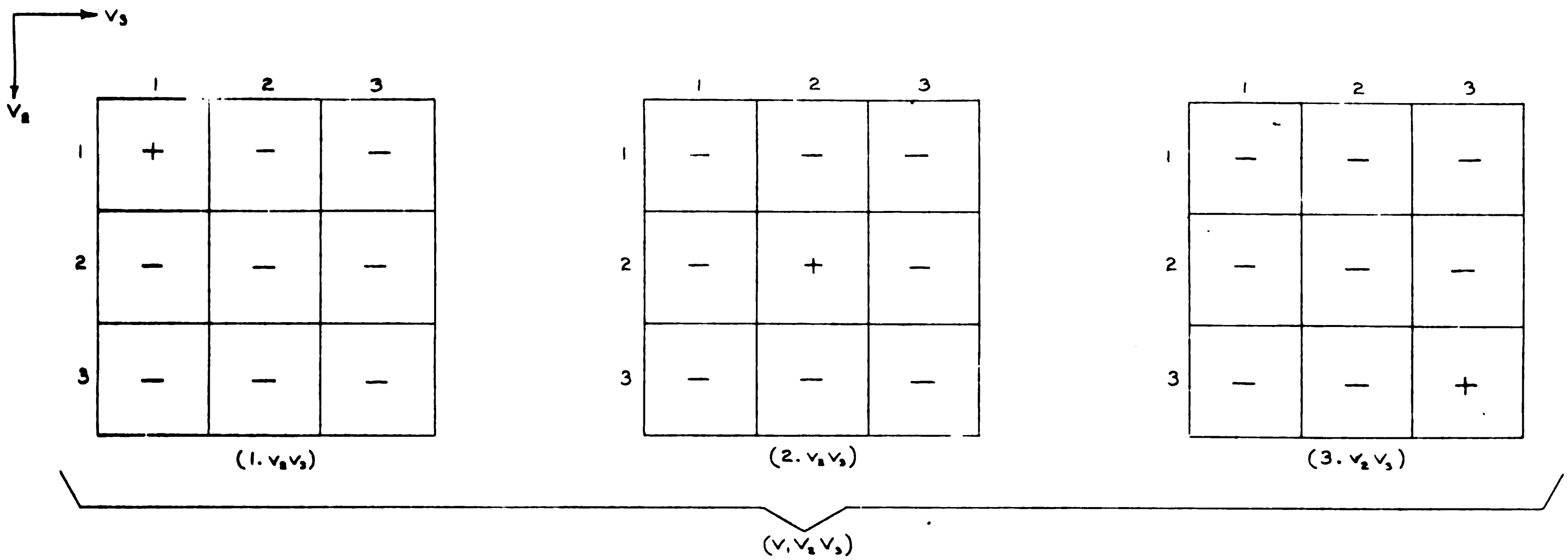


Fig. 18

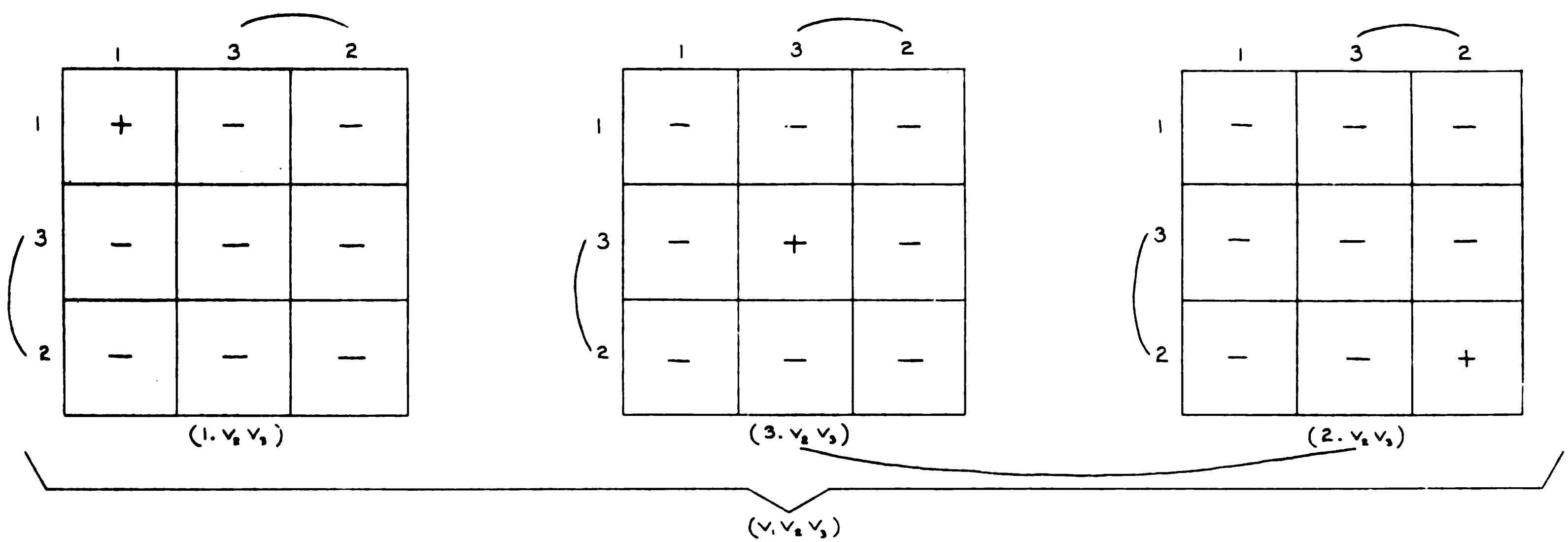


Fig. 19

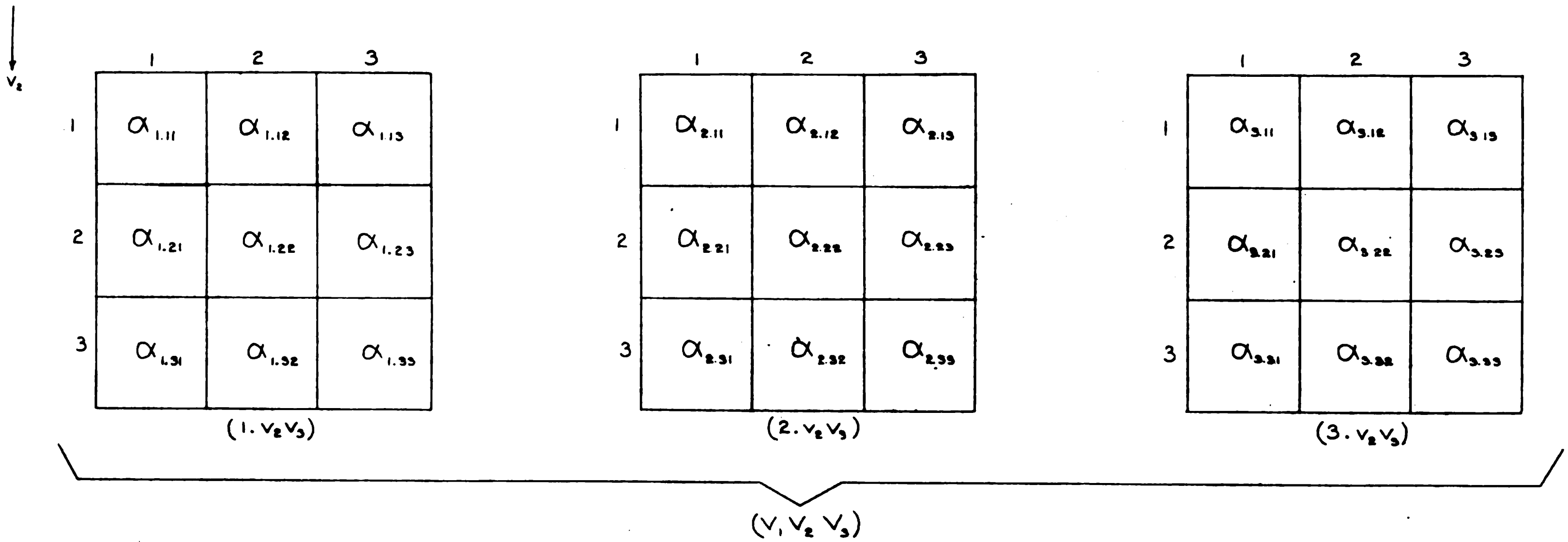


Fig. 20

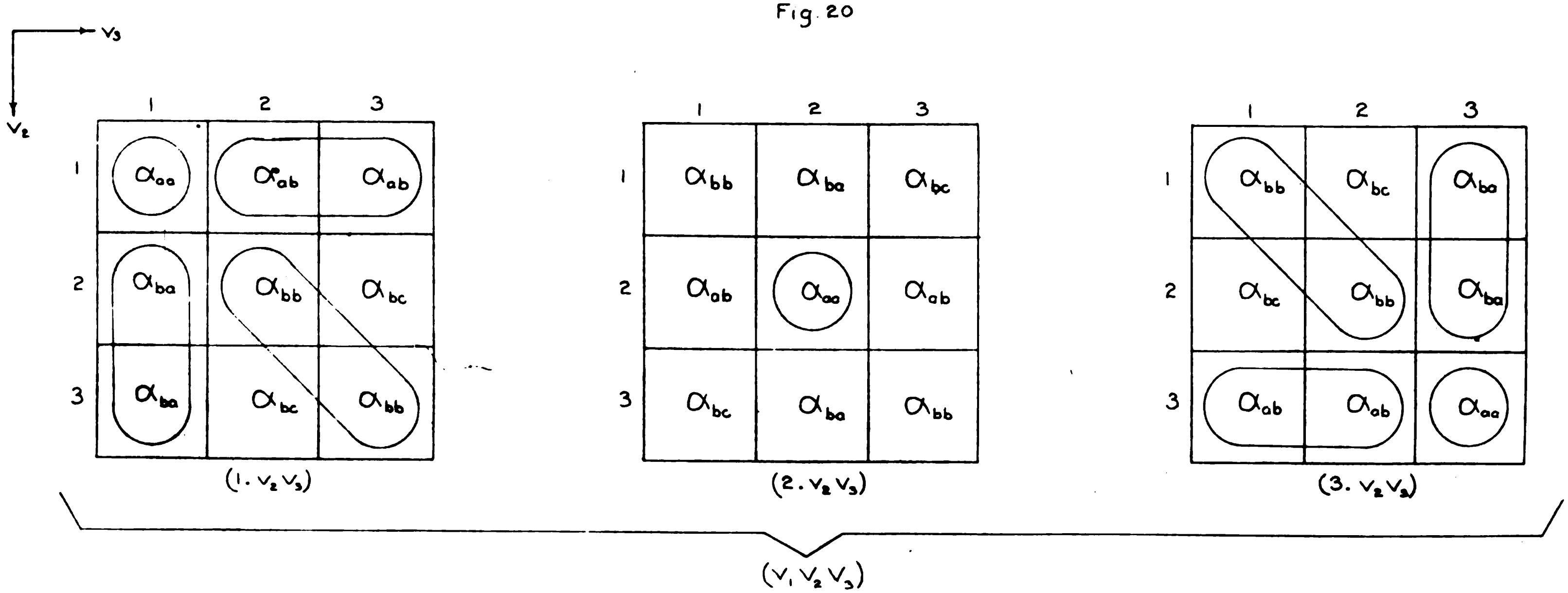
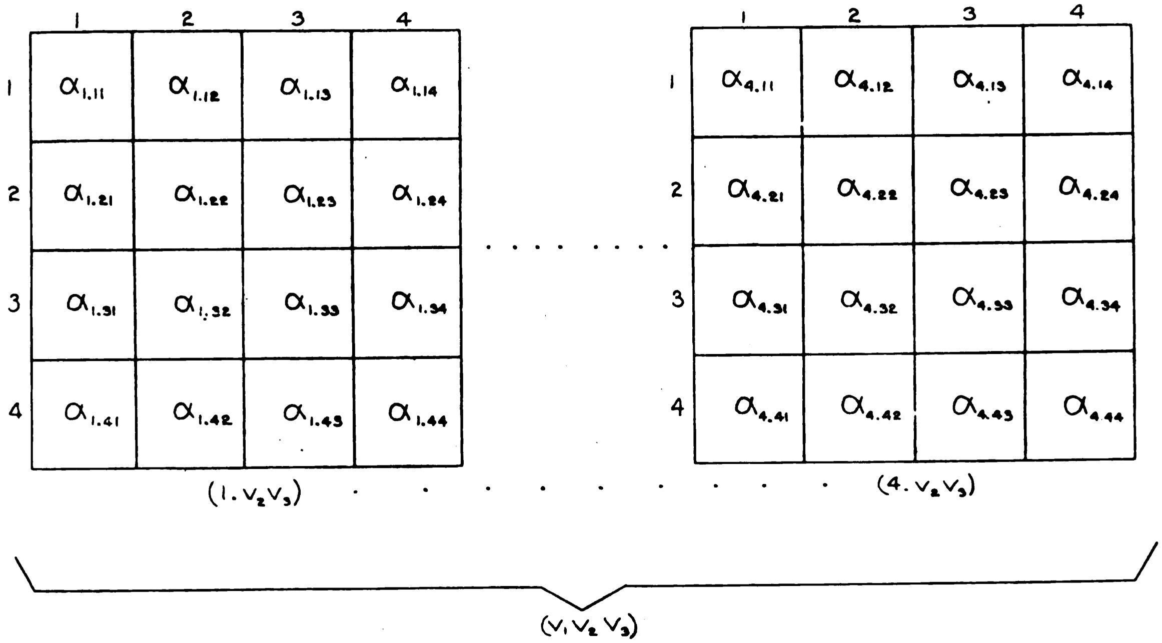
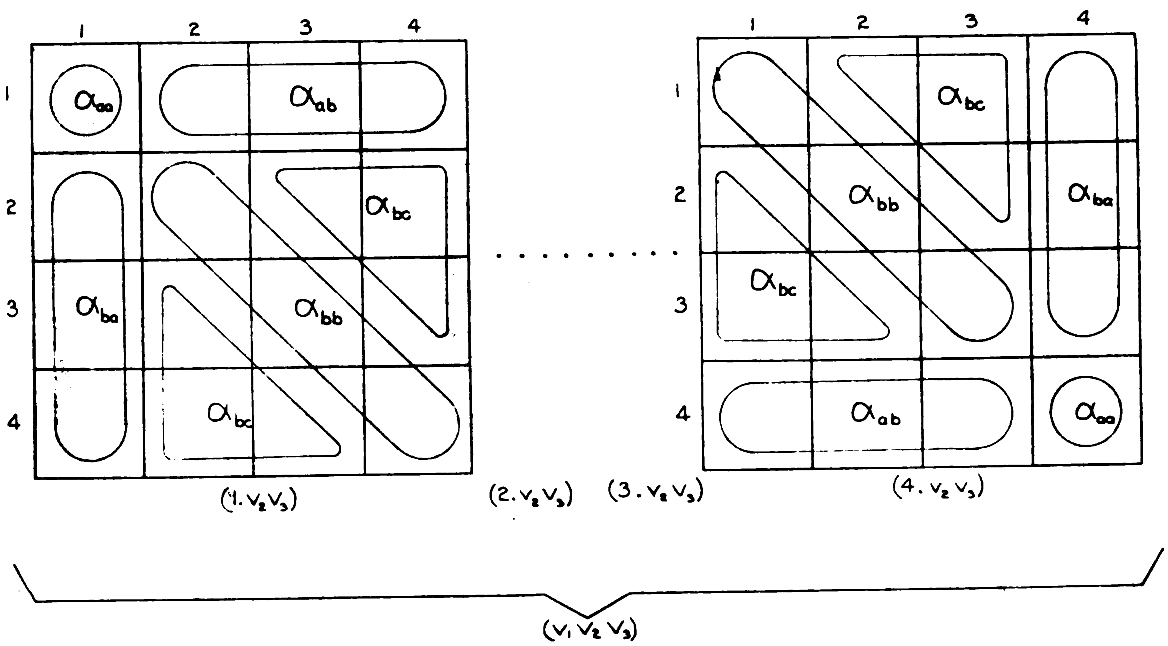


Fig 21



1 2 3 4
α1.11 α1.12 α1.13 α1.14
α1.21 α1.22 α1.23 α1.24
α1.31 α1.32 α1.33 α1.34
α1.41 α1.42 α1.43 α1.44
(1. v2v3)
α4.11 α4.12 α4.13 α4.14
α4.21 α4.22 α4.23 α4.24
α4.31 α4.32 α4.33 α4.34
α4.41 α4.42 α4.43 α4.44
(4. v2v3)
(v1 v2 v3)

Fig 22



1 2 3 4
αaa
αab
αba
αbb
αbc
(1.v2v3)
(2.v2v3)
(3.v2v3)
(4.v2v3)
(v1 v2 v3)

Fig. 23

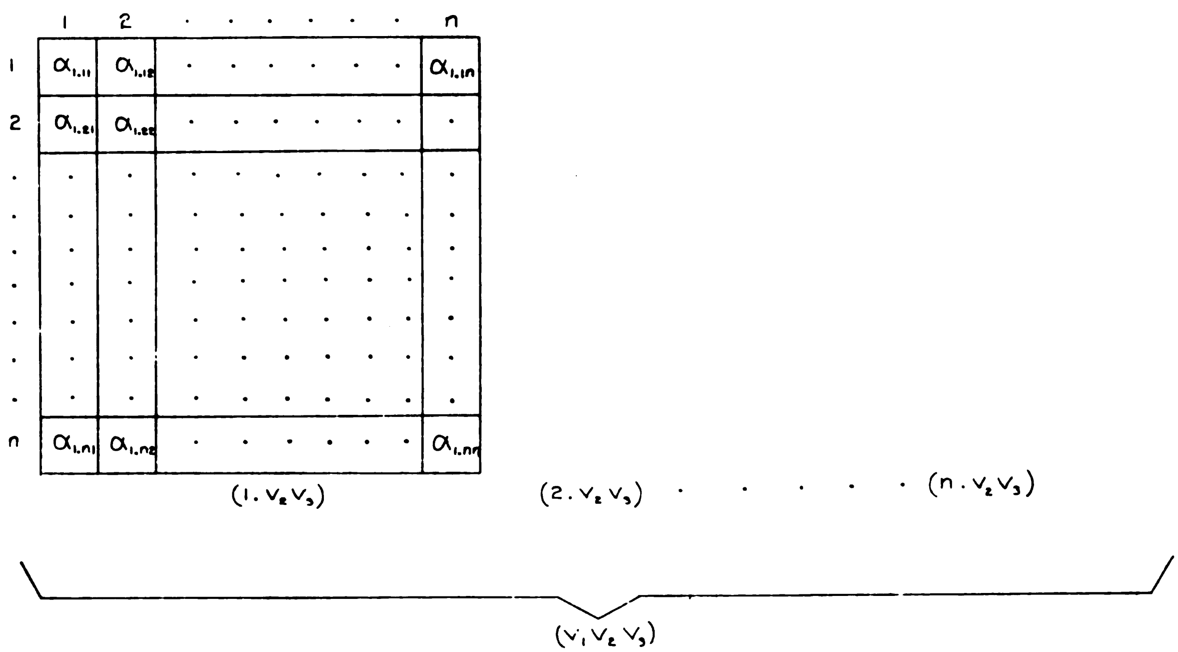

α1.11 α1.12 α1.1n
α1.21 α1.22
α1.n1 α1.n2 α1.nn
(1. v2 v3)
(2. v2 v3)
(n . v2 v3)
(v1 v2 v3)

Fig 24

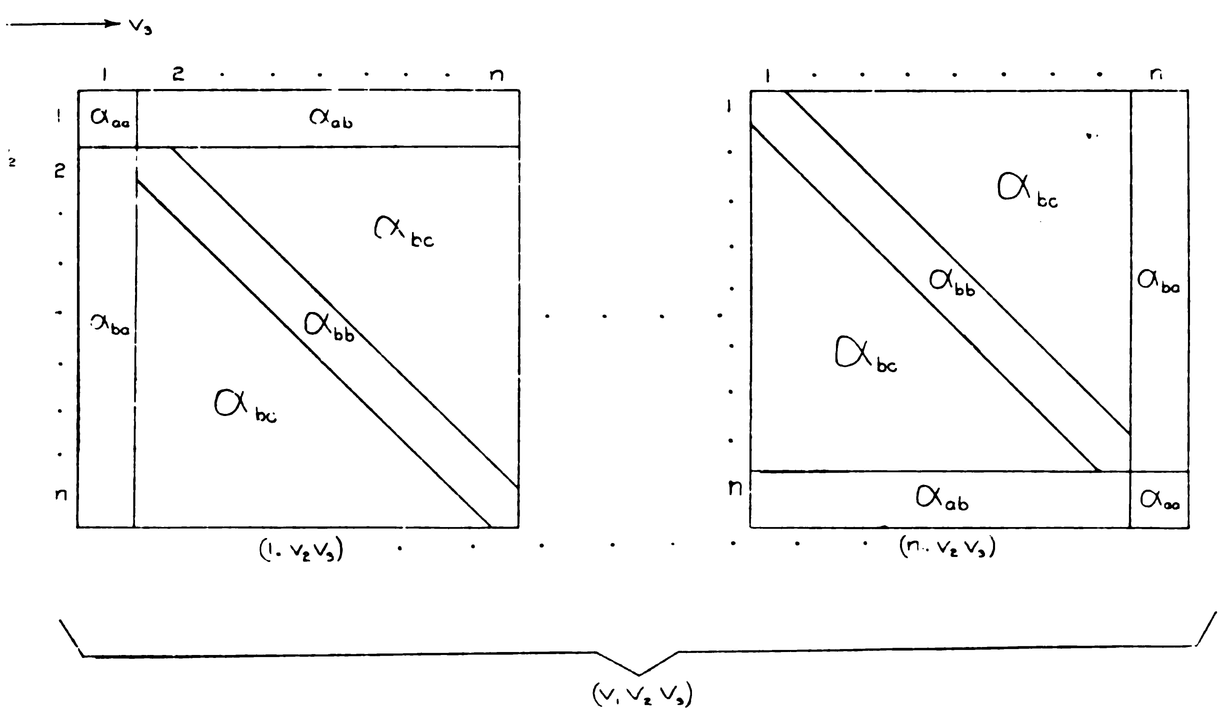
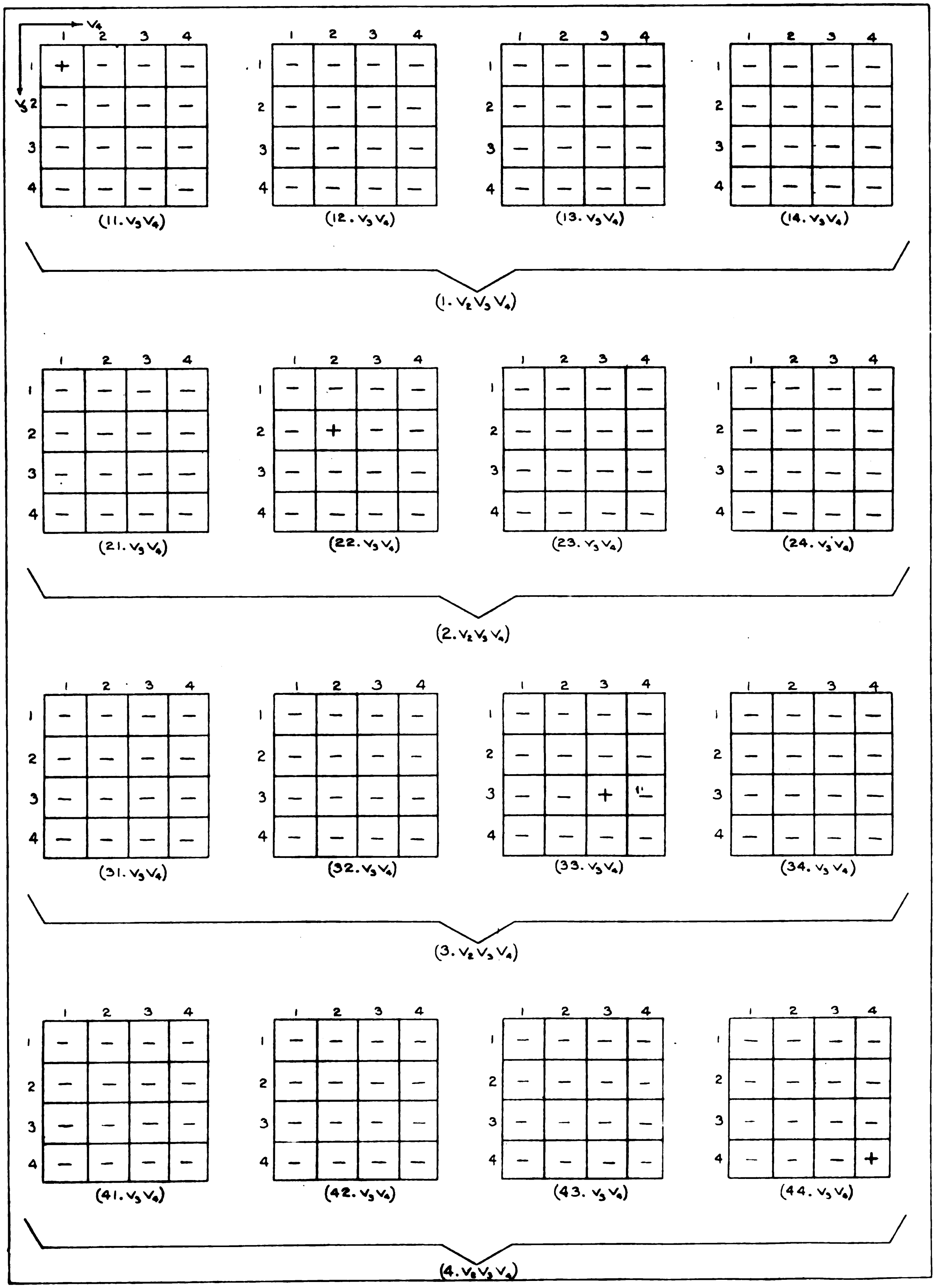

v3
αaa
αab
αba
αbb
αbc
(1. v2 v3)
(n. v2 v3)
(v1 v2 v3)

Fig. 25



$(V_1 V_2 V_3 V_4)$

Fig. 26

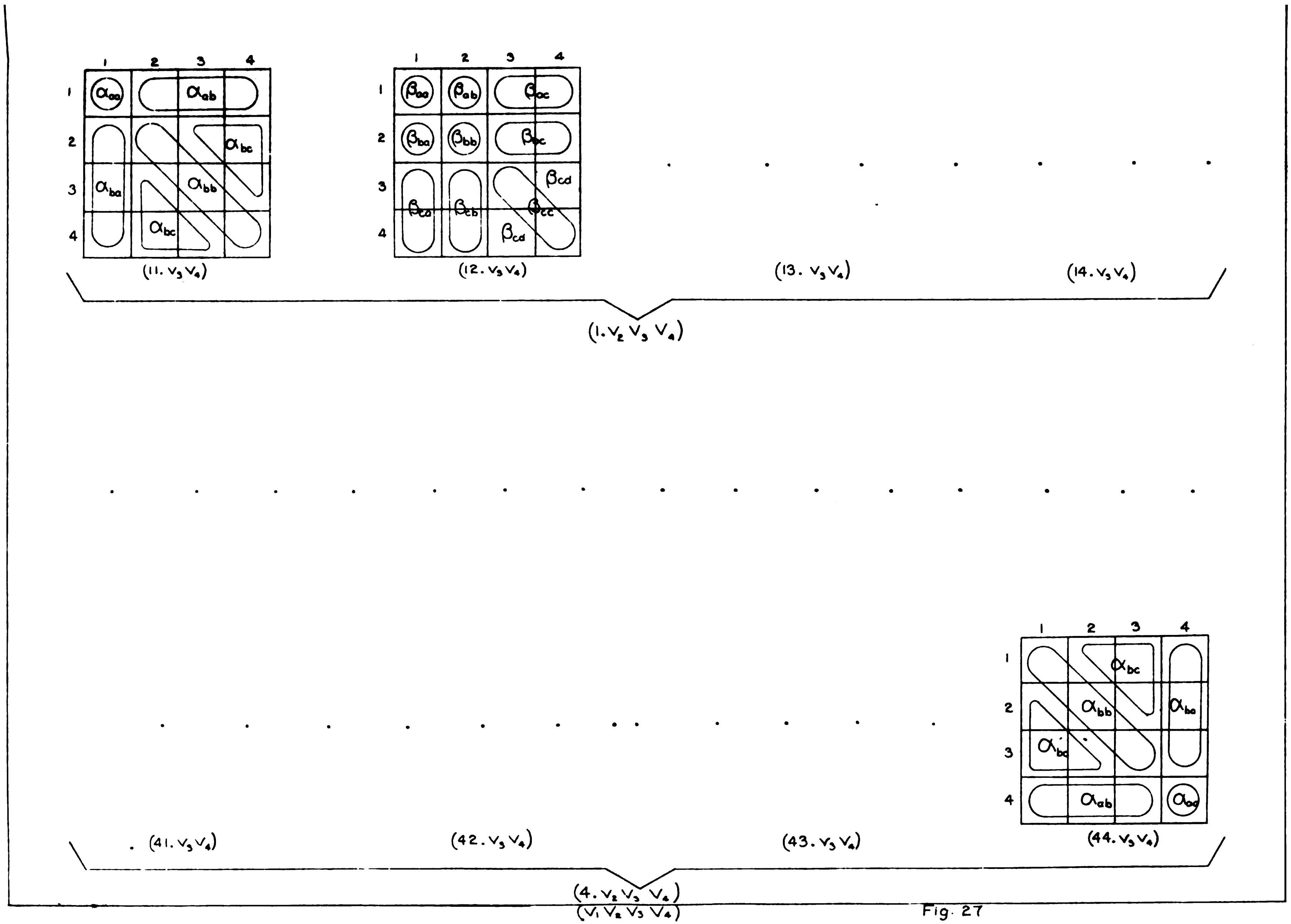
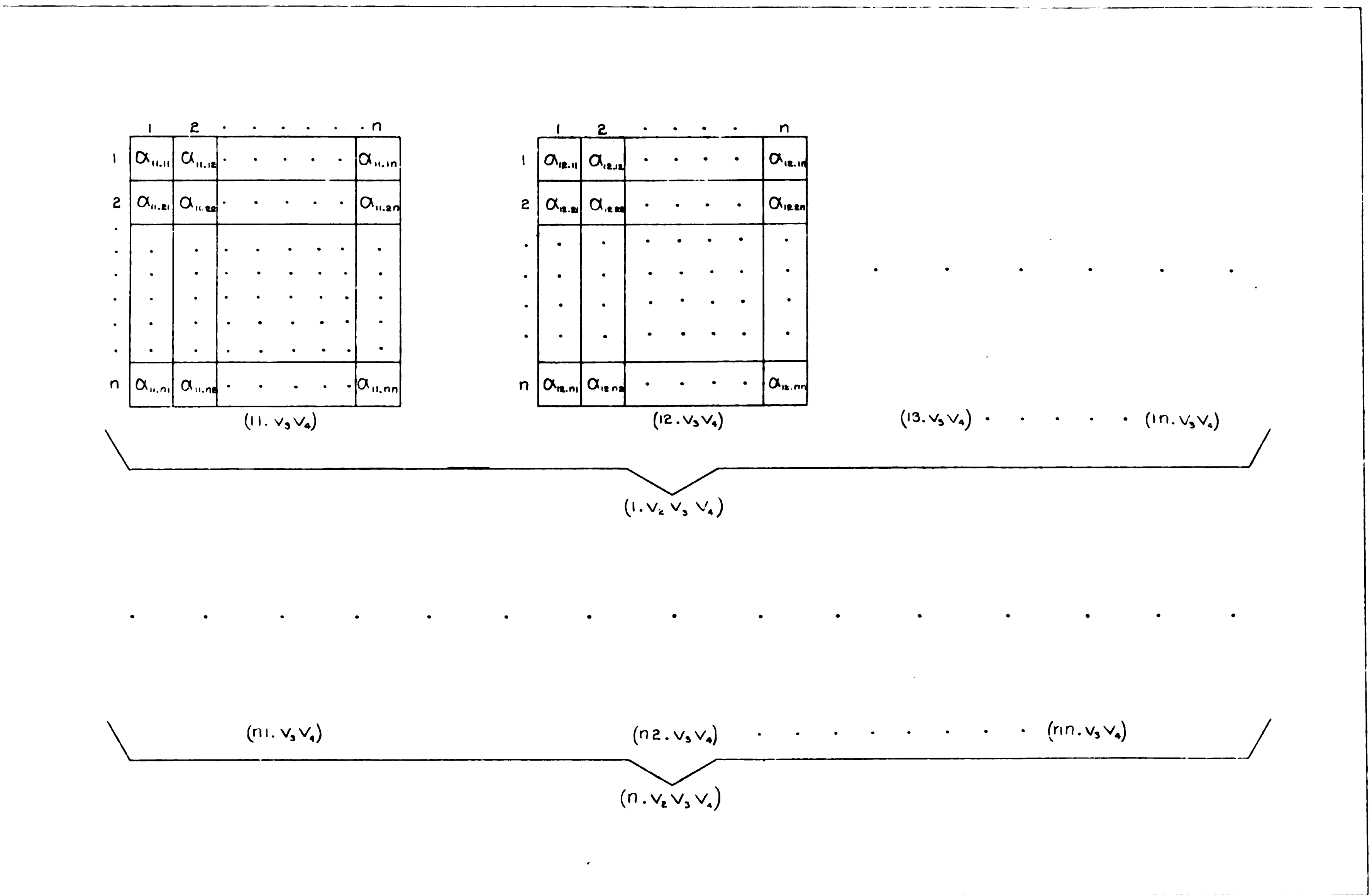
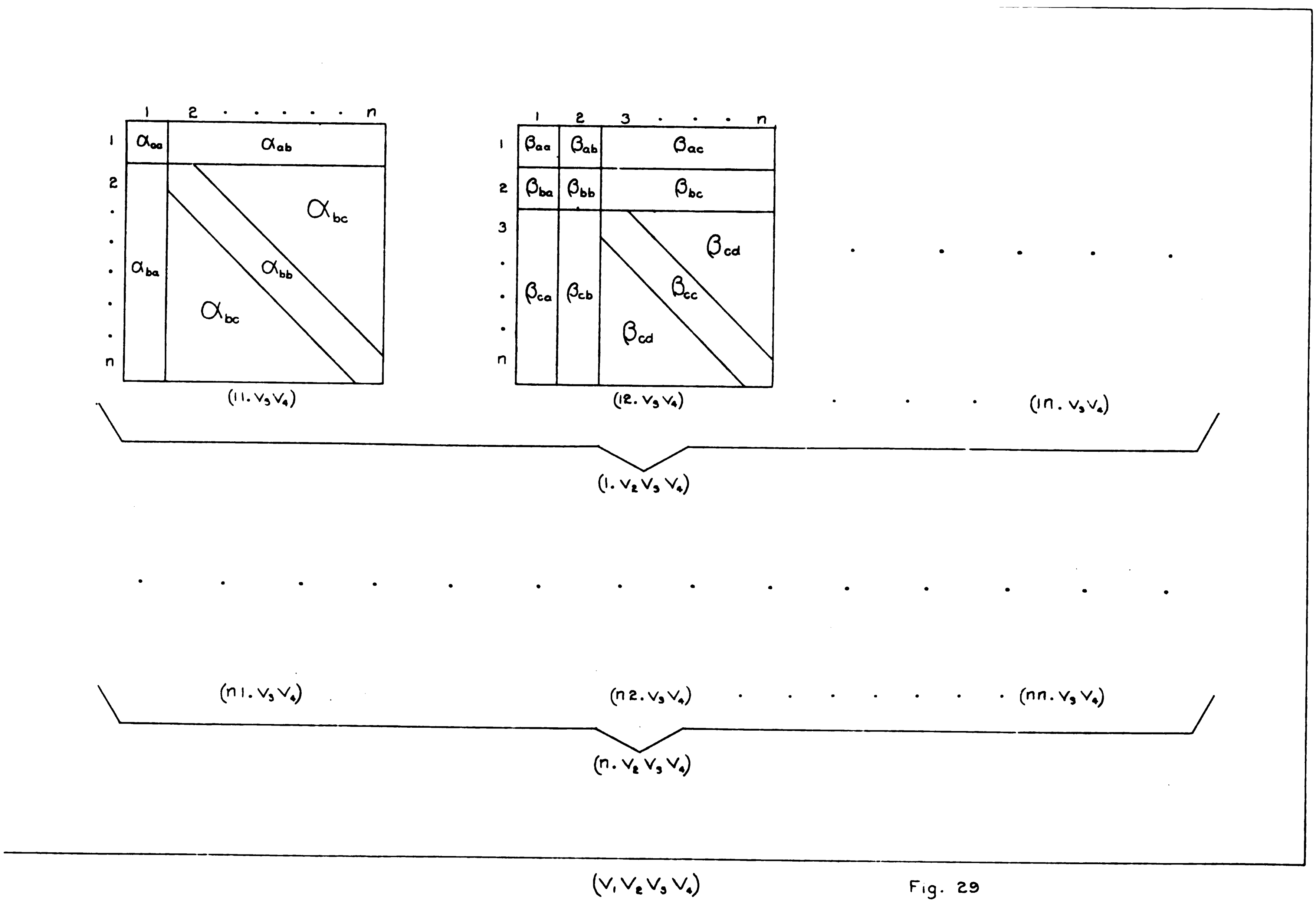

(11. V3 V4)
(12. V3 V4)
(13. V3 V4)
(14. V3 V4)
(1. V2 V3 V4)
(41. V3 V4)
(42. V3 V4)
(43. V3 V4)
(44. V3 V4)
(4. V2 V3 V4)
(V1 V2 V3 V4)

Fig. 27



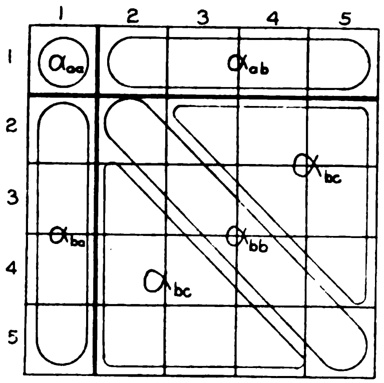
$(V_1 V_2 V_3 V_4)$

Fig. 28

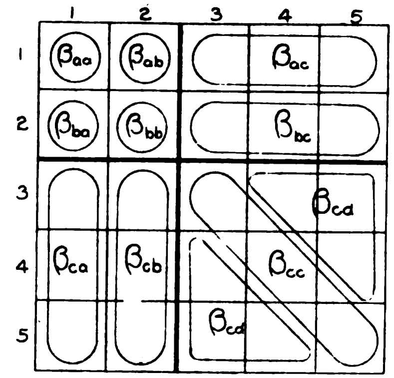


(11. V_3V_4)
(12. V_3V_4)
(1n. V_3V_4)
(1. $V_2V_3V_4$)
(n1. V_3V_4)
(n2. V_3V_4)
(nn. V_3V_4)
(n. $V_2V_3V_4$)
($V_1V_2V_3V_4$)

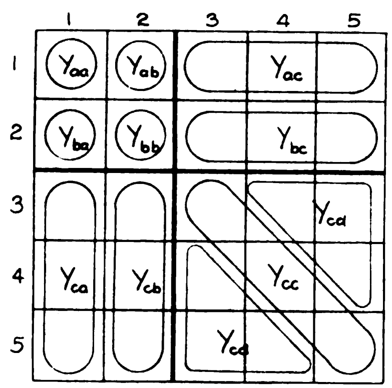
Fig. 29



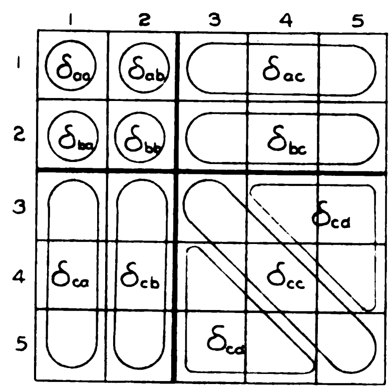
(111. V_4V_5)



(112. V_4V_5)

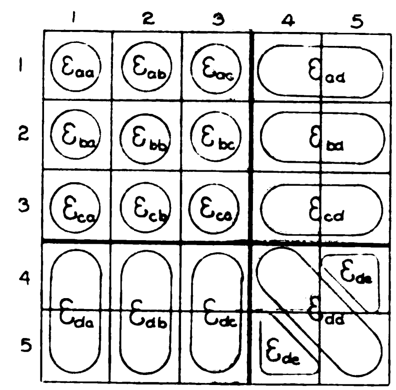


(121. V_4V_5)



(122. V_4V_5)

Fig 30



(123. V_4V_5)

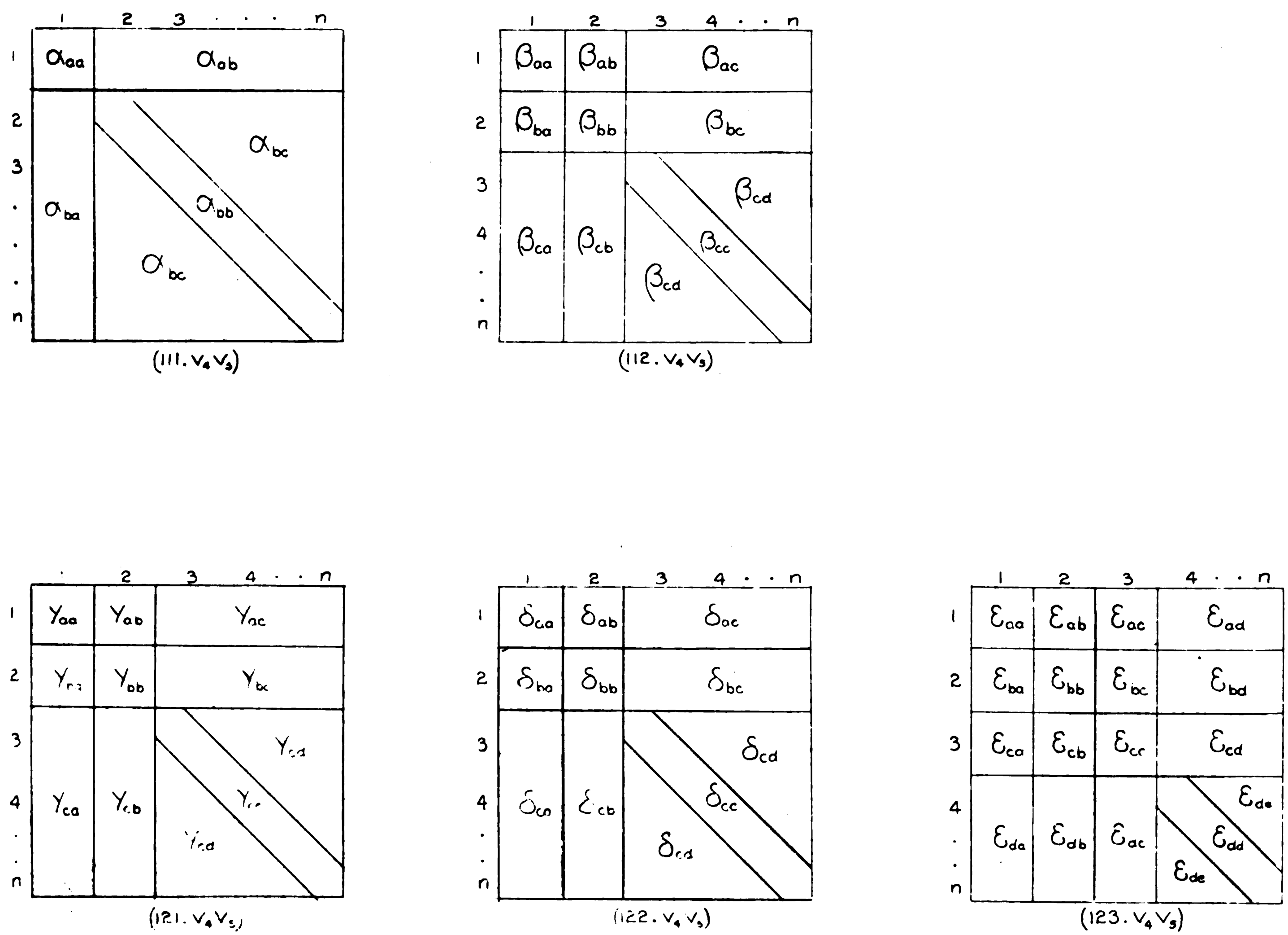
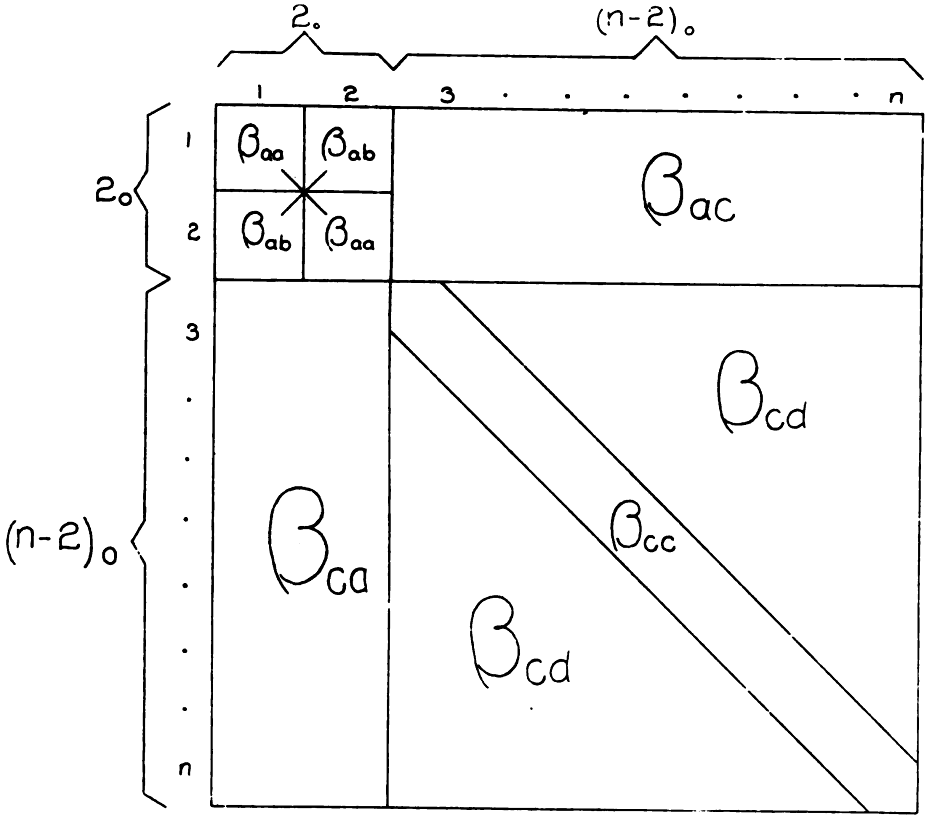
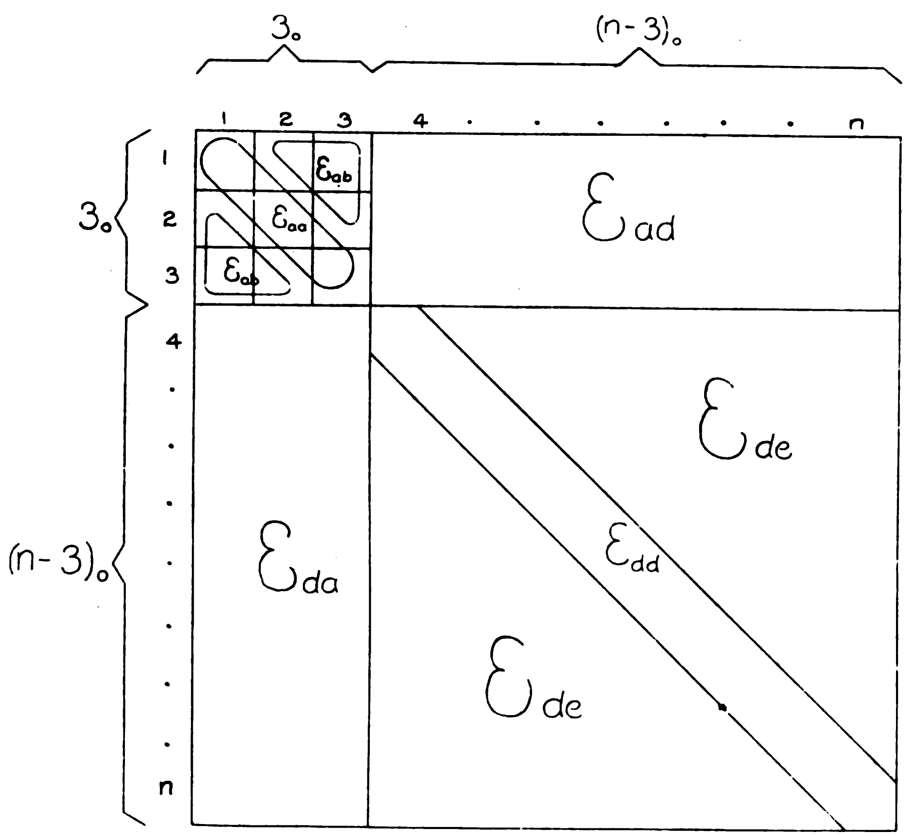


Fig. 31



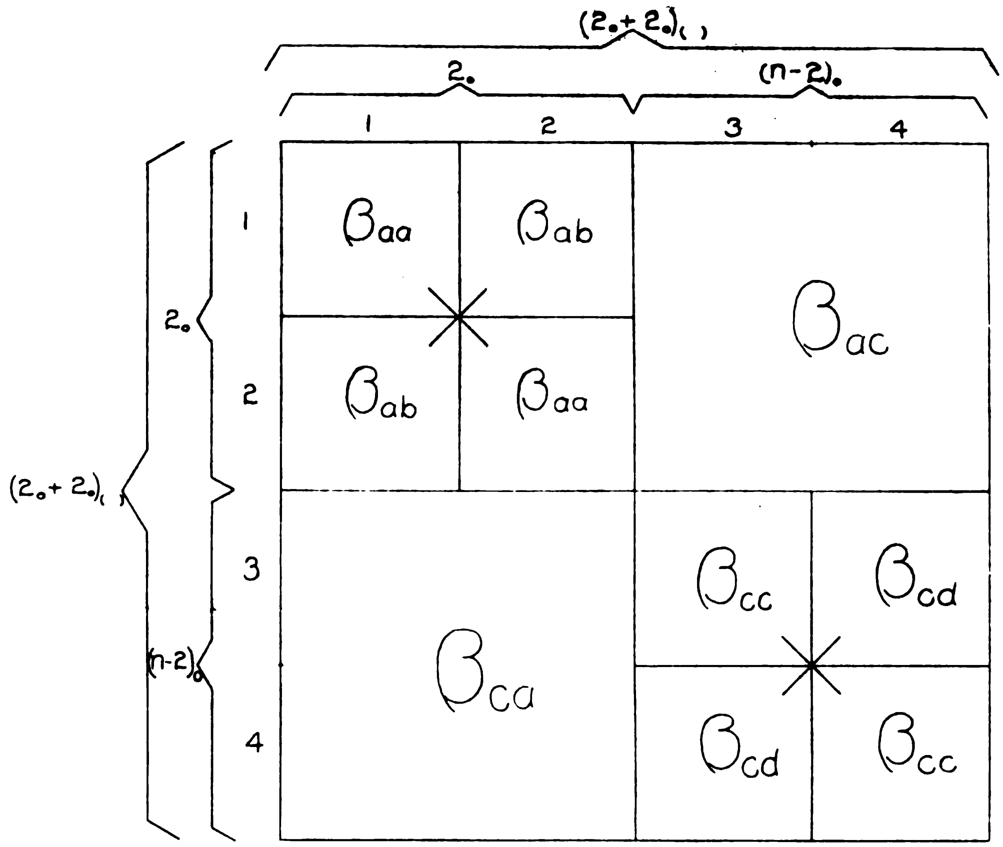
$(p_1 p_2 \cdot v_3 v_4)$

Fig. 32



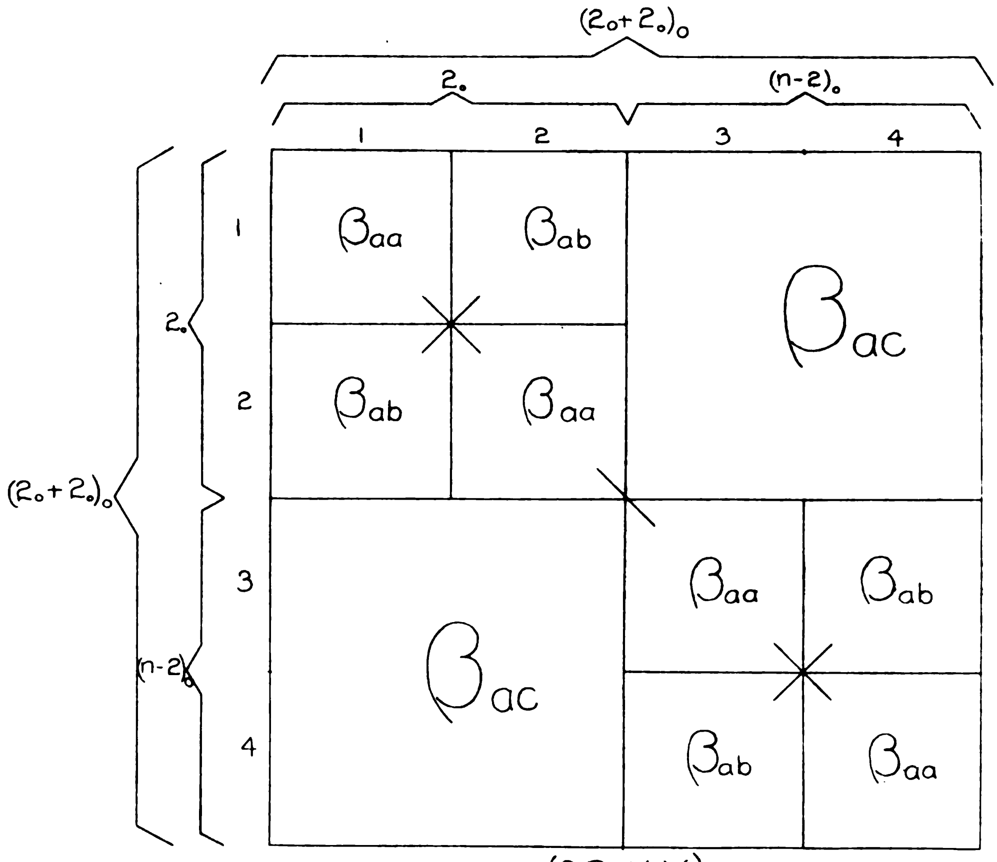
$(p_1 p_2 p_3 \cdot v_4 v_5)$

Fig. 33



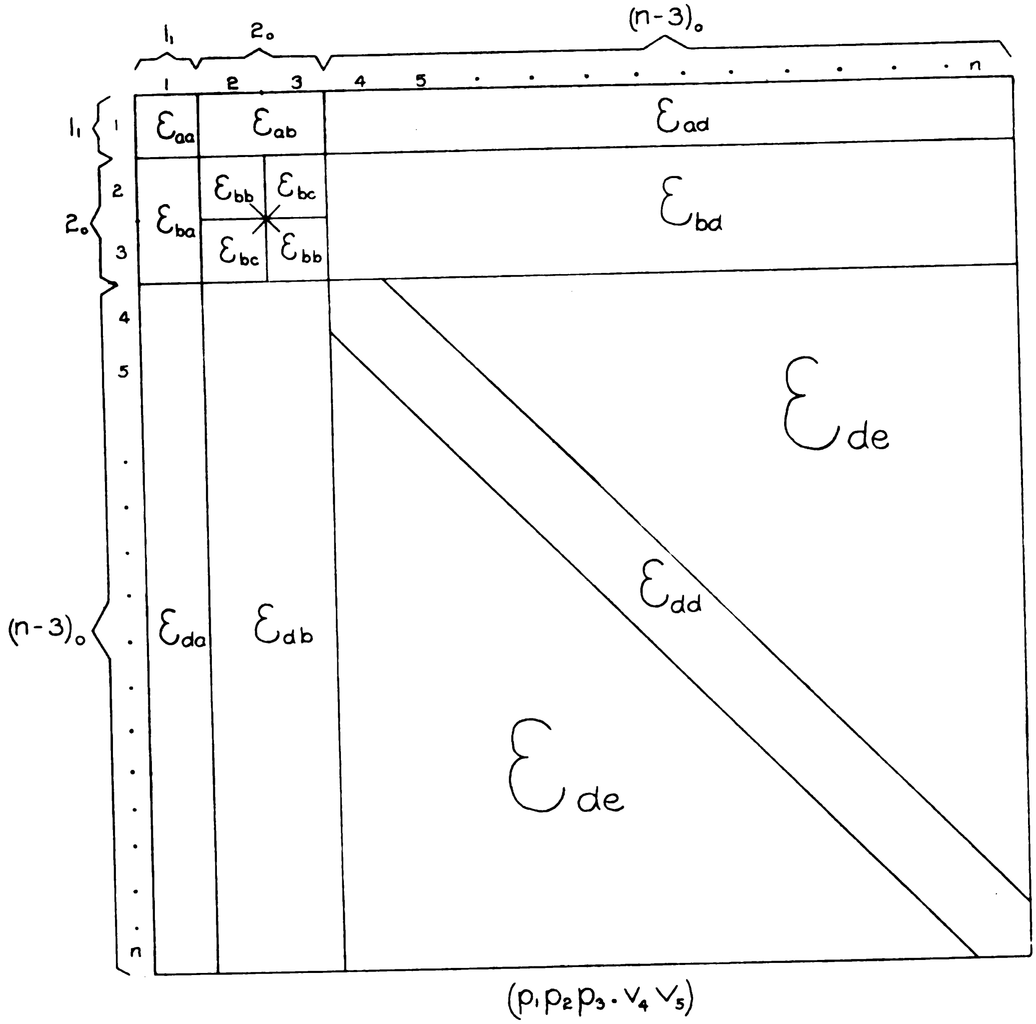
$(\rho_1\rho_2 \cdot V_3 V_4)$

Fig 34



$(\rho_1\rho_2 \cdot V_3 V_4)$

Fig 35



1₁
2₀
(n-3)₀
1 2 3 4 5 . . . . n
1
2
3
4
5
n
εaa
εab
εad
εba
εbb
εbc
εbc
εbb
εbd
εde
εdd
εde
εda
εdb
(p₁p₂p₃ . v₄v₅)

Fig. 36

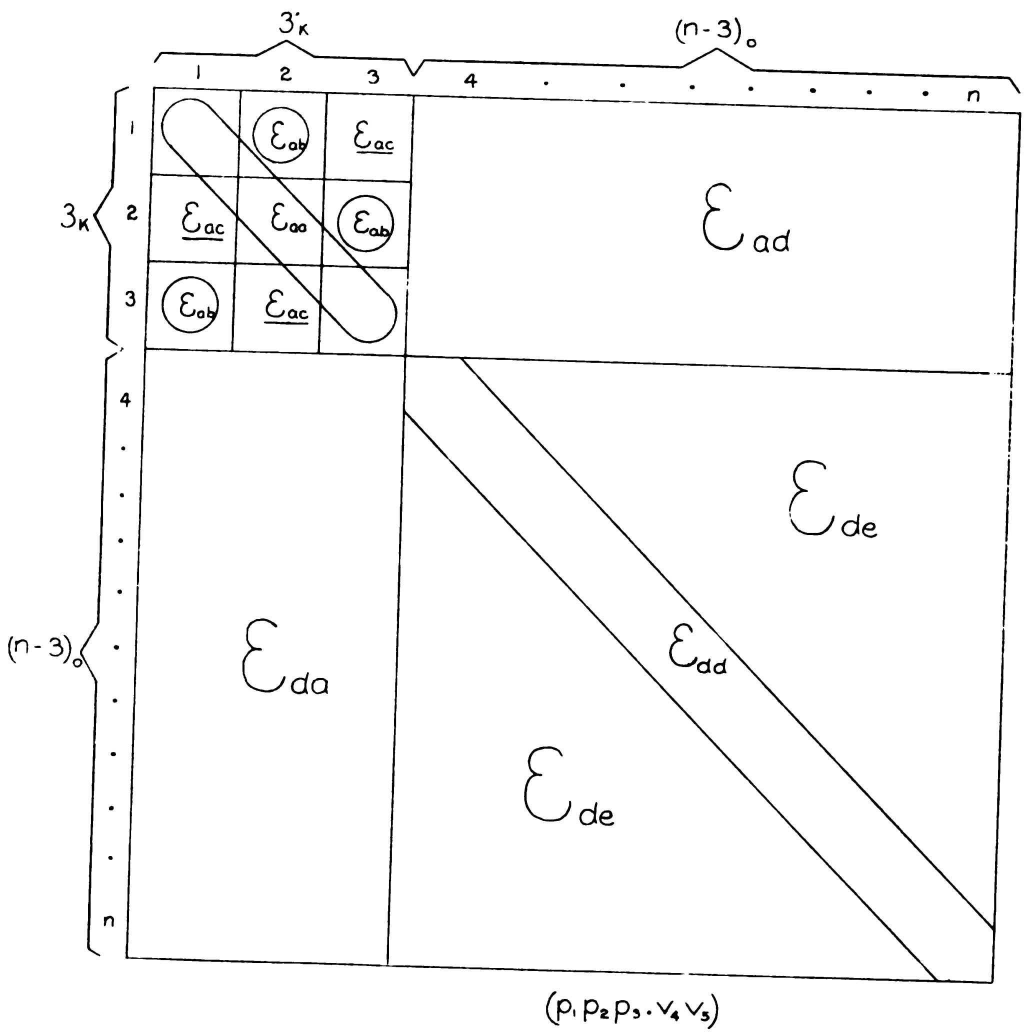


Fig. 37

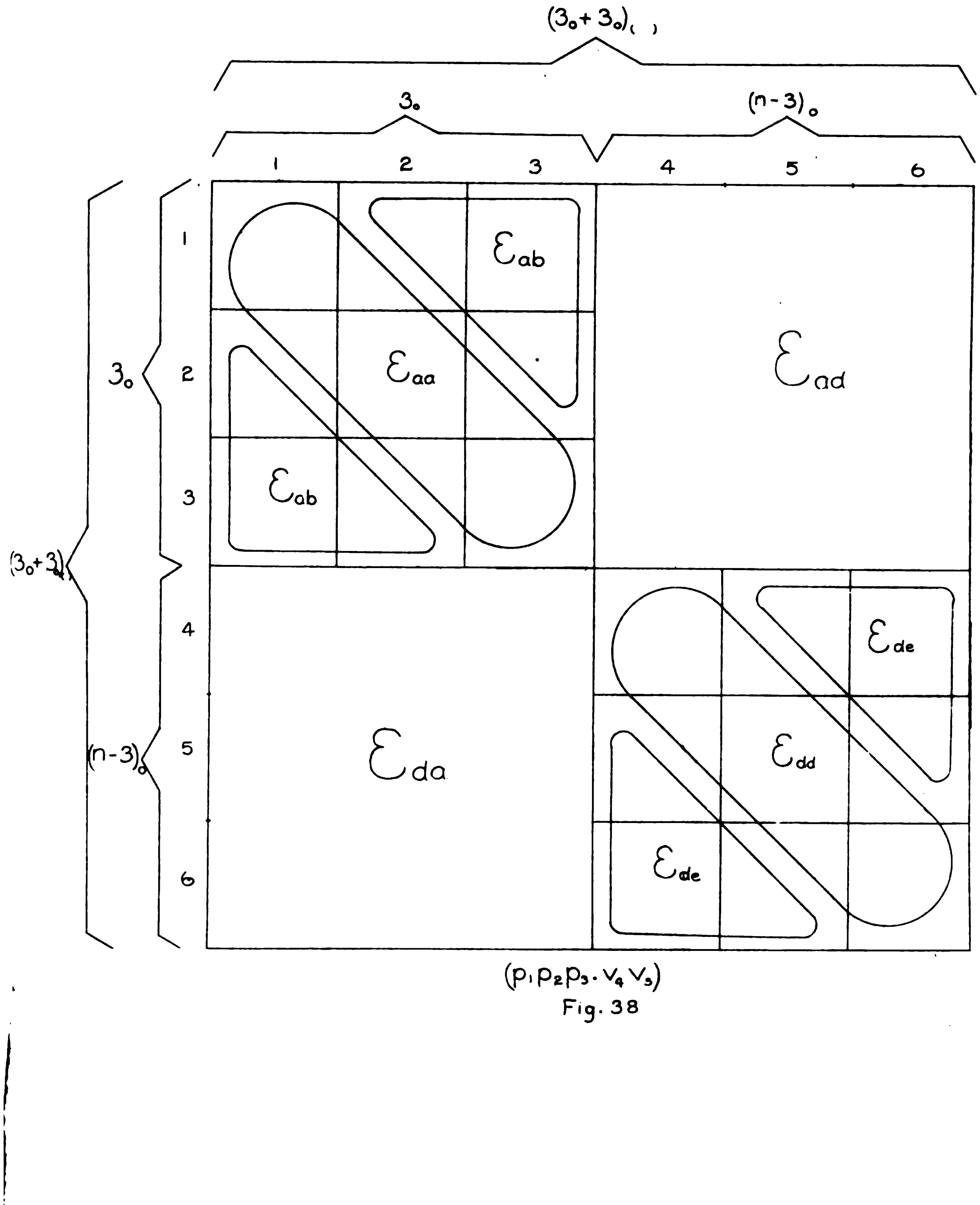
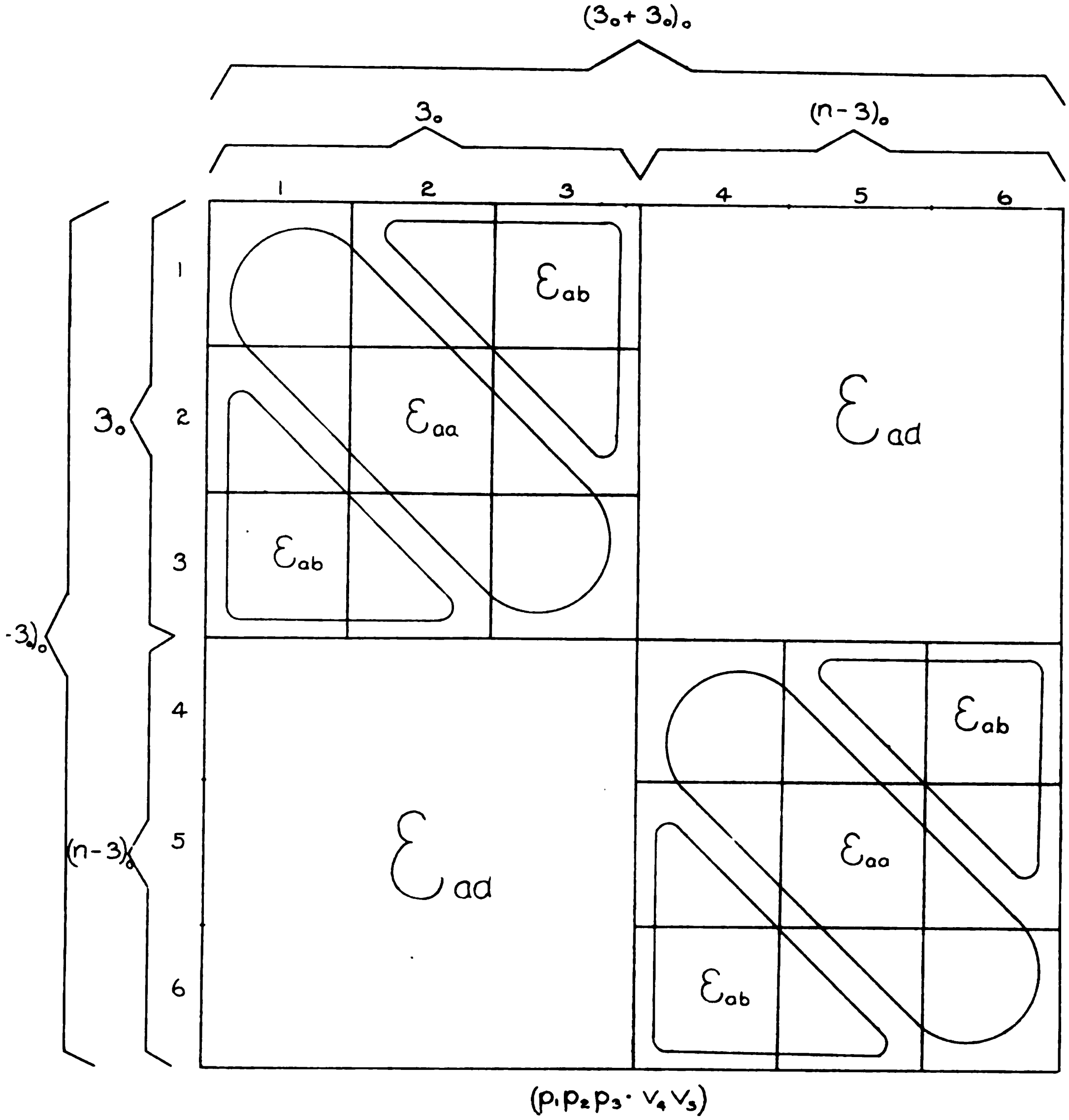
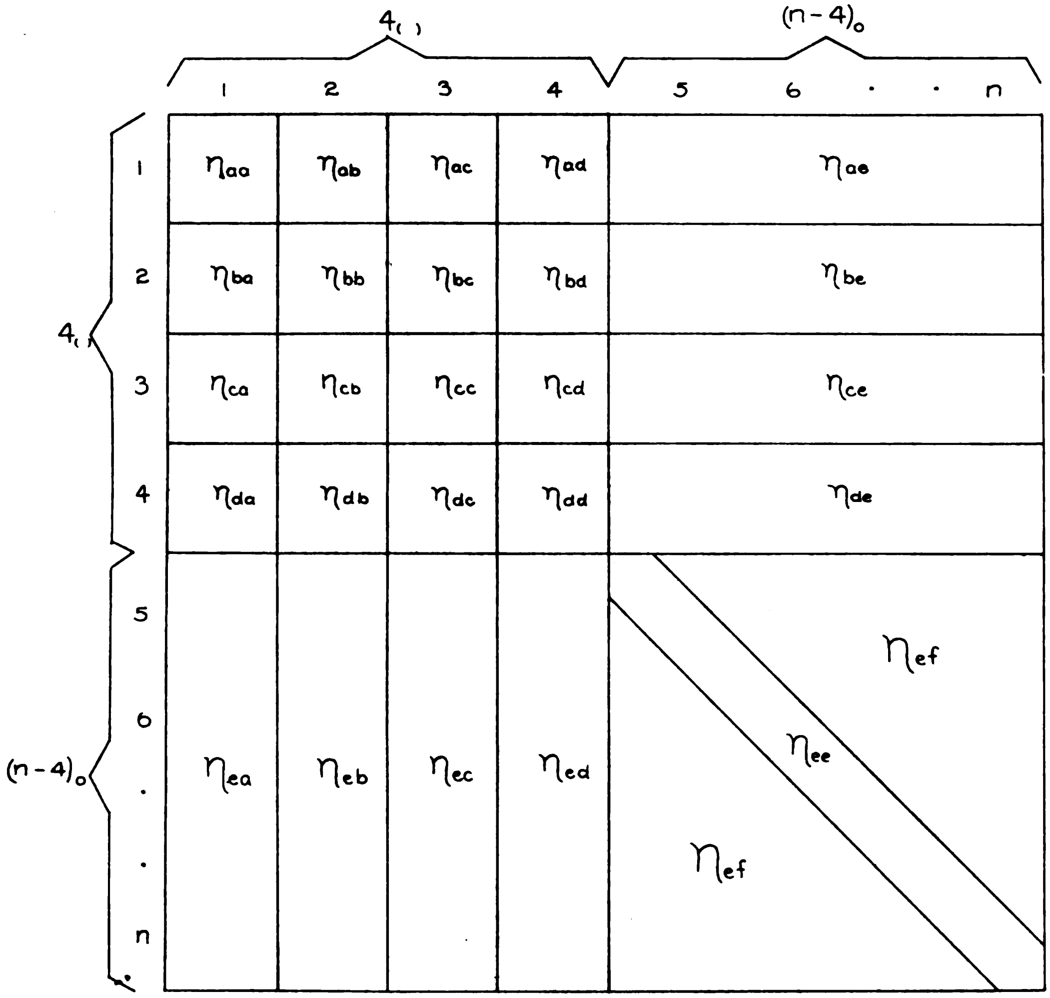


Fig. 38



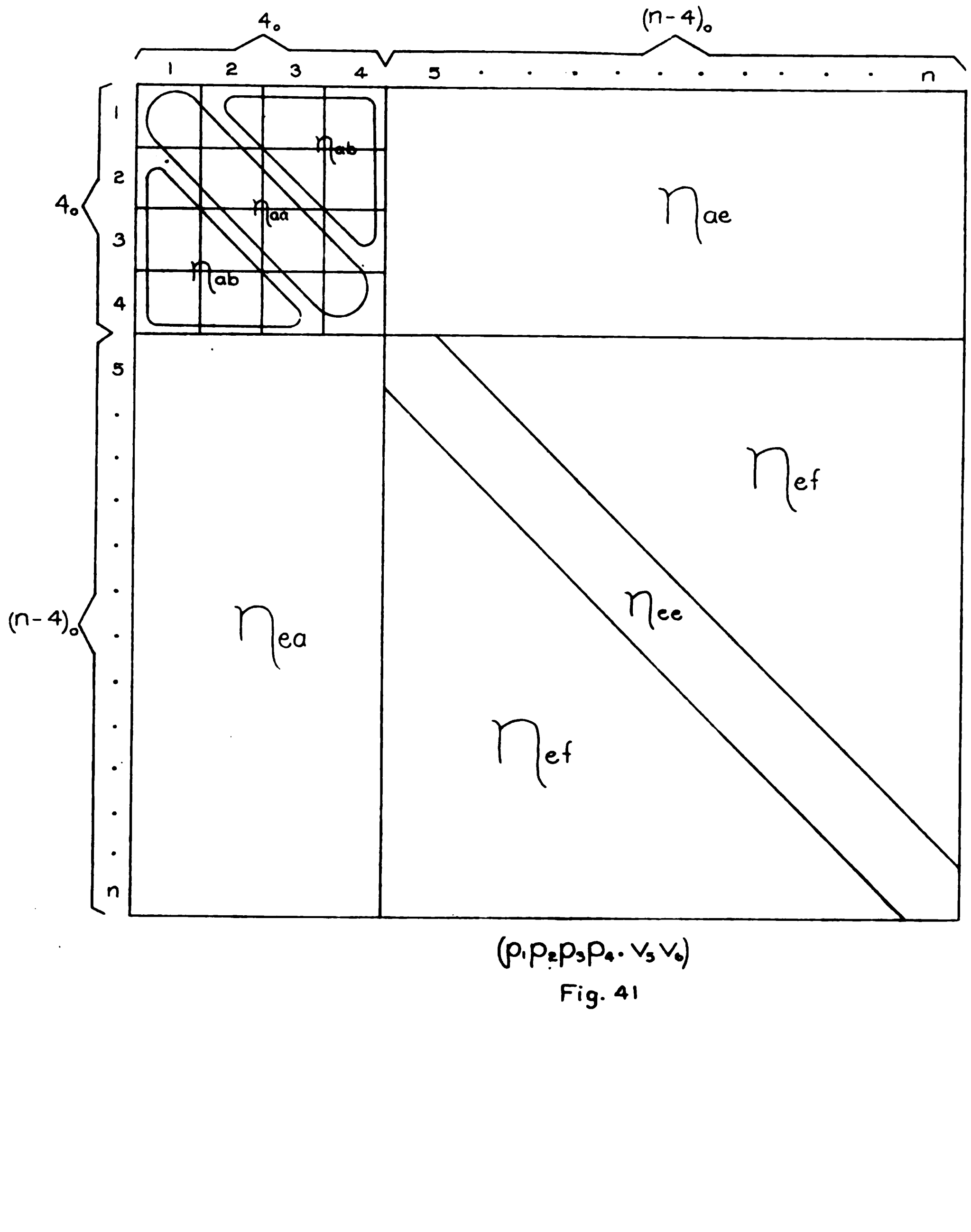
$(3_o + 3_o)_o$
3_o
$(n-3)_o$
1
2
3
4
5
6
$\mathcal{E}_{ab}$
$\mathcal{E}_{aa}$
$\mathcal{E}_{ad}$
$(p_1 p_2 p_3 \cdot V_4 V_5)$

Fig 39



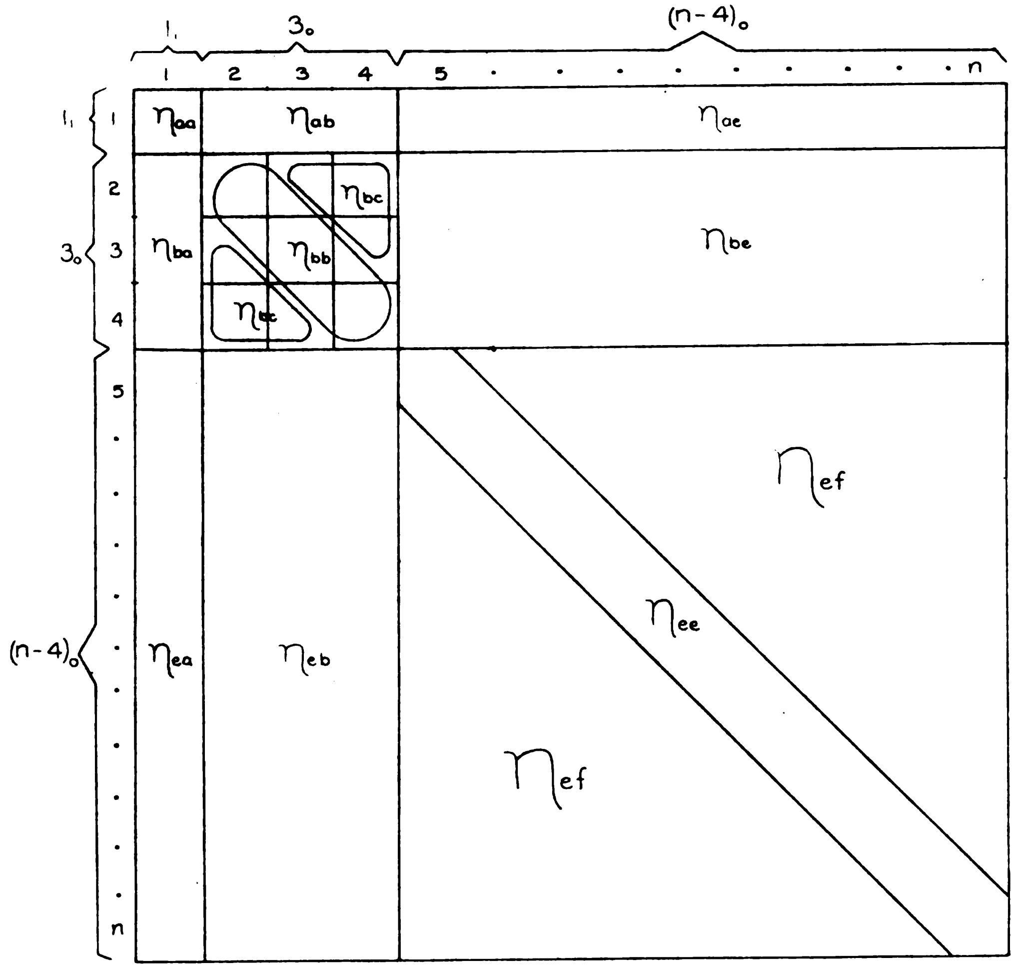
$(1234 .. V_5 V_6)$

Fig. 40



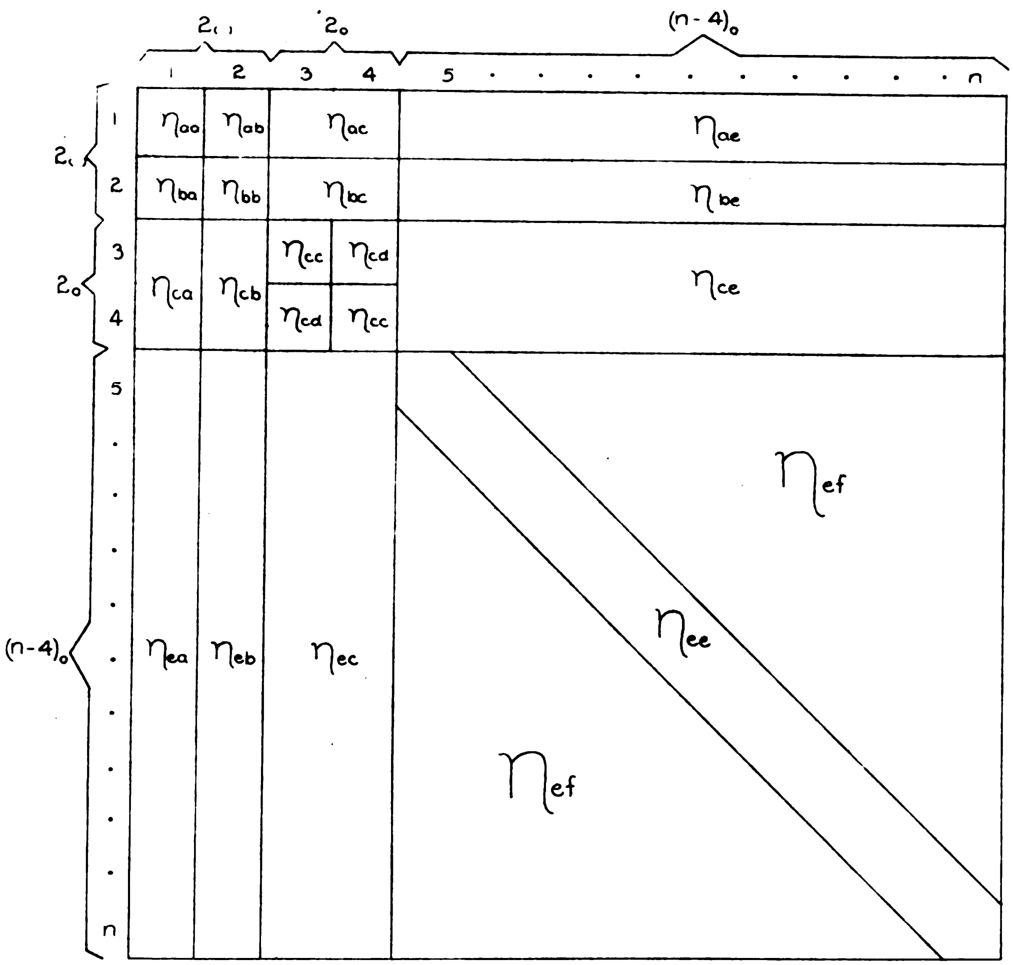
4_o
$(n-4)_o$
1 2 3 4 5 . . . . n
η_{ab}
η_{aa}
η_{ab}
η_{ae}
η_{ef}
η_{ee}
η_{ea}
η_{ef}
$(p_1 p_2 p_3 p_4 \cdot v_5 v_6)$

Fig. 41



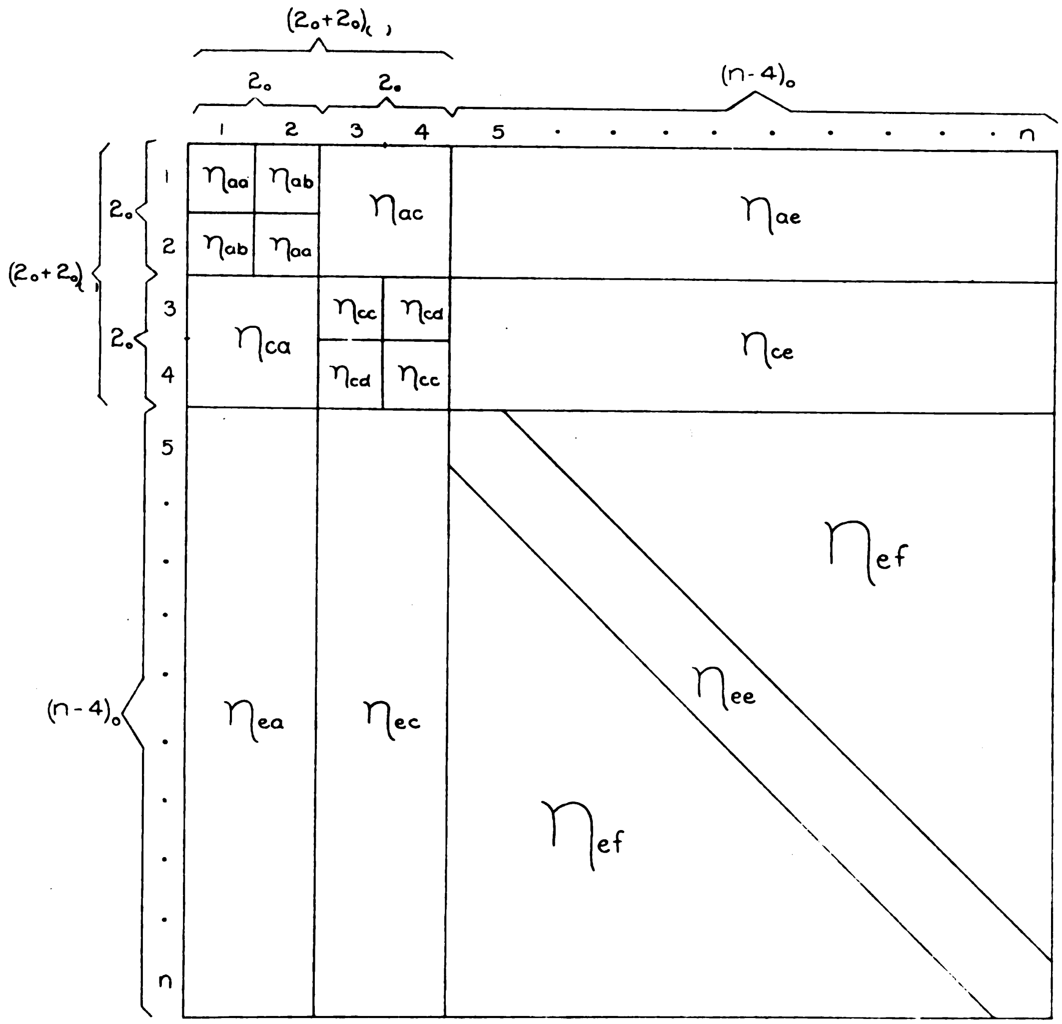
$(p_1 p_2 p_3 p_4 \cdot V_5 V_6)$

Fig. 42



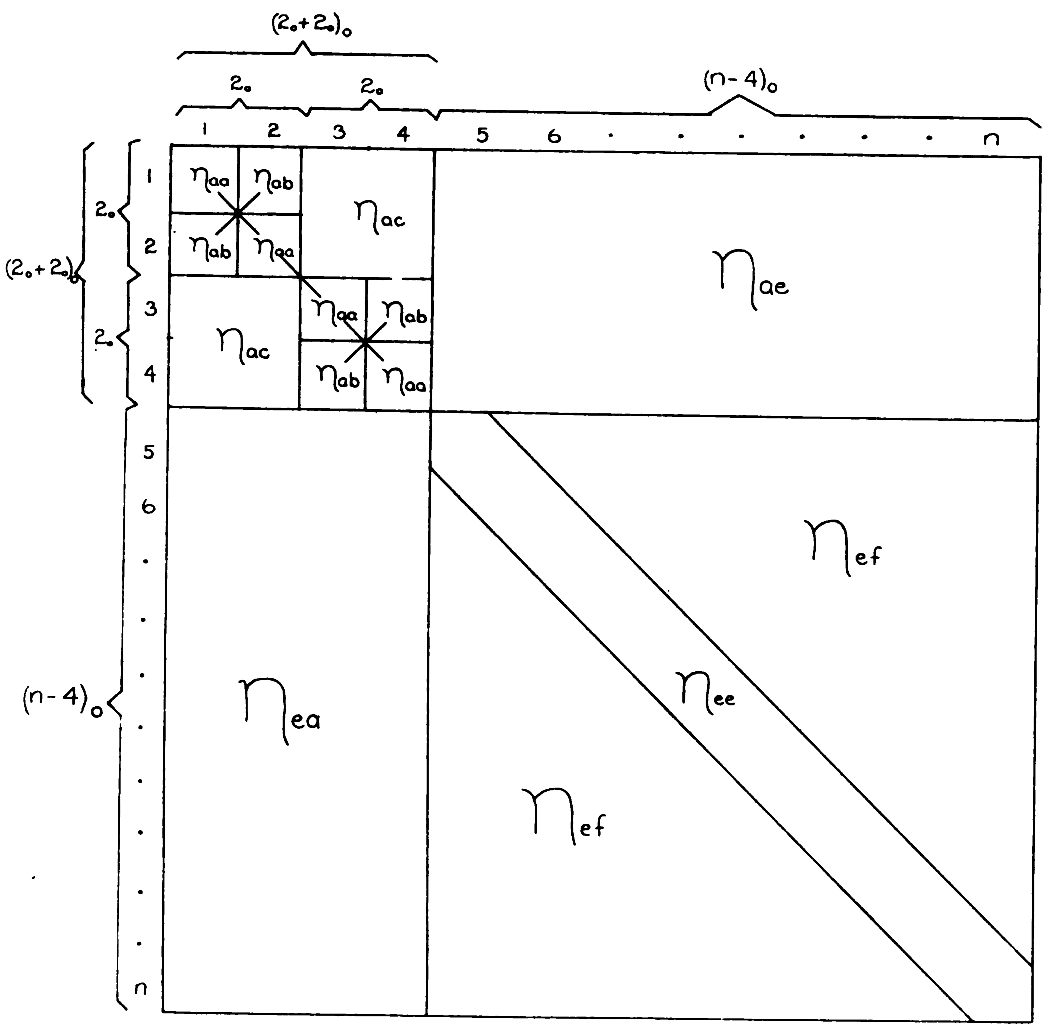
$(p_1 p_2 p_3 p_4 \cdot V_5 V_6)$

Fig. 43



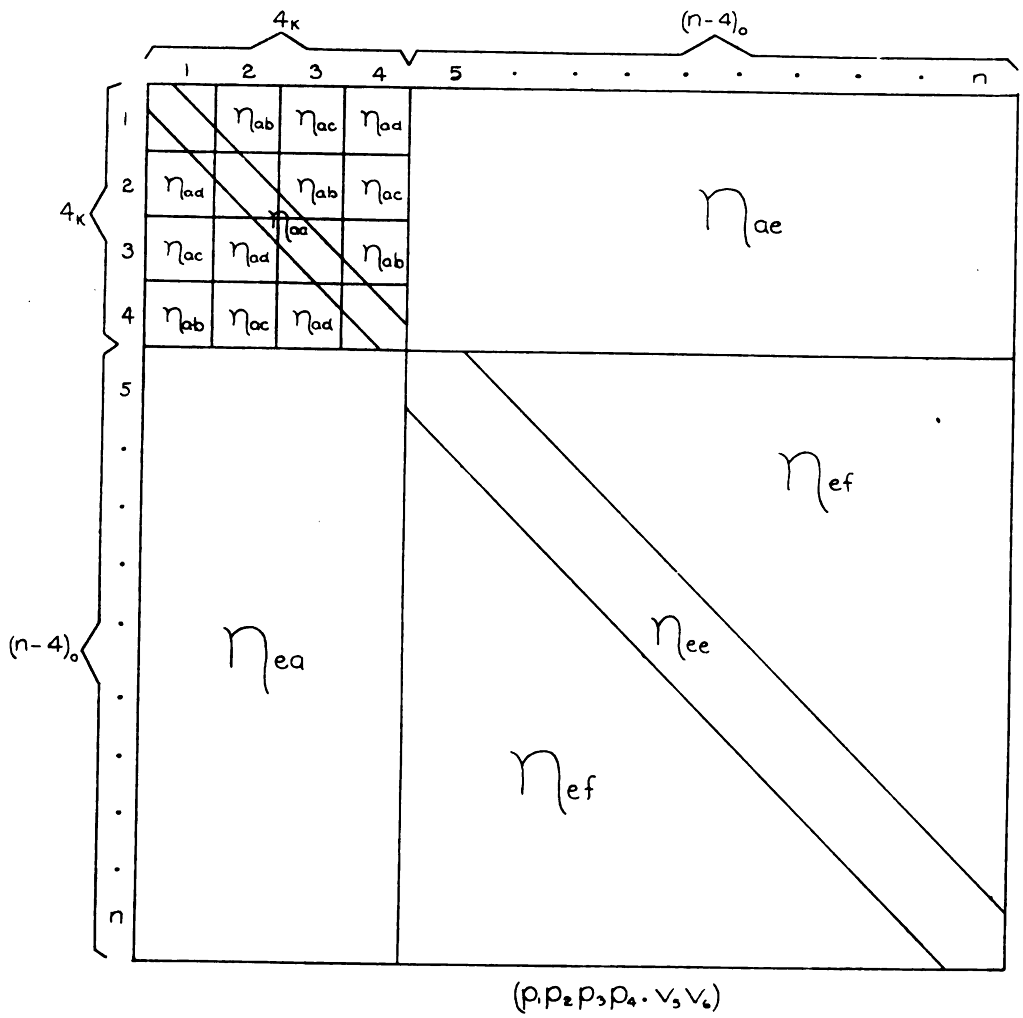
$(p_1 p_2 p_3 p_4 . V_5 V_6)$

Fig. 44



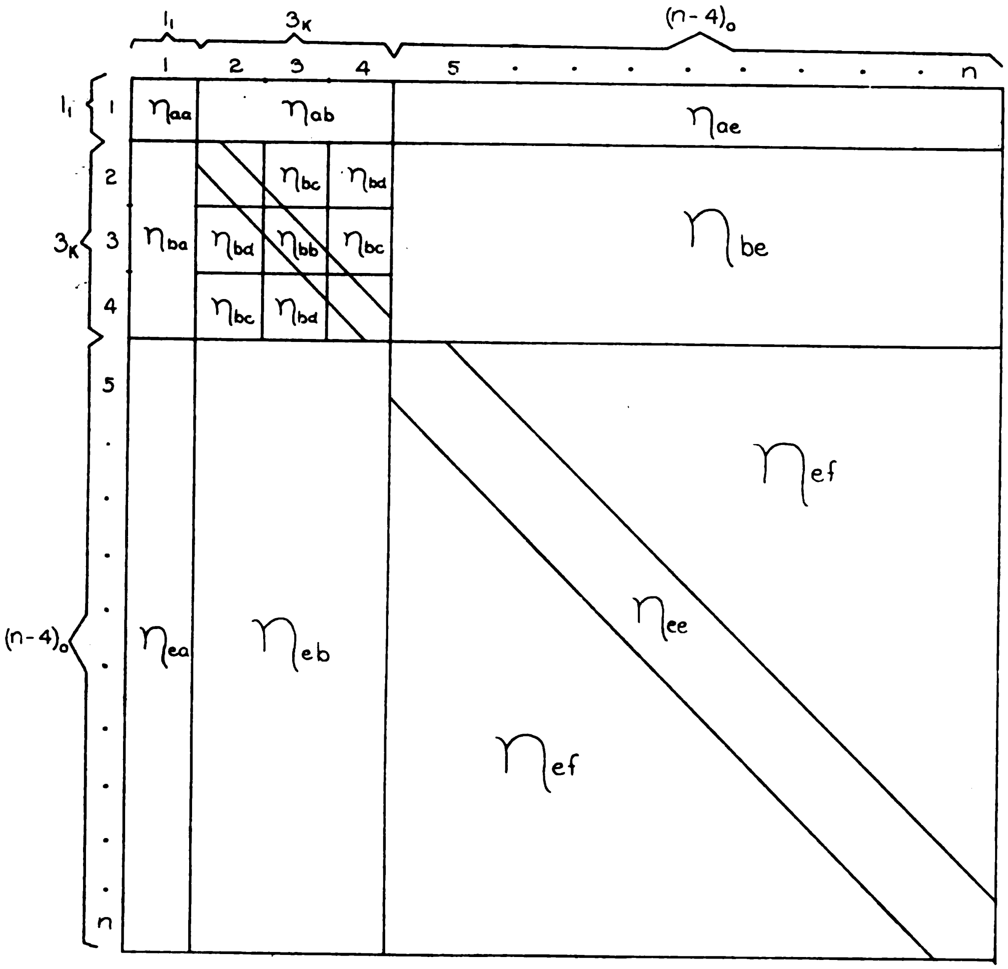
(p₁p₂p₃p₄ . v₅v₆)

Fig. 45



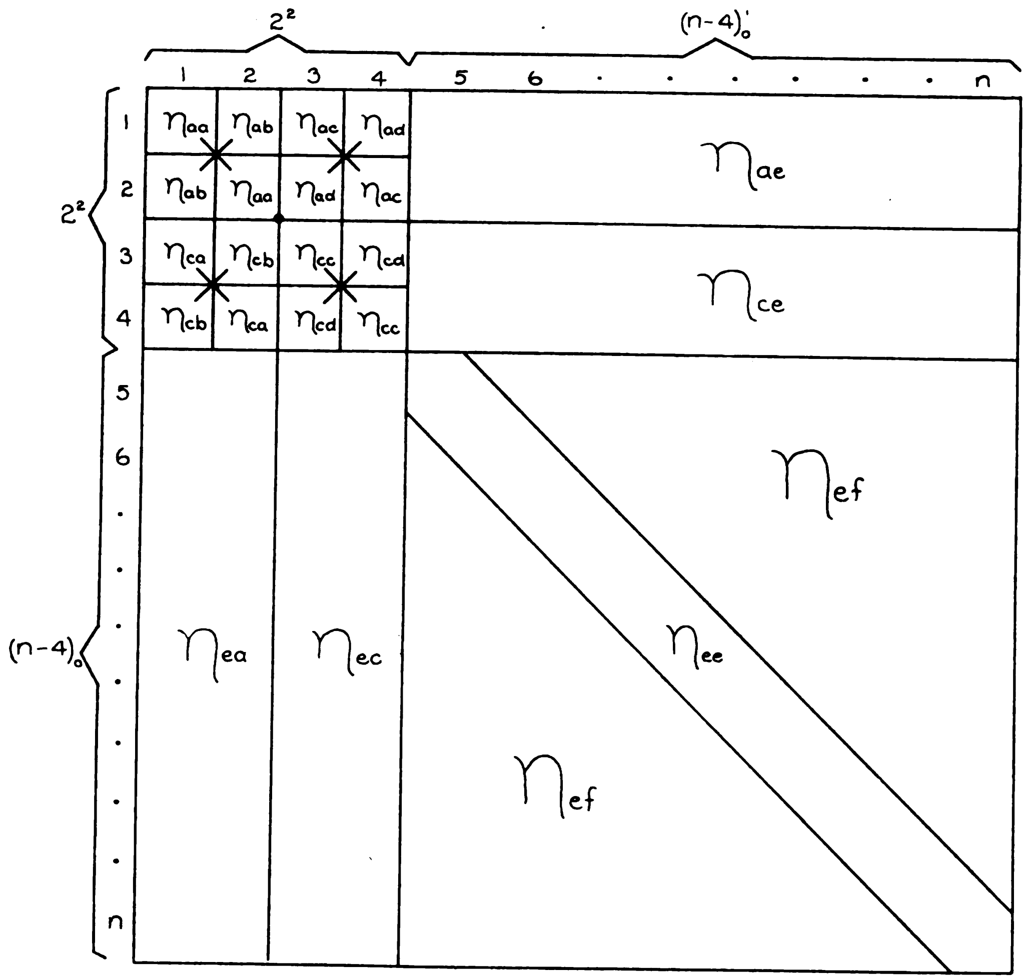
4_K
$(n-4)_o$
1 2 3 4 5 . . . . n
η_{ab} η_{ac} η_{ad}
η_{ad} η_{ab} η_{ac}
η_{aa}
η_{ac} η_{ad} η_{ab}
η_{ab} η_{ac} η_{ad}
η_{ae}
η_{ef}
η_{ea}
η_{ee}
η_{ef}
$(p_1 p_2 p_3 p_4 . v_5 v_6)$

Fig. 46



$(p_1 p_2 p_3 p_4 \cdot v_5 v_6)$

Fig. 47



$(p_1 p_2 p_3 p_4 \cdot V_5 V_6)$

Fig. 48