Tidal forces on the PBH binary

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1 Tidal Forces on the PBHs binary due to the neighbouring PBHs

Consider a PBH binary with a neighbouring PBH outside the binary shown roughly as:-

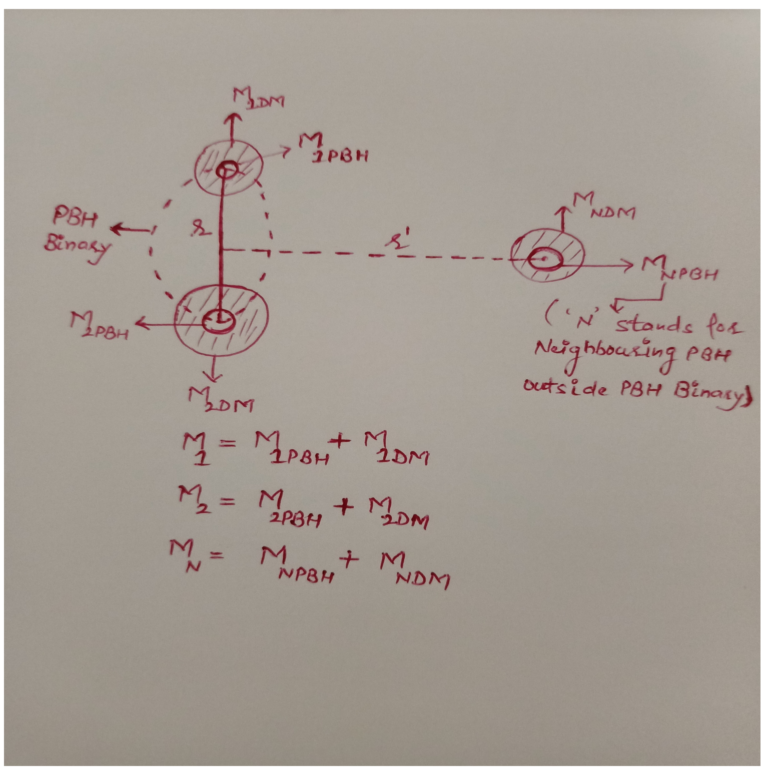

Figure 1: Here, all the PBHs are having dark matter halos being accreted around them.

Now, the tidal field acting on the PBHs in the binary due the differential gravitational force of the neighbouring PBHs is given by:-

$$T_{ij} = -\partial_i \partial_j \phi \tag{1}$$

where,

$$\phi = -\frac{GM_N}{r'} \tag{2}$$

is the gravitational potential due to the neighbouring PBH of mass, $M_N = M_{NPBH} + M_{NDM}$
&

$$r' = \sqrt{\sum_{k=1}^{3} y_k^2} \tag{3}$$

Then

$$\partial_j \phi = \frac{1}{2} \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} \left(\sum_k 2 y_k \delta_{jk}\right) \tag{4}$$

$$= \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} y_j \tag{5}$$

&

$$\partial_i \partial_j \phi = \partial_i \left(\frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} y_j \right) \tag{6}$$

which gives:-

$$\partial_i \partial_j \phi = \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} \delta_{ij} - 3 \frac{GM_N}{\left(\sum_k y_k^2\right)^{5/2}} y_i y_j \tag{7}$$

using eq.(7) in eq.(1); we get:-

$$T_{ij} = -\partial_i \partial_j \phi = 3 \frac{GM_N}{\left(\sum_k y_k^2\right)^{5/2}} y_i y_j - \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} \delta_{ij} \tag{8}$$

using $G = 1$; we get:-

$$\frac{T_{ij}}{M_N} = \frac{3 y_i y_j}{\left(\sum_k y_k^2\right)^{5/2}} - \frac{\delta_{ij}}{\left(\sum_k y_k^2\right)^{3/2}} \tag{9}$$

or

$$\frac{T_{ij}}{M_N} = \frac{3 \hat{y}_i \hat{y}_j}{\left(\sum_k y_k^2\right)^{3/2}} - \frac{\delta_{ij}}{\left(\sum_k y_k^2\right)^{3/2}} \tag{10}$$

using eq.(3) in eq.(10); we can write:-

$$\frac{T_{ij}}{M_N} = \frac{3 \hat{y}_i \hat{y}_j - \delta_{ij}}{r'^3} \tag{11}$$

(eq.(16) in reference [**1**]). the comoving separation of the neighbouring PBHs is constant then:-

$$T = s^{-3} M_N \tag{12}$$

or

$$T = s^{-3} T_{eq} \frac{M_N}{M_N^{eq}} \tag{13}$$

where, T_{eq} is the tidal field being exerted at matter-radiation equality.

The perturbative force per unit mass acting on the PBHs in the binary is:-

$$F = T \cdot r \tag{14}$$

The tidal field produces a torque given as:-

$$\dot{l} = r \times F = r \times [T \cdot r] \tag{15}$$

where, l is the angular momentum produced which prevents the head on collision of the PBHs in the binary.

$$l = \int dt \; r \times [T \cdot r] \tag{16}$$

using eq.(13); we get:-

$$l = \int dt \; \frac{\chi(s;\lambda)}{s^3} \frac{M_N}{M_N^{eq}} \; x \times [T_{eq}.x] \tag{17}$$

(Here, χ is the dimensionless sepration of the PBHs in the binary defined as:-

$$\chi = \frac{r}{x} \tag{18}$$

where, r is the physical distance between the PBHs in the binary & x is their comoving distance.)

As

$$H(s) = \frac{1}{s} \frac{ds}{dt} = \sqrt{\frac{8G\pi\rho_{eq}}{3}} h(s) \tag{19}$$

with s being the scale factor,

$$h(s) = \sqrt{s^{-3} + s^{-4}} \tag{20}$$

using eq.(20) in eq.(17); we get:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}} \right)^{1/2} \int \frac{ds}{sh(s)} \; \frac{\chi(s;\lambda)}{s^3} \; \frac{M_N}{M_N^{eq}} \; x \times [T_{eq} \cdot x] \tag{21}$$

where,

$$M_N = M_{NPBH} + M_{NDM}(s) \tag{22}$$

2 PBH binaries decoupling around matter-radiation equality

As we know that the dynamics of a PBH binary having DM halos and decoupling later around matter-radiation equality is given by:-

$$\ddot{\chi} + \left(s\dot{\chi} - \chi\right) \frac{\left(s\dot{h}(s) + h(s)\right)}{s^2 h(s)} + \frac{1}{\lambda(s)} \frac{1}{\left(sh\right)^2 \chi^2} \frac{\chi}{|\chi|} = 0 \tag{23}$$

where,

$$\lambda(s) = \frac{8\pi \rho_{eq} x^3}{3\left(M_{1PBH} + M_{2PBH} + M_{1DM} + M_{2DM}\right)} \tag{24}$$

or

$$\lambda(s) = \frac{8\pi \rho_{eq} x^3}{3M_{Binary}} \tag{25}$$

where, $M_{Binary} = M_{1PBH} + M_{2PBH} + M_{1DM} + M_{2DM}$

or

$$\lambda(s) = \lambda_0 \times f(s) \tag{26}$$

where,

$$\lambda_0 = \frac{8\pi \rho_{eq} x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} \tag{27}$$

&

$$f(s) = \frac{1}{\left(1 + \frac{M_{DM}}{M_{PBH}}\right)} \tag{28}$$

The neighbouring PBH also decouples from the Hubble flow around matter-radiation equality and accretes a halo of mass given as:-

$$M_{NDM}(s) = \left(\frac{8\pi G \rho_{eq} t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{29}$$

using eq.(29) in eq.(22):-

$$M_N = M_{NPBH} + \left(\frac{8\pi G \rho_{eq} t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{30}$$

using eq.(30) in eq.(21);we get:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2}\left(\frac{M_{NPBH}}{M_N^{eq}}\right)\int\frac{ds}{s^4h(s)}\ \chi^2(s;\lambda)\ x\times[T_{eq}\cdot x]\ +\ \left(\frac{8\pi G\rho_{eq}t_{eq}^6}{3}\right)^{1/4}\left(\frac{M_{NPBH}}{M_N^{eq}}\right)2^{5/4}$$
$$\int\frac{ds}{s^4h(s)}\ \chi^2(s;\lambda)\ \left(\frac{2}{3}(s-2)(s+1)^{1/2}+\frac{4}{3}\right)^{1/2} x\times[T_{eq}\cdot x] \tag{31}$$

3 PBH binary decoupling early in radiation dominated era

As we know that the dynamics of a PBH binary having DM halos and decoupling in radiation dominated era is given by:-

$$\ddot{\chi}-\frac{1}{s^2}\left(s\dot{\chi}-\chi\right)+\frac{1}{\lambda(s)}\frac{s^2}{\chi^2}\frac{\chi}{|\chi|}=0 \tag{32}$$

where,

$$\lambda(s)=\frac{8\pi\rho_{eq}x^3}{3\left(M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}\right)} \tag{33}$$

or

$$\lambda(s)=\frac{8\pi\rho_{eq}x^3}{3M_{Binary}} \tag{34}$$

where, $M_{Binary}=M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}$

or

$$\lambda(s)=\lambda_0\times f(s) \tag{35}$$

where,

$$\lambda_0=\frac{8\pi\rho_{eq}x^3}{3\left(M_{1PBH}+M_{2PBH}\right)} \tag{36}$$

&

$$f(s)=\frac{1}{\left(1+\frac{M_{DM}}{M_{PBH}}\right)} \tag{37}$$

If the neighbouring PBH is also decoupling from the Hubble expansion in radiation dominated era then the mass of it's dark matter halo is given as:-

$$M_{NDM}=\left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4}\times 2^{3/4}\times M_{NPBH}\times s \tag{38}$$

substituting eq.(38) in eq.(22):-

$$M_N = M_{NPBH} + \left(\frac{8\pi G \rho_{eq} t_{eq}^2}{3}\right)^{3/4} \times 2^{3/4} \times M_{NPBH} \times s \tag{39}$$

using eq.(39) in eq.(21); we get:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x]\ +\ \left(\frac{8\pi G\rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) 2^{3/4} \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ s\ x \times [T_{eq} \cdot x] \tag{40}$$

Also, in radiation domination:-

$$h(s) = \frac{1}{s^2} \tag{41}$$

so, we can further simplify eq.(40) as:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) \int \frac{ds}{s^2}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x]\ +\ \left(\frac{8\pi G\rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) 2^{3/4} \int \frac{ds}{s}\ \chi^2(s;\lambda)\ \ x \times [T_{eq} \cdot x] \tag{42}$$

4 Calculation of $\mathbf{M}_N^{eq}$

As we know that M_N^{eq} is the sum of the masses of neighbouring PBH and it's dark matter halo at the matter-radiation equality.
i.e.

$$M_N^{eq} = M_{NPBH}^{eq} + M_{NDM}^{eq} \tag{43}$$

The mass of the PBH remains constant with time so,

$$M_{NPBH}^{eq} = M_{NPBH} \tag{44}$$

so,

$$M_N^{eq} = M_{NPBH} + M_{NDM}^{eq} \tag{45}$$

4.1 M_N^{eq} around matter-radiation equality

From eq.(29); we have:-

$$M_{NDM}^{eq} = \left(\frac{8\pi G\rho_{eq} t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{2}{3}\left(s_{eq} - 2\right)\left(s_{eq} + 1\right)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{46}$$

For matter-radiation equality, $s_{eq} = 1$, so:-

$$M_{NDM}^{eq} = \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{2}{3}(1-2)(1+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{47}$$

or

$$M_{NDM}^{eq} = \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{4-2\sqrt{2}}{3}\right)^{1/2} \tag{48}$$

using eq.(48) in eq.(45); we get:-

$$M_N^{eq} = M_{NPBH} + \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{4-2\sqrt{2}}{3}\right)^{1/2} \tag{49}$$

or

$$\frac{M_N^{eq}}{M_{NPBH}} = \left\{1 + \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times \left(\frac{4-2\sqrt{2}}{3}\right)^{1/2}\right\} \tag{50}$$

(where , $\rho_{eq} = 2.1536\times10^{-16}kgm^{-3}$, $t_{eq} = 1.59246\times10^{12}s$ & $G = 6.67\times10^{-11}kg^{-1}m^3s^{-2}$.)

4.2 M_N^{eq} in radiation domination

From eq.(38); we have:-

$$M_{NDM}^{eq} = \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{3/4} \times M_{NPBH} \times s_{eq} \tag{51}$$

For matter-radiation equality, $s_{eq} = 1$, so:-

$$M_{NDM}^{eq} = \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{3/4} \times M_{NPBH} \tag{52}$$

using eqs.(52) in eq.(45); we get:-

$$M_N^{eq} = M_{NPBH}\left\{1 + \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{3/4}\right\} \tag{53}$$

or

$$\frac{M_{NPBH}}{M_N^{eq}} = \left\{1 + \left(\frac{8\pi G\rho_{eq}t_{eq}^2}{3}\right)^{3/4} \times 2^{3/4}\right\} \tag{54}$$

5 Numerical Calculations(without numerical factors)

5.1 For PBH binaries decoupling in radiation dominated era

I tried to calculate the value of angular momentum, l by solving eq.(14) in [**1**]

$$l = \int \frac{ds}{s^2}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x] \tag{55}$$

where, $h(s) = \frac{1}{s^2}$ and $\chi(s;\lambda)$ is the solution of the eq.(9) in [**1**] given as:-

$$\ddot{\chi} - \frac{1}{s^2}\left(s\dot{\chi} - \chi\right) + \frac{1}{\lambda}\frac{s^2}{\chi^2}\frac{\chi}{|\chi|} = 0 \tag{56}$$

with

$$\lambda = \frac{8\pi\rho_{eq}x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} \tag{57}$$

the solution of eq.(55) for different values of λ is:-

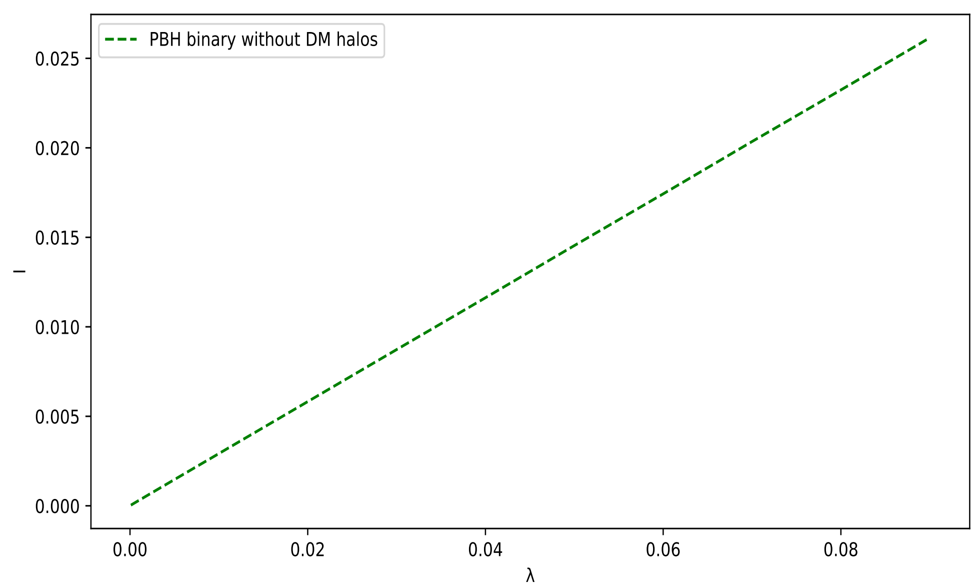

Figure 2: Variation of the angular momentum, l with λ for the PBH binary without DM halos in radiation domination.

then I tried to compare the solution of eq.(42) i.e.

$$l = \int \frac{ds}{s^2}\ \chi^2(s;\lambda)\ x\times[T_{eq}\cdot x]\ +\ \int \frac{ds}{s}\ \chi^2(s;\lambda)\ \ x\times[T_{eq}\cdot x] \tag{58}$$

with the solution of eq.(55);as:-

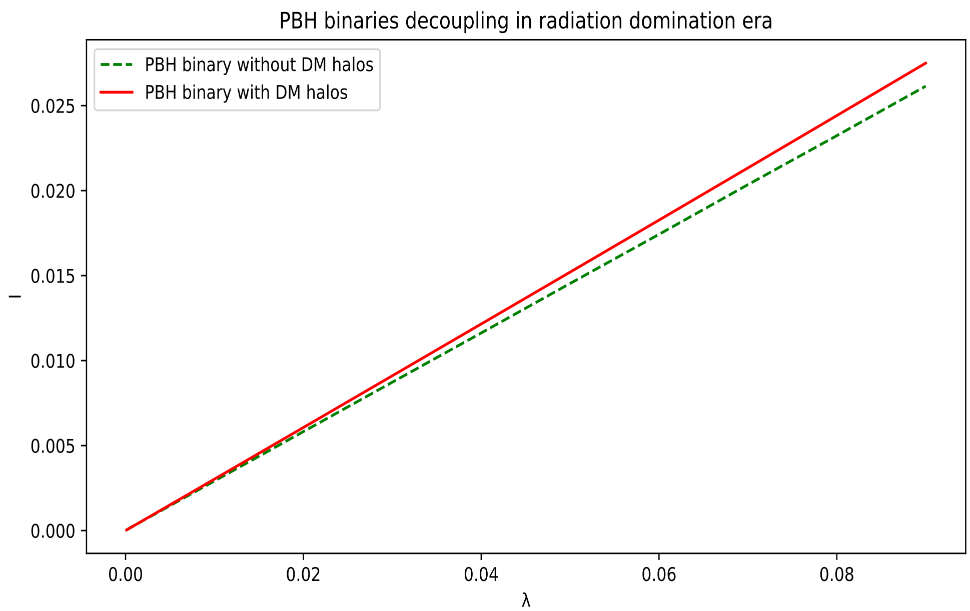

Figure 3: Variation of the angular momentum, l with λ for the PBH binaries with and without DM halos in radiation domination.

This graph shows that in the radiation dominated era, the angular momentum imparted by the neighbouring PBH is slightly more in the binaries having dark matter halos than the binaries with no halos.

5.2 For PBH binaries decoupling around matter-radiation equality

I calculated the value of the angular momentum, l by solving eq.(14) in [**1**]:-

$$l = \int \frac{ds}{sh(s)}\ \frac{\chi^2(s;\lambda)}{s^3}\ x\times[T_{eq}\cdot x] \tag{59}$$

where, $h(s) = \sqrt{s^{-3}+s^{-4}}$ and $\chi(s;\lambda)$ is the solution of the eq.(6) in [**1**] given as:-

$$\ddot{\chi} + \left(s\dot{\chi}-\chi\right)\frac{\left(s\dot{h}(s)+h(s)\right)}{s^2h(s)} + \frac{1}{\lambda}\frac{1}{\left(sh\right)^2\chi^2}\frac{\chi}{|\chi|} = 0 \tag{60}$$

with

$$\lambda = \frac{8\pi\rho_{eq}x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} \tag{61}$$

so, the solution of eq.(59) for different values of λ is shown as follows:- then I

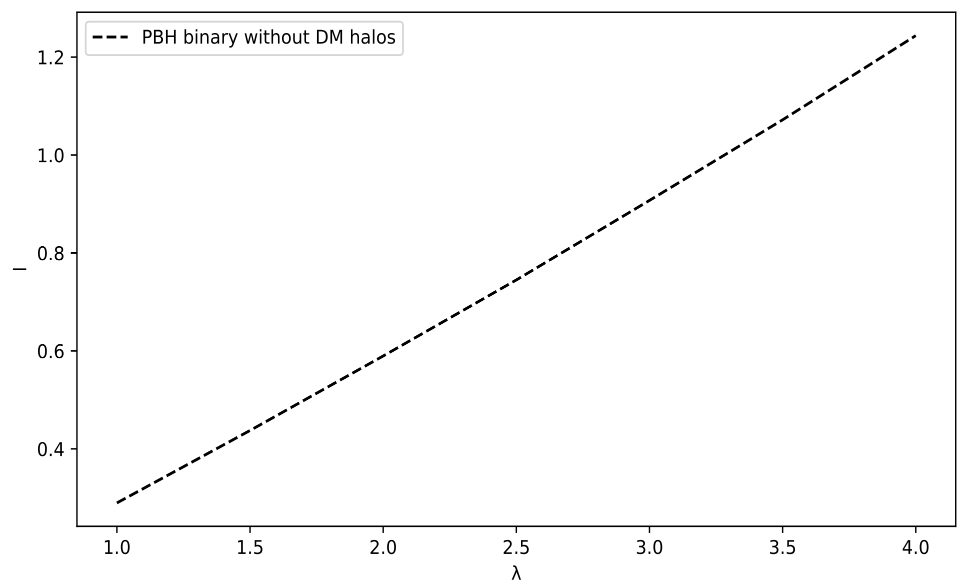

Figure 4: Variation of the angular momentum, l with λ for the PBH binary without DM halos decoupling around matter-radiation equality.

tried to compare the solution of eq.(31)(without numerical factors) i.e.

$$l = \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x]\ +\ \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2}\ x \times [T_{eq} \cdot x] \tag{62}$$

with the solution of eq.(59);as:-

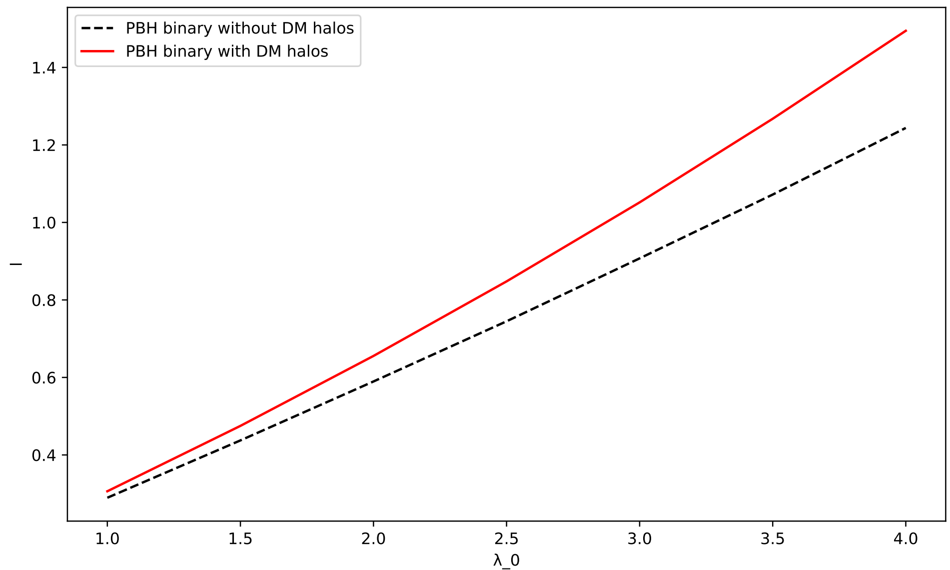

Figure 5: Variation of the angular momentum, l with λ for the PBH binary with and without DM halos decoupling around matter-radiation equality.

This graph shows that the PBH binaries with dark matter halos, decoupling around matter-radiation equality experience more torque due to the neighbouring PBH than the binaries with no halos.

6 Calculations of reduced angular momentum, j of the PBH binaries

6.1 For PBH binaries decoupling in radiation dominated era

I tried to calculate the value of angular momentum, l by solving eq.(13) in [**1**]

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \int \frac{ds}{s^2}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x]\ = A\ x \times [T_{eq} \cdot x] \tag{63}$$

where,

$$A = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \int \frac{ds}{s^2}\ \chi^2(s;\lambda) \tag{64}$$

$h(s) = \frac{1}{s^2}$ & $\chi(s;\lambda)$ is the solution of the eq.(9) in [**1**] given as:-

$$\ddot{\chi} - \frac{1}{s^2}\left(s\dot{\chi} - \chi\right) + \frac{1}{\lambda}\frac{s^2}{\chi^2}\frac{\chi}{|\chi|} = 0 \tag{65}$$

with

$$\lambda = \frac{8\pi\rho_{eq}x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} \tag{66}$$

The solution of eq.(63) for different values of λ is:-

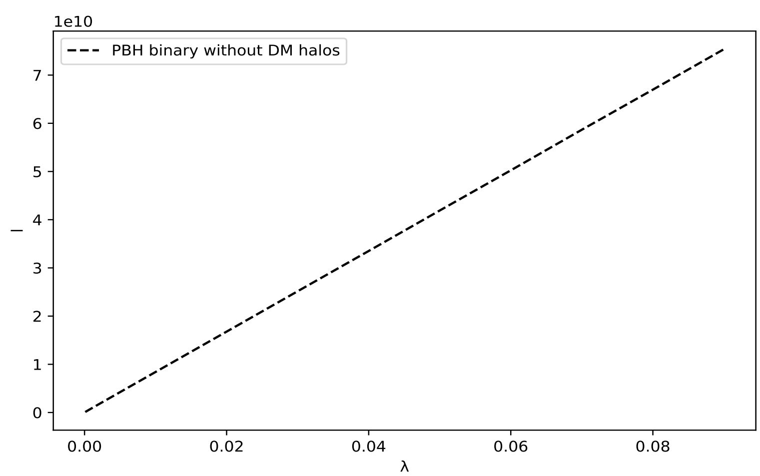

Figure 6: Variation of the angular momentum, l with λ for the PBH binary without DM halos in radiation domination.

Now, for any value of λ, the value of the reduced angular momentum of the binary without dark matter halos is given as:-

$$j = \frac{l}{\sqrt{G\left(M_{1PBH} + M_{2PBH}\right)a}} = \sqrt{1-e^2} \tag{67}$$

where,
a - the semi major axis of the PBH binary orbit
e - eccentricity of the orbit
l - angular momentum per unit reduced mass, μ of the binary.

As we know that the semi major axis, a of the PBH binary orbit without dark matter halos is given as; eq.(11) in [**1**]:-

$$a \approx 0.1\lambda x \tag{68}$$

So, using eqs. (63) and (68) in eq.(67); we get:-

$$j = \frac{A\ x \times [T_{eq} \cdot x]}{\sqrt{0.1G\lambda x\left(M_{1PBH} + M_{2PBH}\right)}} = \sqrt{1-e^2} \tag{69}$$

or

$$j = \frac{A}{\sqrt{0.1G\lambda}}\ x^{3/2}\ \frac{\hat{x} \times [T_{eq} \cdot \hat{x}]}{\sqrt{\left(M_{1PBH} + M_{2PBH}\right)}} = \sqrt{1-e^2} \tag{70}$$

then, as an example for $\lambda = 0.001$, we have:-

$$j = 1.06706 \times 10^{16}\ x^{3/2}\ \frac{\hat{x} \times [T_{eq} \cdot \hat{x}]}{\sqrt{\left(M_{1PBH} + M_{2PBH}\right)}} = \sqrt{1-e^2} \tag{71}$$

After that, I tried to compare the solution of eq.(42) i.e.

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) \int \frac{ds}{s^2}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x]\ +\ \left(\frac{8\pi G\rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) 2^{3/4} \int \frac{ds}{s}\ \chi^2(s;\lambda)\ \ x \times [T_{eq} \cdot x] \tag{72}$$

or

$$l = B\ x \times [T_{eq} \cdot x] \tag{73}$$

where,

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) \int \frac{ds}{s^2}\ \chi^2(s;\lambda)\ +\ \left(\frac{8\pi G\rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) 2^{3/4} \int \frac{ds}{s}\ \chi^2(s;\lambda) \tag{74}$$

with the solution of eq.(63); as:-

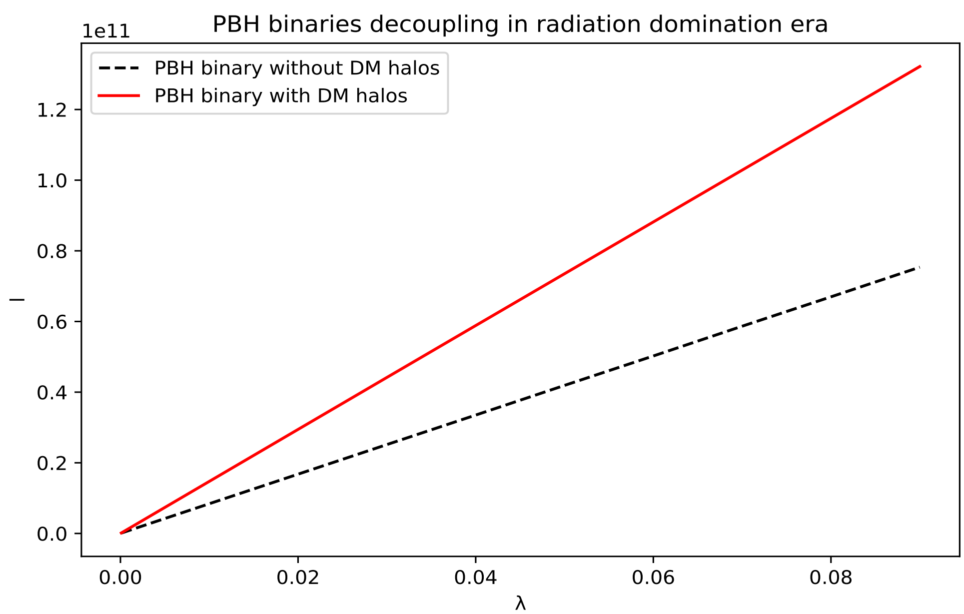

Figure 7: Variation of the angular momentum, l with λ for the PBH binaries with and without DM halos in radiation domination.

This graph shows that in the radiation dominated era, the angular momentum imparted by the neighbouring PBH is very large in the binaries

having dark matter halos than the binaries with no halos.

Similarly, for any value of λ, the value of the reduced angular momentum of the binary with dark matter halos is given as:-

$$j_r = \frac{l}{\sqrt{G\left(M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}\right)a}} = \sqrt{1-e_r^2} \tag{75}$$

Also, the semi major axis, a of the PBH binary orbit with dark matter halos is in approximation with eq.(11) in [**1**]:-

$$a \approx 0.1\lambda_0 x \tag{76}$$

where,

$$\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3\left(M_{1PBH}+M_{2PBH}\right)} \tag{77}$$

So, using eqs. (73) and (76) in eq.(75); we get:-

$$j_r = \frac{B\ x\times[T_{eq}\cdot x]}{\sqrt{0.1G\lambda_0 x\left(M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}\right)}} = \sqrt{1-e_r^2} \tag{78}$$

or

$$j_r = \frac{B}{\sqrt{0.1G\lambda_0}}\ x^{3/2}\ \frac{\hat{x}\times[T_{eq}\cdot\hat{x}]}{\sqrt{\left(M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}\right)}} = \sqrt{1-e_r^2} \tag{79}$$

then, just to compare with the same example of $\lambda = 0.001$, we have:-

$$j_r = 1.80388\times10^{16}\ x^{3/2}\ \frac{\hat{x}\times[T_{eq}\cdot\hat{x}]}{\sqrt{\left(M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}\right)}} = \sqrt{1-e_r^2} \tag{80}$$

Now, if we compare eqs.(71) & (80) for the same comoving distance, x; then:-

$$\frac{j_r}{j} = 1.6905\ \sqrt{\frac{\left(M_{1PBH}+M_{2PBH}\right)}{\left(M_{1PBH}+M_{2PBH}+M_{1DM}+M_{2DM}\right)}} = \frac{\sqrt{1-e_r^2}}{\sqrt{1-e^2}} \tag{81}$$

i.e. $j_r < j$ which is not expected. So, may be I missed something. Please check.