Tidal torque in the PBH binary

1 Tidal Forces on the PBHs binary due to the neighbouring PBHs

Consider a PBH binary with a neighbouring PBH outside the binary shown roughly as:-

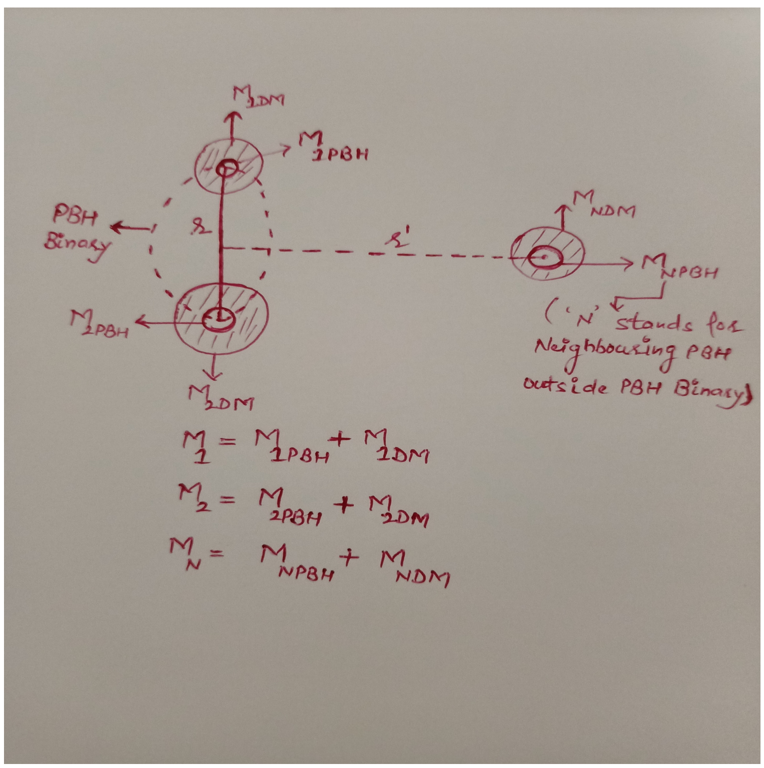

Figure 1: Here, all the PBHs are having dark matter halos being accreted around them.

Now, the tidal field acting on the PBHs in the binary due the differential gravitational force of the neighbouring PBHs is given by:-

$$T_{ij} = -\partial_i \partial_j \phi \tag{1}$$

where,

$$\phi = -\frac{GM_N}{r'} \tag{2}$$

is the gravitational potential due to the neighbouring PBH of mass, $M_N = M_{NPBH} + M_{NDM}$
&

$$r' = \sqrt{\sum_{k=1}^{3} y_k^2} \tag{3}$$

Then

$$\partial_j \phi = \frac{1}{2} \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} \left(\sum_k 2 y_k \delta_{jk}\right) \tag{4}$$

$$= \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} y_j \tag{5}$$

&

$$\partial_i \partial_j \phi = \partial_i \left(\frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} y_j \right) \tag{6}$$

which gives:-

$$\partial_i \partial_j \phi = \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} \delta_{ij} - 3 \frac{GM_N}{\left(\sum_k y_k^2\right)^{5/2}} y_i y_j \tag{7}$$

using eq.(7) in eq.(1); we get:-

$$T_{ij} = -\partial_i \partial_j \phi = 3 \frac{GM_N}{\left(\sum_k y_k^2\right)^{5/2}} y_i y_j - \frac{GM_N}{\left(\sum_k y_k^2\right)^{3/2}} \delta_{ij} \tag{8}$$

$$\frac{T_{ij}}{GM_N} = \frac{3 y_i y_j}{\left(\sum_k y_k^2\right)^{5/2}} - \frac{\delta_{ij}}{\left(\sum_k y_k^2\right)^{3/2}} \tag{9}$$

or

$$\frac{T_{ij}}{GM_N} = \frac{3 \hat{y}_i \hat{y}_j}{\left(\sum_k y_k^2\right)^{3/2}} - \frac{\delta_{ij}}{\left(\sum_k y_k^2\right)^{3/2}} \tag{10}$$

using eq.(3) in eq.(10); we can write:-

$$\frac{T_{ij}}{GM_N} = \frac{3 \hat{y}_i \hat{y}_j - \delta_{ij}}{r'^3} = t_{ij} \quad (say) \tag{11}$$

(eq.(16) in reference [**1**]). the comoving separation of the neighbouring PBHs is constant then:-

$$T = s^{-3} M_N \tag{12}$$

or

$$T = s^{-3} T_{eq} \frac{M_N}{M_N^{eq}} \tag{13}$$

where, T_{eq} is the tidal field being exerted at matter-radiation equality.

The perturbative force per unit mass acting on the PBHs in the binary is:-

$$F = T \cdot r \tag{14}$$

The tidal field produces a torque given as:-

$$\dot{l} = r \times F = r \times [T \cdot r] \tag{15}$$

where, l is the angular momentum per unit reduced mass produced which prevents the head on collision of the PBHs in the binary.

$$l = \int dt \; r \times [T \cdot r] \tag{16}$$

using eq.(13); we get:-

$$l = \int dt \; \frac{\chi(s;\lambda)}{s^3} \frac{M_N}{M_N^{eq}} \; x \times [T_{eq} . x] \tag{17}$$

(Here, χ is the dimensionless separation of the PBHs in the binary defined as:-

$$\chi = \frac{r}{x} \tag{18}$$

where, r is the physical distance between the PBHs in the binary & x is their comoving distance.)

As

$$H(s) = \frac{1}{s}\frac{ds}{dt} = \sqrt{\frac{8G\pi\rho_{eq}}{3}} h(s) \tag{19}$$

with s being the scale factor,

$$h(s) = \sqrt{s^{-3} + s^{-4}} \tag{20}$$

using eq.(20) in eq.(17); we get:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \int \frac{ds}{sh(s)} \; \frac{\chi(s;\lambda)}{s^3} \; \frac{M_N}{M_N^{eq}} \; x \times [T_{eq} \cdot x] \tag{21}$$

where,

$$M_N = M_{NPBH} + M_{NDM}(s) \tag{22}$$

2 PBH binaries decoupling around matter-radiation equality

As we know that the dynamics of a PBH binary having DM halos and decoupling later around matter-radiation equality is given by:-

$$\ddot{\chi} + \left(s\dot{\chi} - \chi\right) \frac{\left(s\dot{h}(s) + h(s)\right)}{s^2 h(s)} + \frac{1}{\lambda(s)} \frac{1}{\left(sh\right)^2 \chi^2} \frac{\chi}{|\chi|} = 0 \tag{23}$$

where,

$$\lambda(s) = \frac{8\pi \rho_{eq} x^3}{3\left(M_{1PBH} + M_{2PBH} + M_{1DM} + M_{2DM}\right)} \tag{24}$$

or

$$\lambda(s) = \frac{8\pi \rho_{eq} x^3}{3M_{Binary}} \tag{25}$$

where, $M_{Binary} = M_{1PBH} + M_{2PBH} + M_{1DM} + M_{2DM}$

or

$$\lambda(s) = \lambda_0 \times f(s) \tag{26}$$

where,

$$\lambda_0 = \frac{8\pi \rho_{eq} x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} \tag{27}$$

&

$$f(s) = \frac{1}{\left(1 + \frac{M_{DM}}{M_{PBH}}\right)} \tag{28}$$

The neighbouring PBH also decouples from the Hubble flow around matter-radiation equality and accretes a halo of mass given as:-

$$M_{NDM}(s) = \left(\frac{8\pi G \rho_{eq} t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{29}$$

using eq.(29) in eq.(22):-

$$M_N = M_{NPBH} + \left(\frac{8\pi G \rho_{eq} t_{eq}^2}{3}\right)^{3/4} \times 2^{5/4} \times M_{NPBH} \times \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{30}$$

using eq.(30) in eq.(21);we get:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ x \times [T_{eq} \cdot x]\ +\ \left(\frac{8\pi G \rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) 2^{5/4} \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} x \times [T_{eq} \cdot x] \tag{31}$$

Now, from eq.(11); we have:-

$$T_{ij}^{eq} = G M_N^{eq} t_{ij} \tag{32}$$

using eq.(32) in eq.(31); we get:-

$$l = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ x \times [G M_N^{eq} t_{ij} \cdot x]\ +\ \left(\frac{8\pi G \rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{M_{NPBH}}{M_N^{eq}}\right) 2^{5/4} \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} x \times [G M_N^{eq} t_{ij} \cdot x] \tag{33}$$

which can be simplified as:-

$$\frac{l}{A G M_{NPBH} \left(x \times [t_{ij}^{eq} \cdot x]\right)} = \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ +\ \left(\frac{8\pi G \rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{8G\pi\rho_{eq}}{3}\right)^{1/2} 2^{5/4} \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda)\ \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{34}$$

where,

$$A = \left(\frac{3}{8G\pi\rho_{eq}}\right)^{1/2} \tag{35}$$

3 Numerical Calculations

Let us define:-

$$l' = \frac{l}{A G M_{NPBH} \left(x \times [t_{ij}^{eq} \cdot x]\right)} \tag{36}$$

I calculated the value of the angular momentum, l' by solving eq.(14) in [**1**]:-

$$l' = \int \frac{ds}{s h(s)}\ \frac{\chi^2(s;\lambda)}{s^3} \tag{37}$$

where, $h(s) = \sqrt{s^{-3} + s^{-4}}$ and $\chi(s;\lambda)$ is the solution of the eq.(6) in [**1**] given as:-

$$\ddot{\chi} + \left(s\dot{\chi} - \chi\right) \frac{\left(s\dot{h}(s) + h(s)\right)}{s^2 h(s)} + \frac{1}{\lambda} \frac{1}{(sh)^2 \chi^2} \frac{\chi}{|\chi|} = 0 \tag{38}$$

with

$$\lambda \equiv \lambda_0 = \frac{8\pi \rho_{eq} x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} \tag{39}$$

so, the solution of eq.(37) for different values of λ is shown as follows:-

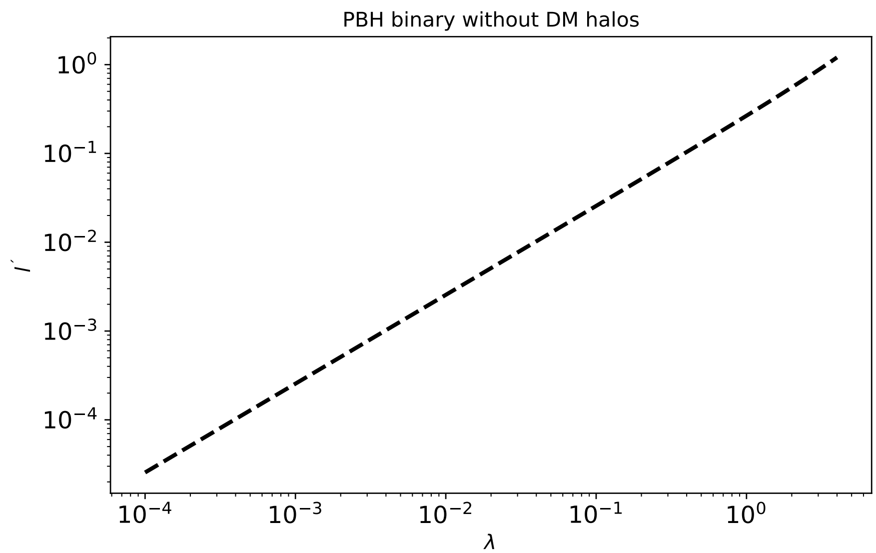

Figure 2: Variation of l' with λ for the PBH binary without DM halos decoupling around matter-radiation equality.

then I tried to compare the solution of eq.(34) i.e.

$$l' = \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda) \ \ + \ \left(\frac{8\pi G \rho_{eq} t_{eq}^6}{3}\right)^{1/4} \left(\frac{8G\pi\rho_{eq}}{3}\right)^{1/2} 2^{5/4} \int \frac{ds}{s^4 h(s)}\ \chi^2(s;\lambda) \left(\frac{2}{3}(s-2)(s+1)^{1/2} + \frac{4}{3}\right)^{1/2} \tag{40}$$

with the solution of eq.(37); as:-

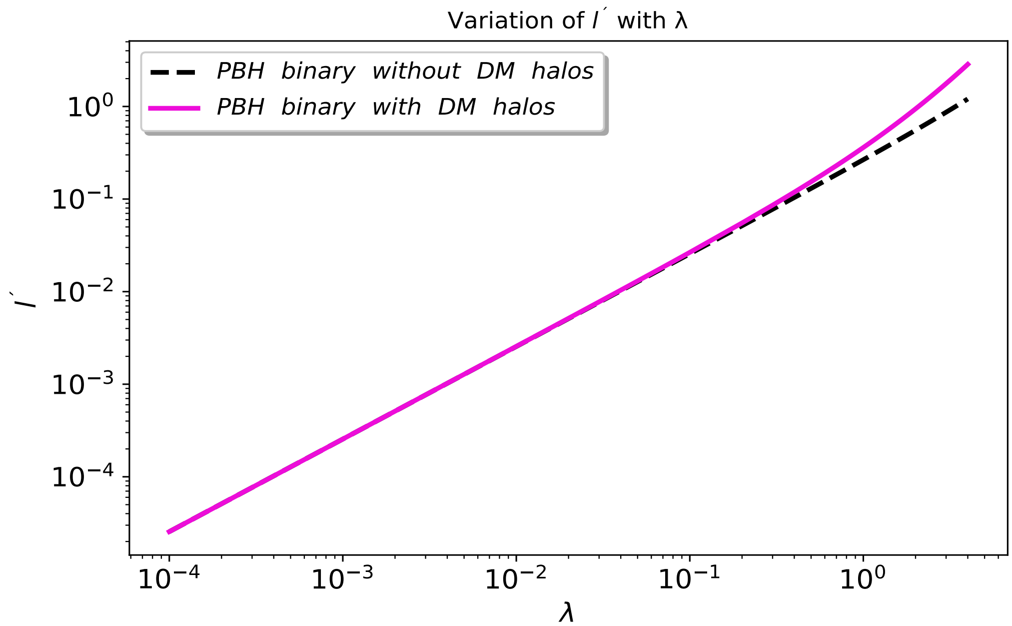

Figure 3: Variation of $l^{'}$ with λ for the PBH binary with and without DM halos decoupling around matter-radiation equality.

This graph shows that the PBH binaries with dark matter halos decoupling around matter-radiation equality corresponding to the higher values of λ experience more torque (or larger angular momentum imparted) due to the neighbouring PBH than the binaries without halos.

4 Calculations of reduced angular momentum, j of the PBH binaries

4.1 For PBH binaries without dark matter halos

For any value of λ, the reduced angular momentum of the binary with mass, $M_{binary} = (M_{1PBH} + M_{2PBH})$ without dark matter halos decoupling around matter-radiation equality is given as:-

$$\mathbf{j} = \frac{\mathbf{l}}{\sqrt{GM_{binary}a}} = \sqrt{1-e^2} \tag{41}$$

using eq.(36) in eq.(41); we get:-

$$\mathbf{j} = \frac{l^{'}AGM_{NPBH}}{\sqrt{GM_{binary}a}} \quad x \ \ \hat{x} \times [t_{ij}^{eq} \cdot \hat{x}] \ \ = \sqrt{1-e^2} \tag{42}$$

from
https://github.com/pratibhajangra591/Ifca-github/blob/main/Tidal%20torque%20in%20the%20PBH%20binary.ipynb, we see that fitted value of $l^{'}$ for PBH binary without dark matter halos is:-

$$l^{'} \approx \left(0.2552\lambda_0 + 0.01131\lambda_0^2\right) \;\; m^{-1} \tag{43}$$

and from
https://github.com/pratibhajangra591/Ifca-github/blob/main/Variation%20of%20semi-major%20axis%20with%20%CE%BB%20.ipynb, we see from the 3rd graph that the semi-major axis, a of the PBH binary without dark matter halos is:-

$$a \approx \left(0.0965\lambda_0 + 0.0165\lambda_0^2\right) x \;\; m \tag{44}$$

using eq.(42) & (43) in eq.(41); we get:-

$$\mathbf{j} \approx \frac{\left(0.2552\lambda_0 + 0.01131\lambda_0^2\right) \, A \, G \, M_{NPBH}}{\sqrt{G\left(0.0965\lambda_0 + 0.0165\lambda_0^2\right) \, x \, M_{binary}}} \quad x \;\; \hat{x} \times [t_{ij}^{eq} \cdot \hat{x}] \;\; = \sqrt{1-e^2} \tag{45}$$

$$\mathbf{j} \approx \frac{\left(0.2552 + 0.01131\lambda_0\right) \, A \, \lambda_0^{1/2}}{\sqrt{G\left(0.0965 + 0.0165\lambda_0\right) M_{binary}}} M_{NPBH} \;\; x^{3/2} \;\; \hat{x} \times [t_{ij}^{eq} \cdot \hat{x}] \;\; = \sqrt{1-e^2} \tag{46}$$

using eqs.(35) & (39) and after simplification; we get:-

$$\mathbf{j} \approx \frac{\left(0.2552 + 0.01131\lambda_0\right)}{\sqrt{\left(0.0965 + 0.0165\lambda_0\right)}} \frac{M_{NPBH}}{M_{binary}} \quad x^3 \;\; \hat{x} \times [t_{ij}^{eq} \cdot \hat{x}] \;\; = \sqrt{1-e^2} \tag{47}$$

or

$$\mathbf{j} \approx 0.8215 \frac{\left(1 + 0.0443\lambda_0\right)}{\sqrt{\left(1 + 0.1709\lambda_0\right)}} \frac{M_{NPBH}}{M_{binary}} \quad x^3 \;\; \hat{x} \times [t_{ij}^{eq} \cdot \hat{x}] \;\; = \sqrt{1-e^2} \tag{48}$$

Now, if we consider the tidal field generated by the neighbouring PBH of mass, M_{NPBH} at comoving separation $y >> x$ (where x is the size of the PBH binary) then using eq.(11); we get:-

$$\mathbf{j} \approx 2.4645 \frac{\left(1 + 0.0443\lambda_0\right)}{\sqrt{\left(1 + 0.1709\lambda_0\right)}} \frac{M_{NPBH}}{M_{binary}} \quad \frac{x^3}{y^3} \left(\hat{x} \cdot \hat{y}\right) \left(\hat{x} \times \hat{y}\right) = \sqrt{1-e^2} \tag{49}$$

Hence, the total reduced angular momentum, j_{total} resulting from all other neighbouring PBHs at comoving separation, $y >> x$ is given by:-

$$\mathbf{j}_{total} \approx 2.4645 \frac{\left(1 + 0.0443\lambda_0\right)}{\sqrt{\left(1 + 0.1709\lambda_0\right)}} \frac{1}{M_{binary}} \sum_p M_{NPBH}^p \, \frac{x^3}{y_p^3} \left(\hat{x} \cdot \hat{y_p}\right) \left(\hat{x} \times \hat{y_p}\right) \tag{50}$$

<u>Note</u>:- Now, if we consider that both the PBHs in the binary are equally massive and the neighbouring PBH is also of the same mass, M_{PBH} ***so***

that $M_{NPBH} = M_{PBH}$ & $M_{binary} = 2 \times M_{PBH}$ and $\lambda_0 \to 0$; then using eq.(50) we get:-

$$\boldsymbol{j} \approx 1.2322 \, \frac{x^3}{y^3} \left(\hat{x} \cdot \hat{y}\right) \left(\hat{x} \times \hat{y}\right) = \sqrt{1 - e^2} \tag{51}$$

which is very roughly in agreement with eq.(17) in [1].

4.2 For PBH binaries with dark matter halos

The value of the reduced angular momentum, j of the binary with dark matter halos and total mass, $M_{halo-binary} = \left(M_{1PBH} + M_{2PBH} + M_{1DM} + M_{2DM}\right)$ is given as:-

$$\mathbf{j}_{halo} = \frac{\mathbf{l}_{halo}}{\sqrt{GM_{halo-binary} a_{halo}}} = \sqrt{1 - e^2_{halo}} \tag{52}$$

As $\left(M_{1PBH} + M_{2PBH} + M_{1DM} + M_{2DM}\right) = M_{hbinary}$; so:-

$$\mathbf{j}_{halo} = \frac{l}{\sqrt{GM_{halo-binary} a_{halo}}} = \sqrt{1 - e^2_{halo}} \tag{53}$$

using eq.(36) in eq.(53); we get:-

$$\mathbf{j}_{halo} = \frac{l'_{halo} A \, G \, M_{NPBH}}{\sqrt{GM_{halo-binary} a_{halo}}} \;\; x \;\; \hat{x} \times [t^{eq}_{ij} \cdot \hat{x}] \;\; = \sqrt{1 - e^2_{halo}} \tag{54}$$

from
https://github.com/pratibhajangra591/Ifca-github/blob/main/Tidal%20torque%20in%20the%20PBH%20binary.ipynb, we see that fitted value of l' for PBH binary with dark matter halos is:-

$$l' \approx \left(0.2529\lambda_0 + 0.1123\lambda_0^2\right) \;\; m^{-1} \tag{55}$$

and from
https://github.com/pratibhajangra591/Ifca-github/blob/main/Variation%20of%20semi-major%20axis%20with%20%CE%BB%20.ipynb, we see that the semi-major axis, a of the PBH binary with dark matter halos is:-

$$a \approx \left(0.0977\lambda_0 + 0.0057\lambda_0^2\right) x \;\; m \tag{56}$$

using eq.(55) & (56) in eq.(54); we get:-

$$\mathbf{j}_{halo} \approx \frac{\left(0.2529\lambda_0 + 0.1123\lambda_0^2\right) \; A \, G \, M_{NPBH}}{\sqrt{G\left(0.0977\lambda_0 + 0.0057\lambda_0^2\right) x \, M_{halo-binary}}} \;\; x \;\; \hat{x} \times [t^{eq}_{ij} \cdot \hat{x}] \;\; = \sqrt{1 - e^2_{halo}} \tag{57}$$

$$\mathbf{j}_{halo} \approx \frac{\left(0.2529 + 0.1123\lambda_0\right) \; A \, G \, \lambda_0^{1/2}}{\sqrt{G\left(0.0977 + 0.0057\lambda_0\right) \, M_{halo-binary}}} M_{NPBH} \;\; x^{3/2} \;\; \hat{x} \times [t^{eq}_{ij} \cdot \hat{x}] \;\; = \sqrt{1 - e^2_{halo}} \tag{58}$$

using eqs.(27) & (35); we get:-

$$\mathbf{j}_{halo} \approx \frac{(0.2529+0.1123\lambda_0)}{\sqrt{(0.0977+0.0057\lambda_0)}}\sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}}\ \frac{M_{NPBH}}{M_{halo-binary}}\ x^3\ \ \hat{x}\times[t^{eq}_{ij}\cdot\hat{x}]\ = \sqrt{1-e^2_{halo}} \tag{59}$$

or

$$\mathbf{j}_{halo} \approx 0.8091\frac{(1+0.4440\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}\sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}}\ \frac{M_{NPBH}}{M_{halo-binary}}\ x^3\ \ \hat{x}\times[t^{eq}_{ij}\cdot\hat{x}]\ = \sqrt{1-e^2_{halo}} \tag{60}$$

Similarly, if we consider the tidal field generated by the neighbouring PBH of mass, M_{NPBH} at comoving separation $y >> x$ (where x is the size of the PBH binary with dark matter halos) then using eq.(11); we get:-

$$\mathbf{j}_{halo} \approx 2.4273\frac{(1+0.4440\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}\sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}}\ \frac{M_{NPBH}}{M_{halo-binary}}\ \frac{x^3}{y^3}\left(\hat{x}\cdot\hat{y}\right)\left(\hat{x}\times\hat{y}\right) = \sqrt{1-e^2} \tag{61}$$

Hence, the total reduced angular momentum, $j_{halo-total}$ resulting from all other neighbouring PBHs at comoving separation, $y >> x$ is given by:-

$$\mathbf{j}_{halo-total} \approx 2.4273\frac{(1+0.4440\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}\sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}}\ \frac{1}{M_{halo-binary}}\sum_p M_{NPBH}\ \frac{x^3}{y_p^3}\left(\hat{x}\cdot\hat{y_p}\right)\left(\hat{x}\times\hat{y_p}\right) \tag{62}$$

Keeping the comoving separation, x and distribution of the neighbouring PBHs same for binaries with and without dark matter halos, we can write:-

$$\frac{\mathbf{j}_{halo-total}}{\mathbf{j}_{total}} = 0.9905\frac{(1+0.4440\lambda_0)}{(1+0.0443\lambda_0)}\sqrt{\frac{(1+0.01691\lambda_0)}{(1+0.0583\lambda_0)}}\sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}}\ \frac{M_{binary}}{M_{halo-binary}} \tag{63}$$

5 Probability Distribution of the initial angular momentum

5.1 Due to the torque exerted by neighbouring PBHs

We consider N neighbouring PBHs of different masses distributed within a volume $V = \frac{4\pi R^3}{3}$ and take the limit $N, V \to \infty$ at constant density $n = \frac{N}{V}$.
The probability distribution function (PDF) of PBHs $P(M)$ is normalized to be

$$\int P(M)\,dM = 1 \tag{64}$$

and the abundance of PBHs in the mass interval $(M, M+dM)$ is given by

$$f P(M)\,dM \tag{65}$$

where f is the total abundance of PBHs in non- relativistic matter, [**2**]. We introduce a cross-grained discrete PDF, namely

$$\int P(M)dM = 1 \quad \rightarrow \sum_{M_{min} \leq M_i \leq M_{max}} P_i \Delta \simeq 1 \tag{66}$$

where, $P(M_i) \rightarrow P_i$ is the binned PDF and $dM_i \rightarrow \Delta$ denotes the resolution of PBH mass. We consider, $f P_i \Delta \equiv f_i \Delta$ as the abundance of PBHs with mass M_i.

Now, the average distance $\bar{x}_i$ between two PBHs with mass M_i is given as:-

$$\bar{x}_i = \frac{3M_i}{4\pi \rho_{eq} f_i \Delta} \tag{67}$$

and the average distance $\langle x_{ij} \rangle$ between two neighboring PBHs with different masses M_i and M_j is:-

$$\langle x_{ij} \rangle = \left(x_i^{-3} + x_j^{-3} \right)^{-1/3} = \mu_{ij}^{1/3} \bar{x}_{ij} \tag{68}$$

where,

$$\mu_{ij} = \frac{2M_i M_j f_b}{M_{binary} \left(f_j M_i + f_i M_j \right)} \tag{69}$$

$$\bar{x}_{ij}^3 = \frac{3M_{binary}}{8\pi \rho_{eq} f_b \Delta} \tag{70}$$

&

$$f_b = f_i + f_j \tag{71}$$

$$M_{binary} = M_i + M_j \tag{72}$$

Ignoring the subscripts ij onwards and using eq.(25) we consider a dimensionless variable, X defined as:-

$$\lambda \equiv \lambda_0 = \frac{8\pi \rho_{eq} x^3}{3\left(M_{1PBH} + M_{2PBH}\right)} = \frac{8\pi \rho_{eq} x^3}{3M_{binary}} = \frac{X}{f_b \Delta} \tag{73}$$

$$\left(Because, for\ PBH\ binary\ without\ dark\ matter\ halos\ ,\ M_{binary} = \left(M_{1PBH} + M_{2PBH}\right). \right)$$

with

$$X = \frac{x^3}{\bar{x}^3} \tag{74}$$

where, $\bar{x}^3$ is given by eq.(70).

5.1.1 For PBH binary without dark matter halos

For PBH binary without dark matter halos, the reduced angular momentum due to the tidal torque exerted by the neighbouring PBHs is given as:-

$$\mathbf{j}_{total} \approx 2.4645 \frac{\left(1+0.0443\lambda_0\right)}{\sqrt{\left(1+0.1709\lambda_0\right)}} \frac{1}{M_{binary}} \sum_p M^p_{NPBH} \frac{x^3}{y_p^3} \left(\hat{x}\cdot\hat{y_p}\right)\left(\hat{x}\times\hat{y_p}\right) \tag{75}$$

$$\approx 2.4645 \frac{\left(1+0.0443\lambda_0\right)}{\sqrt{\left(1+0.1709\lambda_0\right)}} \frac{1}{M_{binary}} \sum_p M^p_{NPBH} \frac{x^3}{y_p^5} \left(\hat{x}\cdot\mathbf{y}_p\right)\left(\hat{x}\times\mathbf{y}_p\right) \tag{76}$$

or

$$\mathbf{j}_{total} \approx 2.4645 \frac{C}{M_{binary}} \sum_p M^p_{NPBH} \frac{x^3}{y_p^5} y_{p\parallel}\, \mathbf{y}_{p\perp} \tag{77}$$

where, $y_{p\parallel} = \left(\hat{x}\cdot\mathbf{y}_p\right)$, $\mathbf{y}_{p\perp} = \left(\hat{x}\times\mathbf{y}_p\right)$ & $C = \frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}$.

Now, the two-dimensional probability distribution for j is given by:-

$$\frac{dP}{d^2j} = \lim_{V\to\infty} \prod_{p=1}^{N} \int_V \frac{d^3y_p}{V} \delta_D\left[\mathbf{j} - j_{total}\right] \tag{78}$$

By definition of the two dimensional delta function; we have:-

$$\delta_D(\mathbf{X}) = \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{X}} \tag{79}$$

so, using eq.(84) in eq.(83); we get:-

$$\frac{dP}{d^2j} = \lim_{V\to\infty} \prod_{p=1}^{N} \int_V \frac{d^3y_p}{V} \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot[\mathbf{j}-j_{total}]} \tag{80}$$

using eq.(82); we get:-

$$\frac{dP}{d^2j} = \lim_{V\to\infty} \prod_{p=1}^{N} \int_V \frac{d^3y_p}{V} \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot\left\{\mathbf{j}-2.4645\frac{C}{M_{binary}}\sum_q M^q_{NPBH} \frac{x^3}{y_q^5} y_{q\parallel}\,\mathbf{y}_{p\perp}\right\}} \tag{81}$$

$$= \lim_{V\to\infty} \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{j}} \left[\prod_{p=1}^{N} \int_V \frac{d^3y_p}{V} e^{i\mathbf{k}\cdot\left\{-2.4645\frac{C}{M_{binary}}\sum_q M^q_{NPBH} \frac{x^3}{y_q^5} y_{q\parallel}\,\mathbf{y}_{p\perp}\right\}}\right] \tag{82}$$

or

$$\frac{dP}{d^2j} = \lim_{V\to\infty} \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{j}}\, I^N \tag{83}$$

where,

$$I = \int_V \frac{d^3y}{V} e^{\left\{-2.4645iC\frac{M_{NPBH}}{M_{binary}} \frac{x^3}{y^5} y_{q\parallel}\,\mathbf{k}\cdot\mathbf{y}_{p\perp}\right\}} \tag{84}$$

$$= 1 - \frac{1}{V}\int_V d^3y \left[1 - exp\left\{-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3}{y^5}y_{q\parallel}\ \mathbf{k}\cdot\mathbf{y}_{p\perp}\right\}\right] \tag{85}$$

(As, $\int_V d^3y = V$)

now,

$$\lim_{V\to\infty} I^N = \lim_{V\to\infty}\left[1 - \frac{1}{V}\int_V d^3y \left\{1 - exp\left(-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3}{y^5}y_{q\parallel}\ \mathbf{k}\cdot\mathbf{y}_{p\perp}\right)\right\}\right]^N \tag{86}$$

As $n = \frac{N}{V}$, so:-

$$\lim_{V\to\infty} I^N = \lim_{V\to\infty}\left[1 - \frac{1}{V}\int_V d^3y \left\{1 - exp\left(-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3}{y^5}y_{q\parallel}\ \mathbf{k}\cdot\mathbf{y}_{p\perp}\right)\right\}\right]^{nV} \tag{87}$$

$$= e^{-nJ} \tag{88}$$

with

$$J = \int_V d^3y \left\{1 - exp\left(-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3}{y^5}y_{q\parallel}\ \mathbf{k}\cdot\mathbf{y}_{p\perp}\right)\right\} \tag{89}$$

$$= \int_V d^3y \left[1 - exp\left\{-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3}{y^3}\left(\hat{x}\cdot\frac{\mathbf{y}}{y}\right)\ \mathbf{k}\cdot\left(\hat{x}\times\frac{\mathbf{y}}{y}\right)\right\}\right] \tag{90}$$

$As,\ y_{\parallel} = (\hat{x}\cdot\mathbf{y})\ \ \&\ \ \mathbf{y}_{\perp} = (\hat{x}\times\mathbf{y}).$

$$= \int_V d^3y \left[1 - exp\left\{-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3}{y^3}\left(\hat{x}\cdot\hat{y}\right)\ \mathbf{k}\cdot\left(\hat{x}\times\hat{y}\right)\right\}\right] \tag{91}$$

$$= \int_V d^3y \left[1 - exp\left\{-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3k}{y^3}\left(\hat{x}\cdot\hat{y}\right)\ \frac{\mathbf{k}}{k}\cdot\left(\hat{x}\times\hat{y}\right)\right\}\right] \tag{92}$$

$$= \int_V d^3y \left[1 - exp\left\{-2.4645iC\frac{M_{NPBH}}{M_{binary}}\ \frac{x^3k}{y^3}\left(\hat{x}\cdot\hat{y}\right)\ \left(\hat{k}\cdot\hat{y}\right)\right\}\right] \tag{93}$$

As, $(\hat{x}\times\hat{y}) = \mathbf{y}_{\perp}$ and $\mathbf{k}\perp\mathbf{x}$ so $\mathbf{y}_{\perp}\parallel\mathbf{k}$.

Let us substitute:-

$$u = -\frac{y}{\left(2.4645Ck\frac{M_{NPBH}}{M_{binary}}\right)^{1/3} x} \tag{94}$$

which gives:-

$$dy = -\left(2.4645Ck\frac{M_{NPBH}}{M_{binary}}\right)^{1/3} xdu \tag{95}$$

using eqs.(99) & (100) in eq.(98); we get:-

$$J = -\left(2.4645Ck\frac{M_{NPBH}}{M_{binary}}\right)x^3 \int_V d^3u \left[1 - exp\left\{\frac{i}{u^3}(\hat{x}\cdot\hat{u})\left(\hat{k}\cdot\hat{u}\right)\right\}\right] \tag{96}$$

Now, let's change the variable from u to some variable y such that $u = y$, then:-

$$J = -\left(2.4645Ck\frac{M_{NPBH}}{M_{binary}}\right)x^3 \int_V d^3y \left[1 - exp\left\{\frac{i}{y^3}(\hat{x}\cdot\hat{y})\left(\hat{k}\cdot\hat{y}\right)\right\}\right] \tag{97}$$

Using spherical polar coordinates; we can also write:-

$$J = -\left(2.4645Ck\frac{M_{NPBH}}{M_{binary}}\right)x^3 \int\int\int y^2 \sin\theta \left[1 - exp\left\{\frac{i}{y^3}(\hat{x}\cdot\hat{y})\left(\hat{k}\cdot\hat{y}\right)\right\}\right] dyd\theta d\phi \tag{98}$$

or

$$J = -\left(9.8580\pi Ck\frac{M_{NPBH}}{M_{binary}}\right)x^3 \int_0^\infty y^2 dy \frac{d^2\hat{y}}{4\pi}\left[1 - e^{\left\{\frac{i}{y^3}(\hat{x}\cdot\hat{y})\left(\hat{k}\cdot\hat{y}\right)\right\}}\right] \tag{99}$$

where, $d^2\hat{y} = \int_0^\pi \int_0^{2\pi}\left[1 - e^{\left\{\frac{i}{y^3}(\hat{x}\cdot\hat{y})\left(\hat{k}\cdot\hat{y}\right)\right\}}\right]\sin\theta d\theta d\phi$ is the angular integral.

Substituting

$$v = \frac{1}{y^3} \tag{100}$$

which gives:-

$$dy = \frac{-1}{3}v^{-4/3}dv \tag{101}$$

using eqs.(103) & (104) in eq.(102); we get:-

$$J = \left(3.2860\pi Ck\frac{M_{NPBH}}{M_{binary}}\right)x^3 \int_0^\infty \frac{dv}{v^2}\frac{d^2\hat{y}}{4\pi}\left[1 - e^{\left\{iv(\hat{x}\cdot\hat{y})\left(\hat{k}\cdot\hat{y}\right)\right\}}\right] \tag{102}$$

$$\equiv \left(3.2860\pi Ck\frac{M_{NPBH}}{M_{binary}}\right)x^3 \int_0^\infty \frac{dv}{v^2}A(v) \tag{103}$$

with

$$A(v) = \int \frac{d^2\hat{y}}{4\pi}\left[1 - e^{\left\{iv(\hat{x}\cdot\hat{y})\left(\hat{k}\cdot\hat{y}\right)\right\}}\right] \tag{104}$$

For polar axis, $\hat{x} \times \hat{k}$:-

$$A(v) = \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 d\mu \left[1 - exp\left\{\frac{iv}{2}\sin(2\phi)\left(1-\mu^2\right)\right\}\right] \tag{105}$$

$$= \int_0^1 d\mu \left[1 - J_0\left\{\frac{v}{2}\left(1-\mu^2\right)\right\}\right] \tag{106}$$

where J_0 is the zeroth-order Bessel function. Since $J_0(x) = 1 + O(x^2)$ for $x \to 0$, so we can compute the integral over v first; using eq.(111) in eq.(108), [**1**]:-

$$J = \left(3.2860\pi Ck \frac{M_{NPBH}}{M_{binary}}\right) x^3 \int_0^1 d\mu \int_0^\infty \frac{dv}{v^2}\left[1 - J_0\left\{\frac{v}{2}\left(1-\mu^2\right)\right\}\right] \tag{107}$$

$$= \left(3.2860\pi Ck \frac{M_{NPBH}}{M_{binary}}\right) x^3 \int_0^1 d\mu \frac{1-\mu^2}{2} \int_0^\infty \frac{du}{u^2}\{1 - J_0(u)\} \tag{108}$$

$$= \left(3.2860\pi Ck \frac{M_{NPBH}}{M_{binary}}\right) x^3 \left(\frac{1}{3}\right) \tag{109}$$

$$= \left(1.0953\pi Ck \frac{M_{NPBH}}{M_{binary}}\right) x^3 \tag{110}$$

using eq.(73); we get:-

$$J = \left(0.4107 Ck \frac{M_{NPBH}}{f_b \Delta \rho_{eq}}\right) X \tag{111}$$

which gives:-

$$nJ = \left(0.4107 Ck \frac{nM_{NPBH}}{f_b \Delta \rho_{eq}}\right) X \tag{112}$$

Since $n^l_{NPBH} M^l_{NPBH} = \rho^l_{NPBH}$ is the energy density of PBHs with mass M^l_{NPBH} and $\sum_l \rho^l_{NPBH} = \rho_{PBH} = f\rho_{eq}$; so:-

$$nJ = \left(0.4107 Ck \frac{f\rho_{eq}}{f_b \Delta \rho_{eq}}\right) X \tag{113}$$

(As, $n \equiv n_{NPBH}$)

which becomes:-

$$nJ = \left(0.4107 Ck \frac{f}{f_b \Delta}\right) X \tag{114}$$

Using eq.(119) in eq.(88); we get:-

$$\frac{dP}{d^2 j} = \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{j}}\ e^{-\left(0.4107 Ck \frac{f}{f_b \Delta} X\right)} \tag{115}$$

Hence, the probability distribution is:-

$$\frac{dP}{dj} = 2\pi j \frac{dP}{d^2 j} = j \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{2\pi} \;\; e^{i\mathbf{k}\cdot\mathbf{j} - \left(0.4107C\frac{f}{f_b\Delta}X\right)k} \tag{116}$$

$$\equiv j \int_{\mathbf{k}\perp\hat{x}} \frac{d^2k}{2\pi} \;\; e^{i\mathbf{k}\cdot\mathbf{j} - j_X k} \tag{117}$$

or

$$\frac{dP}{dj} = j \int_{\mathbf{k}\perp\hat{x}} k\,dk\,J_0(kj) \;\; e^{-j_X k} \tag{118}$$

where,

$$j_X = \left(0.4107C\frac{f}{f_b\Delta}X\right) \tag{119}$$

Using $C = \frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}$; we get:-

$$j_X = \left(0.4107\frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}\frac{f}{f_b\Delta}X\right) \tag{120}$$

Integrating over eq.(124); we get:-

$$j\frac{dP}{dj}\bigg|_X = P\left(j/j_X\right) \equiv P\left(\gamma\right) \tag{121}$$

where, $j/j_X = \gamma$ and

$$P\left(\gamma\right) = \frac{\gamma^2}{\left(1+\gamma^2\right)^{3/2}} \tag{122}$$

This is the probability distribution of the reduced angular momentum (for a given X hence j_X), j of a PBH binary without dark matter halos accounting for tidal torquing by all other PBHs (not just the nearest neighbor).

Note:- Now, if we consider $\lambda_0 \to 0$***; then eq.(120) becomes:-***

$$j_X = \left(0.4107\frac{f}{f_b\Delta}X\right) \tag{123}$$

which is very roughly in agreement with eq.(25) in [2].

5.1.2 For PBH binary with dark matter halos

For PBH binary with dark matter halos, the reduced angular momentum due to the tidal torque exerted by the neighbouring PBHs is given as:-

$$\mathbf{j}_{halo-total} \approx 2.4273\frac{(1+0.4440\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}\sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}}\;\frac{1}{M_{halo-binary}}\sum_p M^p_{NPBH}\frac{x^3}{y_p^3}\left(\hat{x}\cdot\frac{\mathbf{y}_p}{y_p}\right)\left(\hat{x}\times\frac{\mathbf{y}_p}{y_p}\right) \tag{124}$$

Using the similar calculations done previously; we get the probability distribution of the reduced angular momentum (for a given X hence j_X), j accounting for tidal torquing by all other PBHs (not just the nearest neighbor) is given as:-

$$P\left(\gamma_{halo}\right) = \frac{\gamma_{halo}^2}{\left(1+\gamma_{halo}^2\right)^{3/2}} \tag{125}$$

where, $j_{halo}/j_{X_{halo}} = \gamma_{halo}$ and

$$j_{X_{halo}} = \left\{ 0.4045 \frac{(1+0.4440\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}} \sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}} \; \frac{f}{f_b \Delta} X \right\} \tag{126}$$

or

$$j_{X_{halo}} = \left(0.4045 \; D \sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}} \; \frac{f}{f_b \Delta} X \right) \tag{127}$$

with $D = \frac{(1+0.4440\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}$.

<u>Note</u>:- Now, if we consider $\lambda_0 \to 0$; then eq.(127) becomes:-

$$j_X = \left\{ 0.4045 \; \sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}} \; \frac{f}{f_b \Delta} X \right\} \tag{128}$$

which is very different from eq.(25) in [2] from the PBH binaries without dark matter halos.

5.2 Due to the torque exerted by linear density perturbations

If the PBH fraction is smaller than the characteristic large-scale matter density perturbation δ_m, then tidal torques are dominated by large-scale linear perturbations, $T_{ij}^{eq} = -\partial_i \partial_j \phi = -4\pi G \rho_{eq} \partial_i \partial_j \partial^{-2} \delta_m$. Let us now consider torques by linear density perturbations in the case where PBHs do not make all of the dark matter. The linear density field, hence tidal tensor, are Gaussian, and so is the resulting **j**,[**1**].

5.2.1 For PBH binary without dark matter halos

For PBH binary without dark matter halos, the reduced angular momentum due to the tidal torque exerted by a neighbouring PBH of mass, M_{NPBH} is given as (eq.(48)):-

$$\mathbf{j} \approx 0.8215 \frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}} \frac{M_{NPBH}}{M_{binary}} \quad x^3 \;\; \hat{x} \times [t_{ij}^{eq} \cdot \hat{x}] \;\; = \sqrt{1-e^2} \tag{129}$$

Using $\frac{T_{ij}^{eq}}{GM_{NPBH}} = t_{ij}^{eq}$; we get:-

$$\mathbf{j} \approx 0.8215\left(\frac{Cx^3}{GM_{binary}}\right) \quad \hat{x} \times [T_{ij}^{eq} \cdot \hat{x}] \; = \sqrt{1-e^2} \tag{130}$$

where, $C = \frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}$.

In tensor notation, eq.(130) can be written as:-

$$\mathbf{j} \approx 0.8215\left(\frac{Cx^3}{GM_{binary}}\right) \; \epsilon_{ijk}\hat{x}_j T_{kl}\hat{x}_l = \sqrt{1-e^2} \tag{131}$$

Now, using eq.(131), the variance of $\mathbf{j}$ is calculated as follows:-

$$\left\langle j^2 \right\rangle = 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle \epsilon_{ijk}\hat{x}_j T_{kl}\hat{x}_l \epsilon_{ipq}\hat{x}_p T_{qm}\hat{x}_m \right\rangle \tag{132}$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle \epsilon_{ijk}\epsilon_{ipq}\hat{x}_j T_{kl}\hat{x}_l \hat{x}_p T_{qm}\hat{x}_m \right\rangle \tag{133}$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle \left(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right)\hat{x}_j T_{kl}\hat{x}_l \hat{x}_p T_{qm}\hat{x}_m \right\rangle \tag{134}$$

$$\left(As, \;\; \epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\right)$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle \delta_{jp}\delta_{kq}\hat{x}_j T_{kl}\hat{x}_l \hat{x}_p T_{qm}\hat{x}_m - \delta_{jq}\delta_{kp}\hat{x}_j T_{kl}\hat{x}_l \hat{x}_p T_{qm}\hat{x}_m \right\rangle \tag{135}$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle \left(\delta_{jp}\hat{x}_p\right)\hat{x}_j T_{kl}\hat{x}_l \left(\delta_{kq}T_{qm}\right)\hat{x}_m - \hat{x}_j T_{kl}\hat{x}_l \left(\delta_{kp}\hat{x}_p\right)\left(\delta_{jq}T_{qm}\right)\hat{x}_m \right\rangle \tag{136}$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle \hat{x}_j\hat{x}_j T_{kl}\hat{x}_l T_{km}\hat{x}_m - \hat{x}_j T_{kl}\hat{x}_l \hat{x}_k T_{jm}\hat{x}_m \right\rangle \tag{137}$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle T_{kl}\hat{x}_l T_{km}\hat{x}_m - \hat{x}_k T_{kl}\hat{x}_l \hat{x}_j T_{jm}\hat{x}_m \right\rangle \tag{138}$$

$$\left(As, \;\; \hat{x}_j\hat{x}_j = \tfrac{x}{x} = 1\right)$$

$$= 0.6806\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle T_{kl}\hat{x}_l T_{km}\hat{x}_m - \left(\hat{x}_k T_{kl}\hat{x}_l\right)^2 \right\rangle \tag{139}$$

Averaging over the direction of $\hat{x}$, we get:-

$$\left\langle j^2 \right\rangle = \frac{0.6806}{5}\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left\langle T_{ij}T_{ij} - \frac{1}{3}T_{ii}T_{jj} \right\rangle \tag{140}$$

In Fourier space,

$$T^{eq}_{ij} = -k_i k_j \phi = \hat{k_i}\hat{k_j} 4\pi G \rho_{eq}\delta_{eq} \tag{141}$$

with δ_{eq} being the larger scale matter density perturbations of the dark matter at matter-radiation equality and ρ_{eq} being the density of the universe at matter-radiation equality.

Using eq.(141) in eq.(140); we get:-

$$\left\langle j^2 \right\rangle = \frac{0.6806}{5}\left(\frac{Cx^3}{GM_{binary}}\right)^2 \left(\frac{4\pi G}{3}\right)^2 \rho_{eq}^2 \left\langle \delta_{eq}^2 \right\rangle \tag{142}$$

which gives:-

$$\left\langle j^2 \right\rangle^{1/2} = \sqrt{\frac{0.6806}{5\times 4}}\left(\frac{Cx^3}{M_{binary}}\right)\left(\frac{8\pi}{3}\right)\rho_{eq}\left\langle \delta_{eq}^2 \right\rangle^{1/2} \tag{143}$$

So, the variance of $\mathbf{j}$ due to the torques by density perturbations is:-

$$\left\langle j^2 \right\rangle^{1/2} = 0.4519\left(\frac{Cx^3}{M_{binary}}\right)\left(\frac{8\pi}{3}\right)\rho_{eq}\sigma_{eq} \tag{144}$$

where, $\sigma_{eq} = \left\langle \delta_{eq}^2 \right\rangle^{1/2}$ is the variance of density perturbations of the rest of dark matter on scale of order $(10^0 \sim 10^3)M_\odot$ at equality.

Using $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{33(M_{1PBH}+M_{2PBH})} = \frac{X}{f_b\Delta}$; we get:-

$$\left\langle j^2 \right\rangle^{1/2} = 0.4519 C \frac{\sigma_{eq}}{f_b\Delta} X \tag{145}$$

or

$$\left\langle j^2 \right\rangle^{1/2} = 0.4519 \frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}\frac{\sigma_{eq}}{f_b\Delta} X \tag{146}$$

Note**:- Now, if we consider* $\lambda_0 \to 0$; then eq.(146) becomes:-***

$$\left\langle j^2 \right\rangle^{1/2} = 0.4519 \frac{\sigma_{eq}}{f_b\Delta} X \tag{147}$$

which is in agreement with eq.(27) in [2] for the PBH binaries without dark matter halos.

5.2.2 For PBH binary with dark matter halos

For PBH binary without dark matter halos, the reduced angular momentum due to the tidal torque exerted by a neighbouring PBH of mass, M_{NPBH} is given as (eq.(60)):-

$$\mathbf{j}_{halo} \approx 0.8091 \frac{(1+0.0444\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}} \sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}} \frac{M_{NPBH}}{M_{halo-binary}} \; x^3 \; \hat{x}\times[t^{eq}_{ij}\cdot\hat{x}] \; = \sqrt{1-e^2_{halo}} \tag{148}$$

Using $\frac{T^{eq}_{ij}}{GM_{NPBH}} = t^{eq}_{ij}$; we get:-

$$\mathbf{j}_{halo} \approx 0.8091 \sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}} \left(\frac{Dx^3}{GM_{halo-binary}}\right) \; \hat{x}\times[T^{eq}_{ij}\cdot\hat{x}] \; = \sqrt{1-e^2_{halo}} \tag{149}$$

where, $D = \frac{(1+0.0444\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}$.

In tensor notation, eq.(149) can be written as:-

$$\mathbf{j}_{halo} \approx 0.8091 \sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}} \left(\frac{Dx^3}{GM_{halo-binary}}\right) \; \epsilon_{ijk}\hat{x}_j T_{kl}\hat{x}_l = \sqrt{1-e^2_{halo}} \tag{150}$$

Using the similar approach, we get the variance of, $\mathbf{j}$ due to the torques by density perturbations is:-

$$\left\langle j^2_{halo}\right\rangle^{1/2} = 0.4432 \sqrt{\frac{M_{halo-binary}}{(M_{1PBH}+M_{2PBH})}} \left(\frac{Dx^3}{M_{halo-binary}}\right)\left(\frac{8\pi}{3}\right)\rho_{eq}\sigma_{eq} \tag{151}$$

Using $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3(M_{1PBH}+M_{2PBH})} = \frac{X}{f_b\Delta}$; we get:-

$$\left\langle j^2_{halo}\right\rangle^{1/2} = 0.4432\, D \sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}} \left(\frac{\sigma_{eq}}{f_b\Delta}\right) X \tag{152}$$

or

$$\left\langle j^2_{halo}\right\rangle^{1/2} = 0.4432 \frac{(1+0.0444\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}} \sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}} \left(\frac{\sigma_{eq}}{f_b\Delta}\right) X \tag{153}$$

<u>Note</u>:- Now, if we consider $\lambda_0 \to 0$; then eq.(153) becomes:-

$$\left\langle j^2_{halo}\right\rangle^{1/2} = 0.4432 \sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}} \left(\frac{\sigma_{eq}}{f_b\Delta}\right) X \tag{154}$$

which is different from eq.(27) in [2] for the PBH binaries without dark matter halos.

6 Characteristic initial properties of binaries merging today

6.1 For PBH binaries without dark matter halos

For PBH binaries without dark matter halos, the characteristic initial reduced angular momentum, j as a function of the dimensionless parameter, X is given by (adding eqs.(120) and (146)): -

$$j_X^2 = \left(0.4107\frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}\frac{f}{f_b\Delta}X\right)^2 + \left(0.4519\frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}\frac{\sigma_{eq}}{f_b\Delta}X\right)^2 \tag{155}$$

$$= \left\{(0.4107)^2 f^2 + (0.4509)^2\sigma_{eq}^2\right\}\left(\frac{C\,X}{f_b\Delta}\right)^2 \tag{156}$$

or

$$j_X = \left\{(0.4107)^2 f^2 + (0.4509)^2\sigma_{eq}^2\right\}^{1/2}\left(\frac{C\,X}{f_b\Delta}\right) \tag{157}$$

For the PBH binaries without dark matter halos having initial eccentricities close to unity, i.e. $j << 1$, the coalescence time through GW emission is given by; [**1**]:-

$$t(a;j) = \frac{3}{85}\left(\frac{a^4 j^7}{G^3 M_{1PBH}M_{2PBH}\left(M_{1PBH}+M_{2PBH}\right)}\right) \tag{158}$$

or

$$t(a;j) = \frac{3}{85}\left(\frac{a^4 j^7}{G^3 M_{1PBH}M_{2PBH}M_{binary}}\right) \tag{159}$$

$$\left(As,\ M_{binary} = M_{1PBH}+M_{2PBH}\right)$$

For a given X hence a, there is a unique j such that the merger time is t, so:-

$$j(t;a) = \left\{\frac{85}{3}\ \frac{G^3 M_{1PBH}M_{2PBH}M_{binary}\ t}{a^4}\right\}^{1/7} \tag{160}$$

Using semi-major axis, $a = \left(0.0965\lambda_0 + 0.0165\lambda_0^2\right)x$, $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3M_{binary}} = \frac{X}{f_b\Delta}$ & $X = \frac{x^3}{\bar{x}^3}$; we get:-

$$j(t;X) = \left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{binary}\ t}{\left(0.0965+0.0165\lambda_0\right)^4\bar{x}^4 X^{16/3}}\right\}^{1/7} \tag{161}$$

The differential probability distribution of (X,t) is then given by:-

$$\frac{d^2P}{dX\,dt} = \frac{dP}{dX}\left.\frac{dP}{dt}\right|_X = \frac{dP}{dX}\times\left[\frac{\partial j}{\partial t}\left.\frac{dP}{dj}\right|_X\right]_{j(t;X)} \tag{162}$$

Let's assume that PBHs possess a random distribution, probability distribution of the separation x between two nearest PBHs with mass M_{1PBH} and M_{2PBH} without other PBHs in the volume of $\frac{4}{3}\pi x^3$ becomes as; [**2**]:-

$$\frac{dP}{d\tilde{X}} = e^{-\frac{4}{3}x^3 n_T} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T} \tag{163}$$

where, $\tilde{X} \equiv \frac{x^3}{\langle x_{ij}\rangle^3} = \frac{X}{\mu}$ and $n_T = f\rho_{eq}\int_0^\infty \frac{P(M)}{M}dM$

So, using eq.(163) and $\partial j/\partial t = j/7t$ (from eq. (160)); we can write eq.(162) as:-

$$\frac{d^2P}{d\tilde{X}dt} = \frac{1}{7t}e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\,P(\gamma_X) = \frac{1}{7t}e^{-\frac{X}{\mu}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\,P(\gamma_X) \tag{164}$$

$$\left(As,\ P(\gamma) \equiv P(j/j_X) = j\frac{dP}{dj}\bigg|_X\right)$$

Using Bayes' theorem, the probability distribution of $\tilde{X}$ for binaries merging after a time t_0 is, [**1**]:-

$$\frac{dP}{d\tilde{X}}\bigg|_{t_0} \propto \frac{d^2P}{d\tilde{X}dt}\bigg|_{t_0} \propto \frac{1}{7t}e^{-\frac{X}{\mu}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\,P(\gamma_X), \qquad t = t_0 \tag{165}$$

Now, for $X << 1$ and hence $e^{-X} \approx 1$; the above eq. gives:-

$$\frac{\partial}{\partial\tilde{X}}\left[\frac{dP}{d\tilde{X}}\bigg|_{t_0}\right]_{X_*} \propto \mu P^{'}(\gamma_{X_*})\frac{\partial\gamma_X}{\partial X} = 0 \tag{166}$$

and for γ_X to be strictly monotonic, $\left(\partial\gamma_X/\partial X\right) \neq 0$, so:-

$$P^{'}(\gamma_{X_*}) = 0 \tag{167}$$

Using $P\left(\gamma\right) = \frac{\gamma^2}{\left(1+\gamma^2\right)^{3/2}}$ in the above eq.; we get:-

$$\gamma_{X_*} = \sqrt{2} \tag{168}$$

which ultimately gives:-

$$j\left(t_0; X_*\right) = \sqrt{2}j_{X_*} \tag{169}$$

$$\left(As,\ \gamma_X = j(t;X)/j_X\right)$$

Using eqs.(157) and (161) in the above eq.; we get:-

$$\left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{binary}\ t_0}{\left(0.0965+0.0165\lambda_0\right)^4\bar{x}^4 X_*^{16/3}}\right\}^{1/7} = \sqrt{2}\left\{(0.4107)^2 f^2 + (0.4509)^2\sigma_{eq}^2\right\}^{1/2}\left(\frac{C\,X_*}{f_b\Delta}\right) \tag{170}$$

Using $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3(M_{1PBH}+M_{2PBH})} = \frac{X}{f_b\Delta}$ and $X = \frac{x^3}{\bar{x}^3}$; in the above eq. and solving for X_*; we get:-

$$X_* = 0.0238\ t_0^{3/37}\ \left(M_{1PBH}M_{2PBH}\right)^{3/37}\ M_{binary}^{-1/37}\ \rho_{eq}^{4/37}\ f_b\Delta\ \frac{\left(f^2+1.2107\sigma_{eq}^2\right)^{-21/74}}{\left(1+0.1709\lambda_0\right)^{12/37}}\ C^{-21/37} \tag{171}$$

or

$$X_* = 0.0238\ t_0^{3/37}\ \left(M_{1PBH}M_{2PBH}\right)^{3/37}\ M_{binary}^{-1/37}\ \rho_{eq}^{4/37}\ f_b\Delta\ \frac{\left(f^2+1.2107\sigma_{eq}^2\right)^{-21/74}}{\left(1+0.1709\lambda_0\right)^{12/37}}\left(\frac{\left(1+0.0443\lambda_0\right)}{\sqrt{\left(1+0.1709\lambda_0\right)}}\right)^{-21/37} \tag{172}$$

$$\left(As,\ C = \frac{(1+0.0443\lambda_0)}{\sqrt{(1+0.1709\lambda_0)}}\right)$$

So, the most probable value of X for PBH binaries without dark matter halos are merging today is:-

$$X_* = 0.0238\ \left(M_{1PBH}M_{2PBH}\right)^{3/37}\ M_{binary}^{-1/37}\ \rho_{eq}^{4/37}\ f_b\Delta\ \frac{\left(f^2+1.2107\sigma_{eq}^2\right)^{-21/74}}{\left(1+0.1709\lambda_0\right)^{12/37}}\ \left(\frac{\left(1+0.0443\lambda_0\right)}{\sqrt{\left(1+0.1709\lambda_0\right)}}\right)^{-21/37} \tag{173}$$

and the characteristic angular momentum j_* is simply $j\left(t_0,X_*\right) = \sqrt{2}j_{X_*}$ i.e.(using eq.(157)):-

$$j_* \equiv j\left(t_0,X_*\right) = \sqrt{2}\left\{(0.4107)^2 f^2 + (0.4509)^2\,\sigma_{eq}^2\right\}^{1/2}\left(\frac{C\,X_*}{f_b\Delta}\right) \tag{174}$$

using eq.(173) in eq.(174):-

$$j_* = 0.0057\ \left(M_{1PBH}M_{2PBH}\right)^{3/37}\ M_{binary}^{-1/37}\ \rho_{eq}^{4/37}\ \frac{\left(f^2+1.2107\sigma_{eq}^2\right)^{8/37}}{\left(1+0.1709\lambda_0\right)^{12/37}}\ \left(\frac{\left(1+0.0443\lambda_0\right)}{\sqrt{\left(1+0.1709\lambda_0\right)}}\right)^{16/37} \tag{175}$$

6.1.1 Probability Distribution for the semi-major axis of the binary without dark matter halos

The differential probability distribution for the semi-major axis of the PBH binary without dark matter halos is given as:-

$$\frac{dP}{da} = \frac{dP}{d\tilde{X}}\frac{d\tilde{X}}{da} \tag{176}$$

Using $a \approx \left(0.0965\lambda_0 + 0.0165\lambda_0^2\right) x$ and $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3(M_{1PBH}+M_{2PBH})}$; we have:-

$$x^4 = \left\{ \frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\} a \tag{177}$$

which gives:-

$$\frac{dx}{da} = \frac{1}{4}\left\{ \frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\}^{1/4} a^{-3/4} \tag{178}$$

Now, as $X = \frac{x^3}{\bar{x}^3} \equiv \frac{x(a)^3}{\bar{x}^3}$; so we can write:-

$$\tilde{X} = \frac{X}{\mu} = \frac{1}{\mu}\frac{x(a)^3}{\bar{x}^3} \tag{179}$$

which gives:-

$$\frac{d\tilde{X}}{da} = \frac{3}{\left(\mu\bar{x}^3\right)} x(a)^2 \frac{dx(a)}{da} \tag{180}$$

using eqs.(177) & (178) in the above eq; we get:-

$$\frac{d\tilde{X}}{da} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{ \frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\}^{3/4} a^{-1/4} \tag{181}$$

Using $\frac{dP}{d\tilde{X}} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}$ and eq.(181) in eq.(176); we get:-

$$\frac{dP}{da} = \frac{dP}{d\tilde{X}}\frac{d\tilde{X}}{da} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\ \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{ \frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\}^{3/4} a^{-1/4} \tag{182}$$

i.e.

$$\frac{dP}{da} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{ \frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\}^{3/4} a^{-1/4}\ e^{-\frac{X}{\mu}\frac{4}{3}\langle x_{ij}\rangle^3 n_T} \tag{183}$$

using $\langle x_{ij}\rangle = \mu^{1/3}\,\bar{x}$ and $X = \frac{x(a)^3}{\bar{x}^3}$ in the above eq.; we get:-

$$\frac{dP}{da} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{ \frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\}^{3/4} a^{-1/4}\ e^{\frac{-4}{3}x(a)^3 n_T} \tag{184}$$

Since for PBH binary without dark matter halos $M_{binary} = \left(M_{1PBH}+M_{2PBH}\right)$; so the differential probability distribution for the semi-major axis of the binary as:-

$$\frac{dP}{da} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{ \frac{3M_{binary}}{8\pi\rho_{eq}\left(0.0965+0.0165\lambda_0\right)} \right\}^{3/4} a^{-1/4}\ e^{\frac{-4}{3}x(a)^3 n_T} \tag{185}$$

6.1.2 Probability Distribution for the eccentricity of the orbit of the PBH binary without dark matter halos

The differential probability distribution for the eccentricity of the orbit of the PBH binary without dark matter halos is given as:-

$$\frac{dP}{de} = \frac{dP}{d\tilde{X}}\frac{d\tilde{X}}{de} \tag{186}$$

For a given X hence a, there is a unique j such that the merger time is t, so:-

$$j(t;a) = \left\{\frac{85}{3}\,\frac{G^3 M_{1PBH} M_{2PBH} M_{binary}\; t}{a^4}\right\}^{1/7} = \left(1-e^2\right)^{1/2} \tag{187}$$

Using semi-major axis, $a = \left(0.0965\lambda_0 + 0.0165\lambda_0^2\right)x$, $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3M_{binary}} = \frac{X}{f_b\Delta}$ & $X = \frac{x^3}{\bar{x}^3}$; we get:-

$$j(t;X) = \left\{\frac{85}{3}\,\frac{G^3\left(f_b\Delta\right)^4 M_{1PBH} M_{2PBH} M_{binary}\; t}{\left(0.0965+0.0165\lambda_0\right)^4 \bar{x}^4 X^{16/3}}\right\}^{1/7} = \left(1-e^2\right)^{1/2} \tag{188}$$

which gives:-

$$X = \left\{\frac{85}{3}\,\frac{G^3\left(f_b\Delta\right)^4 M_{1PBH} M_{2PBH} M_{binary}\; t}{\left(0.0965+0.0165\lambda_0\right)^4 \bar{x}^4}\right\}^{3/16} \frac{1}{\left(1-e^2\right)^{21/32}} \tag{189}$$

from the above eq.; we get:-

$$\frac{dX}{de} = \frac{21}{16}\left\{\frac{85}{3}\,\frac{G^3\left(f_b\Delta\right)^4 M_{1PBH} M_{2PBH} M_{binary}\; t}{\left(0.0965+0.0165\lambda_0\right)^4 \bar{x}^4}\right\}^{3/16} \frac{e}{\left(1-e^2\right)^{53/32}} \tag{190}$$

or

$$\frac{d\tilde{X}}{de} = \frac{1}{\mu}\frac{dX}{de} = \frac{21}{16}\frac{1}{\mu}\left\{\frac{85}{3}\,\frac{G^3\left(f_b\Delta\right)^4 M_{1PBH} M_{2PBH} M_{binary}\; t}{\left(0.0965+0.0165\lambda_0\right)^4 \bar{x}^4}\right\}^{3/16} \frac{e}{\left(1-e^2\right)^{53/32}} \tag{191}$$

Using $\frac{dP}{d\tilde{X}} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}$ and eq.(191) in eq.(186); we get:-

$$\frac{dP}{de} = \frac{dP}{d\tilde{X}}\frac{d\tilde{X}}{de} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\frac{21}{16}\frac{1}{\mu}\left\{\frac{85}{3}\,\frac{G^3\left(f_b\Delta\right)^4 M_{1PBH} M_{2PBH} M_{binary}\; t}{\left(0.0965+0.0165\lambda_0\right)^4 \bar{x}^4}\right\}^{3/16} \frac{e}{\left(1-e^2\right)^{53/32}} \tag{192}$$

using $\langle x_{ij}\rangle = \mu^{1/3}\,\bar{x}$ and $\tilde{X} = \frac{X}{\mu}$ in the above eq.; we get the differential probability distribution for the eccentricity of the PBH binary orbit as:-

$$\frac{dP}{de} = e^{-X\frac{4}{3}\bar{x}^3 n_T}\frac{21}{16}\frac{1}{\mu}\left\{\frac{85}{3}\,\frac{G^3\left(f_b\Delta\right)^4 M_{1PBH} M_{2PBH} M_{binary}\; t}{\left(0.0965+0.0165\lambda_0\right)^4 \bar{x}^4}\right\}^{3/16} \frac{e}{\left(1-e^2\right)^{53/32}} \tag{193}$$

6.2 For PBH binaries with dark matter halos

For PBH binaries without dark matter halos, the characteristic initial reduced angular momentum, j as a function of the dimensionless parameter, X is given by (adding eqs.(127) and (153)): -

$$j^2_{X_{halo}} = \left(0.4045\, D\sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}}\ \frac{f}{f_b\Delta}X\right)^2 + \left(0.4432\, D\sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}}\left(\frac{\sigma_{eq}}{f_b\Delta}\right)X\right)^2 \tag{194}$$

$$= \left\{(0.4045)^2 f^2 + (0.4432)^2\sigma^2_{eq}\right\}\left(\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}\right)\left(\frac{D\,X}{f_b\Delta}\right)^2 \tag{195}$$

or

$$j_{X_{halo}} = \left\{(0.4045)^2 f^2 + (0.4432)^2\sigma^2_{eq}\right\}^{1/2}\sqrt{\frac{(M_{1PBH}+M_{2PBH})}{M_{halo-binary}}}\left(\frac{D\,X}{f_b\Delta}\right) \tag{196}$$

Similarly, for the PBH binaries with dark matter halos having initial eccentricities close to unity, i.e. $j << 1$, the coalescence time through GW emission is given by; [**1**]:-

$$t(a;j) = \frac{3}{85}\left(\frac{a^4 j^7}{G^3 M_1 M_2 M_{halo-binary}}\right) \tag{197}$$

(where, $M_1 = M_{1PBH} + M_{1DM}$, $M_2 = M_{2PBH} + M_{2DM}$ & $M_{halo-binary} = M_{1PBH} + M_{2PBH} + M_{1DM} + M_{1DM}$.)

then, for a given X hence a_{halo}, there is a unique j_{halo} such that the merger time is t_{halo}, so:-

$$j_{halo}(t;a_{halo}) = \left\{\frac{85}{3}\ \frac{G^3 M_1 M_2 M_{halo-binary}\ t}{a^4_{halo}}\right\}^{1/7} \tag{198}$$

Using semi-major axis, $a_{halo} = \left(0.0977\lambda_0 + 0.0057\lambda_0^2\right)x$, $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3(M_{1PBH}+M_{2PBH})} = \frac{X}{f_b\Delta}$ & $X = \frac{x}{\bar{x}}$; we get:-

$$j_{halo}(t;X) = \left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_1 M_2 M_{halo-binary}\ t}{\left(0.0977+0.0057\lambda_0\right)^4\bar{x}^4 X^{16/3}}\right\}^{1/7} \tag{199}$$

Let's assume that PBHs possess a random distribution, probability distribution of the separation x between two nearest PBHs with mass $M_1 = M_{1PBH} + M_{1DM}$ and $M_2 = M_{2PBH} + M_{2DM}$ without other PBHs in the volume of $\frac{4}{3}\pi x^3$ becomes as; [**2**]:-

$$\frac{dP_{halo}}{d\tilde{X}} = e^{-\frac{4}{3}x^3 n_T} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T} \tag{200}$$

where, $\tilde{X} \equiv \frac{x^3}{\langle x_{ij}\rangle^3} = \frac{X}{\mu}$ and $n_T = f\rho_{eq}\int_0^\infty \frac{P(M)}{M}dM$

then, using eq.(192) and $\partial j/\partial t = j/7t$ (from eq. (191)); we can write:-

$$\frac{d^2P_{halo}}{d\tilde{X}dt} = \frac{1}{7t}e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\,P_{halo}(\gamma_X) = \frac{1}{7t}e^{-\frac{X}{\mu}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\,P_{halo}(\gamma_X) \tag{201}$$

$$\left(As,\ P(\gamma) \equiv P(j/j_X) = j\frac{dP}{dj}\bigg|_X\right)$$

Using Bayes' theorem, the probability distribution of $\tilde{X}$ for binaries merging after a time t_0 is, [**1**]:-

$$\frac{dP_{halo}}{d\tilde{X}}\bigg|_{t_0} \propto \frac{d^2P_{halo}}{d\tilde{X}dt}\bigg|_{t_0} \propto \frac{1}{7t}e^{-\frac{X}{\mu}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\,P_{halo}(\gamma_X), \qquad t = t_0 \tag{202}$$

Now, for $X << 1$ and hence $e^{-X} \approx 1$; the above eq. gives:-

$$\frac{\partial}{\partial\tilde{X}}\left[\frac{dP_{halo}}{d\tilde{X}}\bigg|_{t_0}\right]_{X_*halo} \propto\ \mu P'_{halo}(\gamma_{X_{*halo}})\frac{\partial\gamma_X}{\partial X} = 0 \tag{203}$$

and for γ_X to be strictly monotonic, $\left(\partial\gamma_X/\partial X\right) \neq 0$, so:-

$$P'_{halo}(\gamma_{X_{*halo}}) = 0 \tag{204}$$

Using $P\left(\gamma\right) = \frac{\gamma^2}{\left(1+\gamma^2\right)^{3/2}}$ in the above eq.; we get:-

$$\gamma_{X_{*halo}} = \sqrt{2} \tag{205}$$

which ultimately gives:-

$$j_{halo}\left(t_0; X_{*halo}\right) = \sqrt{2}\,j_{X_{*halo}} \tag{206}$$

$$\left(As,\ \gamma_X = j(t;X)/j_X\right)$$

Using eqs.(196) and (199) in the above eq.; we get:-

$$\left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_1M_2M_{halo-binary}\ t_0}{\left(0.0977+0.0057\lambda_0\right)^4\bar{x}^4X_{*halo}^{16/3}}\right\}^{1/7} = \sqrt{2}\left\{(0.4045)^2f^2+(0.4432)^2\sigma_{eq}^2\right\}^{1/2}\sqrt{\frac{\left(M_{1PBH}+M_{2PBH}\right)}{M_{halo-binary}}}\left(\frac{D\ X}{f_b\Delta}\right) \tag{207}$$

Using $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3(M_{1PBH}+M_{2PBH})} = \frac{X}{f_b\Delta}$ and $X = \frac{x^3}{\bar{x}^3}$; in the above eq. and solving for X_{*halo}; we get the most probable value of X for PBH binaries with dark matter halos merging today as:-

$$X_{*halo} = 0.0089\ t_0^{3/37}\ \left(M_1M_2\right)^{3/37}\ M_{halo-binary}^{-15/74}\ \left(M_{1PBH}+M_{2PBH}\right)^{13/74}\ \rho_{eq}^{4/37}\ f_b\Delta\ \frac{\left(f^2+1.2005\sigma_{eq}^2\right)^{-21/74}}{\left(1+0.0583\lambda_0\right)^{12/37}}\ D^{-21/37} \tag{208}$$

$$\left(where,\ D = \frac{(1+0.0444\lambda_0)}{\sqrt{(1+0.0583\lambda_0)}}\right)$$

and the characteristic angular momentum j_{*halo} is simply $j\left(t_0, X_{*halo}\right) = \sqrt{2} j_{X_{*halo}}$ i.e.(using eq.(157)):-

$$j_{*halo} \equiv j\left(t_0, X_{*halo}\right) = \sqrt{2}\left\{(0.4045)^2 f^2 + (0.4432)^2 \sigma_{eq}^2\right\}^{1/2} \sqrt{\frac{\left(M_{1PBH}+M_{2PBH}\right)}{M_{halo-binary}}} \left(\frac{D\ X_{*halo}}{f_b \Delta}\right) \tag{209}$$

using eq.(208) in eq.(209):-

$$j_* = 0.0021\ \left(M_1 M_2\right)^{3/37}\ M_{halo-binary}^{-52/74}\ \left(M_{1PBH}+M_{2PBH}\right)^{50/74}\ \rho_{eq}^{4/37}\ \frac{\left(f^2+1.2107\sigma_{eq}^2\right)^{8/37}}{\left(1+0.1709\lambda_0\right)^{12/37}}\ D^{16/37} \tag{210}$$

or

$$j_* = 0.0021\ \left(M_1 M_2\right)^{3/37}\ M_{halo-binary}^{-52/74}\ \left(M_{1PBH}+M_{2PBH}\right)^{50/74}\ \rho_{eq}^{4/37}\ \frac{\left(f^2+1.2107\sigma_{eq}^2\right)^{8/37}}{\left(1+0.1709\lambda_0\right)^{12/37}} \left(\frac{\left(1+0.0443\lambda_0\right)}{\sqrt{\left(1+0.1709\lambda_0\right)}}\right)^{16/37} \tag{211}$$

6.2.1 Probability Distribution for the semi-major axis of the binary with dark matter halos

The differential probability distribution for the the semi-major axis of the PBH binary with dark matter halos is given as:-

$$\frac{dP_{halo}}{da_{halo}} = \frac{dP_{halo}}{d\tilde{X}} \frac{d\tilde{X}}{da_{halo}} \tag{212}$$

Using $a_{halo} \approx \left(0.0977\lambda_0 + 0.0057\lambda_0^2\right) x$ and $\lambda_0 = \frac{8\pi\rho_{eq} x^3}{3(M_{1PBH}+M_{2PBH})}$; we have:-

$$x^4 = \left\{\frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0977+0.0057\lambda_0\right)}\right\} a_{halo} \tag{213}$$

which gives:-

$$\frac{dx}{da_{halo}} = \frac{1}{4}\left\{\frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0977+0.0057\lambda_0\right)}\right\}^{1/4} a_{halo}^{-3/4} \tag{214}$$

Now, as $X = \frac{x^3}{\bar{x}^3} \equiv \frac{x(a_{halo})^3}{\bar{x}^3}$; so we can write:-

$$\tilde{X} = \frac{X}{\mu} = \frac{1}{\mu} \frac{x(a_{halo})^3}{\bar{x}^3} \tag{215}$$

which gives:-

$$\frac{d\tilde{X}}{da_{halo}} = \frac{3}{\left(\mu \bar{x}^3\right)} x(a_{halo})^2 \frac{dx(a_{halo})}{da_{halo}} \tag{216}$$

using eqs.(213) & (214) in the above eq; we get:-

$$\frac{d\tilde{X}}{da_{halo}} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{\frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0977+0.0057\lambda_0\right)}\right\}^{3/4} a_{halo}^{-1/4} \tag{217}$$

Using $\frac{dP_{halo}}{d\tilde{X}} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3_{halo} n_T}$ and eq.(181) in eq.(176); we get:-

$$\frac{dP_{halo}}{da_{halo}} = \frac{dP_{halo}}{d\tilde{X}}\frac{d\tilde{X}}{da_{halo}} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\ \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{\frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0977+0.0057\lambda_0\right)}\right\}^{3/4} a_{halo}^{-1/4} \tag{218}$$

i.e.

$$\frac{dP_{halo}}{da_{halo}} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{\frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0977+0.0057\lambda_0\right)}\right\}^{3/4} a_{halo}^{-1/4}\ e^{-\frac{X}{\mu}\frac{4}{3}\langle x_{ij}\rangle^3 n_T} \tag{219}$$

using $\langle x_{ij}\rangle = \mu^{1/3}\,\bar{x}$ and $X = \frac{x(a_{halo})^3}{\bar{x}^3}$ in the above eq.; we get:-

$$\frac{dP_{halo}}{da_{halo}} = \frac{3}{4}\frac{1}{\left(\mu\bar{x}^3\right)}\left\{\frac{3\left(M_{1PBH}+M_{2PBH}\right)}{8\pi\rho_{eq}\left(0.0977+0.0057\lambda_0\right)}\right\}^{3/4} a_{halo}^{-1/4}\ e^{\frac{-4}{3}x(a_{halo})^3 n_T} \tag{220}$$

6.2.2 Probability Distribution for the eccentricity of the orbit of the PBH binary with dark matter halos

The differential probability distribution for the eccentricity of the orbit of the PBH binary with dark matter halos is given as:-

$$\frac{dP_{halo}}{de_{halo}} = \frac{dP_{halo}}{d\tilde{X}}\frac{d\tilde{X}}{de_{halo}} \tag{221}$$

for a given X hence a_{halo}, there is a unique j_{halo} such that the merger time is t_{halo}, so:-

$$j_{halo}(t;a_{halo}) = \left\{\frac{85}{3}\ \frac{G^3 M_1 M_2 M_{halo-binary}\ t}{a_{halo}^4}\right\}^{1/7} = \left(1-e_{halo}^2\right)^{1/2} \tag{222}$$

Using semi-major axis, $a_{halo} = \left(0.0977\lambda_0 + 0.0057\lambda_0^2\right)x$, $\lambda_0 = \frac{8\pi\rho_{eq}x^3}{3(M_{1PBH}+M_{2PBH})} = \frac{X}{f_b\Delta}$ & $X = \frac{x}{\bar{x}}$; we get:-

$$j_{halo}(t;X) = \left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_1 M_2 M_{halo-binary}\ t}{\left(0.0977+0.0057\lambda_0\right)^4\bar{x}^4 X^{16/3}}\right\}^{1/7} = \left(1-e_{halo}^2\right)^{1/2} \tag{223}$$

which gives:-

$$X = \left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{halo-binary}\ t}{\left(0.0977+0.0057\lambda_0\right)^4\bar{x}^4}\right\}^{3/16}\frac{1}{\left(1-e^2\right)^{21/32}} \tag{224}$$

from the above eq.; we get:-

$$\frac{dX}{de_{halo}} = \frac{21}{16}\left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{halo-binary}\ t}{\left(0.0977+0.0057\lambda_0\right)^4\bar{x}^4}\right\}^{3/16} \frac{e_{halo}}{\left(1-e_{halo}^2\right)^{53/32}} \tag{225}$$

or

$$\frac{d\tilde{X}}{de_{halo}} = \frac{1}{\mu}\frac{dX}{de_{halo}} = \frac{21}{16}\frac{1}{\mu}\left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{halo-binary}\ t}{\left(0.0977+0.0057\lambda_0\right)^4\bar{x}^4}\right\}^{3/16} \frac{e_{halo}}{\left(1-e_{halo}^2\right)^{53/32}} \tag{226}$$

Using $\frac{dP_{halo}}{d\tilde{X}} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}$ and eq.(226) in eq.(221); we get:-

$$\frac{dP_{halo}}{de_{halo}} = \frac{dP_{halo}}{d\tilde{X}}\frac{d\tilde{X}}{de_{halo}} = e^{-\tilde{X}\frac{4}{3}\langle x_{ij}\rangle^3 n_T}\frac{21}{16}\frac{1}{\mu}\left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{halo-binary}\ t}{\left(0.0965+0.0165\lambda_0\right)^4\bar{x}^4}\right\}^{3/16} \frac{e_{halo}}{\left(1-e_{halo}^2\right)^{53/32}} \tag{227}$$

using $\langle x_{ij}\rangle = \mu^{1/3}\,\bar{x}$ and $\tilde{X} = \frac{X}{\mu}$ in the above eq.; we get the differential probability distribution for the eccentricity of the PBH binary orbit as:-

$$\frac{dP_{halo}}{de_{halo}} = e^{-X\frac{4}{3}\bar{x}^3 n_T}\frac{21}{16}\frac{1}{\mu}\left\{\frac{85}{3}\ \frac{G^3\left(f_b\Delta\right)^4 M_{1PBH}M_{2PBH}M_{halo-binary}\ t}{\left(0.0965+0.0165\lambda_0\right)^4\bar{x}^4}\right\}^{3/16} \frac{e_{halo}}{\left(1-e_{halo}^2\right)^{53/32}} \tag{228}$$

References

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