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CSE 544, Spring 2021: Probability and Statistics for Data Science

Assignment 5: Hypothesis Testing Due: 4/20, 1:15pm, via Blackboard

(6 questions, 70 points total)

I/We understand and agree to the following:

(a) Academic dishonesty will result in an 'F' grade and referral to the Academic Judiciary.

(b) Late submission, beyond the 'due' date/time, will result in a score of 0 on this assignment.

(write down the name of all collaborating students on the line below)

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1. Hypothesis Testing for a single population **(Total 7 points)**

Consider the 10 samples: {2.78, 0.84, 1.88, 2.23, 1.99, 0.04, 2.65, 0.74, 1.19, 2.57}. Use the K-S test to check whether these samples are from the Uniform(0, 3) distribution. First, set up the hypotheses. Then, create a 10 X 6 table with entries: $[x, F_Y(x), \hat{F}_X^-(x), \hat{F}_X^+(x), |\hat{F}_X^-(x) - F_Y(x)|, |\hat{F}_X^+(x) - F_Y(x)|]$, where $\hat{F}_X^-(x)$ and $\hat{F}_X^+(x)$ are the values of the eCDF to the left and right of x, and $F_Y(x)$ is the CDF of Uniform(0, 3) at x; this is the same notation as in class. Finally, compare the max difference with the threshold of 0.25 to Reject/Accept. Show all rows and columns.

Ans. $D = \{2.78, 0.84, 1.88, 2.23, 1.99, 0.04, 2.65, 0.74, 1.19, 2.57\}$

y = Critical threshold, $c = 0.25$

$$F_{uniform(0,3)}(x) = \frac{x-0}{3-0} = \frac{x}{3}$$

$H_0 : F_D = F_y$ vs $H_1 : F_D \neq F_y$

x	$F_y(x)$	F_x^-	F_x^+	$\lvert \hat{F}_x^- - F_y(x)\rvert$	$\lvert F_x^+ - \hat{F}_y(x)\rvert$
0.04	0.0133	0.0	0.1	0.0133	0.0867
0.74	0.2466	0.1	0.2	0.1466	0.0466
0.84	0.28	0.2	0.3	0.08	0.02
1.19	0.3966	0.3	0.4	0.0966	0.0034
1.88	0.6266	0.4	0.5	$\boxed{0.2266}$	0.1266
1.99	0.6633	0.5	0.6	0.1633	0.0633
2.23	0.7433	0.6	0.7	0.1433	0.0433
2.57	0.8566	0.7	0.8	0.1566	0.0566
2.65	0.8833	0.8	0.9	0.0833	0.0167
2.78	0.9266	0.9	1	0.0266	0.0734

From the above table, we can see that maximum difference is 0.2266. Since $0.2266 < (c = 0.25)$. Hence we accept H_0. (Null Hypothesis)

2. Toy Example for Permutation Test (Total 5 points)

Let X = {5} and Y = {2, 7}. The null hypothesis is that X and Y are from the same distribution. Use the permutation test to decide this using a p-value threshold of 0.05. Please show all steps for each permutation clearly.

Ans. $H_0 : X \equiv Y$

$X = \{5\}$

$Y = \{2, 7\}$

the difference of Means for the given Dataset :

$T_{obs} = |4.5 - 5| = 0.5$

The difference of Means for 6 possible permutations are as follows :-

Sl. No	Permutation	$T_i : \lvert\bar{X} - \bar{Y}\rvert$
1.	$\{5\}\{2, 7\}$	0.5
2.	$\{5\}\{7, 2\}$	0.5
3.	$\{2\}\{5, 7\}$	4
4.	$\{2\}\{7, 5\}$	4
5.	$\{7\}\{2, 5\}$	3.5
6.	$\{7\}\{5, 2\}$	3.5

$$\text{Pvalue} = \frac{1}{N!}\sum_{i=1}^{N!} I\,(T_i > T_{obs})$$

$$= \frac{1}{3!} \times \left[0+0+1+1+1+1\right] = \frac{1}{6} \times 4 = \frac{2}{3}$$

$$= 0.66$$

As pvalue = 0.66 > 0.05 (threshold).
So we will reject Ho (Null Hypothesis).

3. Independence Tests to Save Your Casino (Total 15 points)

Being the owner of Casino 544, you are concerned that you are losing a lot of money because of the dealers at the blackjack tables. The Null hypothesis is that the outcome of the tables should be independent of the dealer, but you aren't sure.

(a) Validate your claim based on the dealer observations for a day, using the χ^2 test. Use α=0.05. You can use tools/online resources to find the CDF of χ^2; one such tool is https://www.danielsoper.com/statcalc/calculator.aspx?id=62. (10 points)

	Dealer A	Dealer B	Dealer C
Win	48	54	19
Draw	7	5	4
Loose	55	50	25

(b) You want to be more certain about the loyalty of your dealers, so you collect more data: number of wins from each dealer for 10 days. Find the Pearson correlation coefficient for each pair of dealers. What can you conclude? (5 points)

	Day-1	Day-2	Day-3	Day-4	Day-5	Day-6	Day-7	Day-8	Day-9	Day-10
Dealer A	48	40	58	53	65	25	52	34	30	45
Dealer B	54	48	51	47	62	35	70	20	25	40
Dealer C	19	40	35	41	38	32	32	37	37	15

3.(a) Given: Observed Values:

	Dealer A	Dealer B	Dealer C	Total
Win	48	54	19	121
Draw	7	5	4	16
Loose	55	50	25	130
Total	110	109	48	267

Expected Values:

	Dealer A	Dealer B	Dealer C
Win	$\frac{110}{267} \times 121 = 49.85$	$\frac{109}{267} \times 121 = 49.40$	$\frac{48}{267} \times 121 = 21.75$
Draw	$\frac{110}{267} \times 16 = 6.59$	$\frac{109}{267} \times 16 = 6.53$	$\frac{48}{267} \times 16 = 2.88$
Loose	$\frac{110}{267} \times 130 = 53.56$	$\frac{109}{267} \times 130 = 53.07$	$\frac{48}{267} \times 130 = 23.37$

Evaluating the χ^2 test:

$$Q_{obs} = \sum_r \sum_c \frac{(E_{rc} - O_{rc})^2}{E_{rc}}$$

$$= \frac{(E_{11} - O_{11})^2}{E_{11}} + \frac{(E_{12} - O_{12})^2}{E_{12}} + \frac{(E_{13} - O_{13})^2}{E_{13}}$$

$$+ \frac{(E_{21} - O_{21})^2}{E_{21}} + \frac{(E_{22} - O_{22})^2}{E_{22}} + \frac{(E_{23} - O_{23})^2}{E_{23}}$$

$$+ \frac{(E_{31} - O_{31})^2}{E_{31}} + \frac{(E_{32} - O_{32})^2}{E_{32}} + \frac{(E_{33} - O_{33})^2}{E_{33}}$$

$$\left.\begin{aligned} &= \frac{(49.85 - 48)^2}{49.85} + \frac{(6.59 - 7)^2}{6.59} + \frac{(53.56 - 55)^2}{53.56} \\ &+ \frac{(49.40 - 54)^2}{49.40} + \frac{(6.53 - 5)^2}{6.53} + \frac{(53.07 - 50)^2}{53.07} \\ &+ \frac{(21.75 - 19)^2}{21.75} + \frac{(2.88 - 4)^2}{2.88} + \frac{(23.37 - 25)^2}{48} \end{aligned}\right\} \text{putting values from expected \& observed tables}$$

$$= 0.0687 + 0.0255 + 0.0387 + 0.4283 + 0.3584 + 0.43556 + 0.1137 + 0.1776 + 0.3477$$

$\therefore\ Q_{obs} = 1.9942$

degrees of freedom (df) = (# datarows − 1) (# data column − 1)
$= (3-1) \times (3-1) = 2 \times 2 = 4$

$$\text{p-value} = Pr(\chi^2_4 > 1.9942) = 1 - 0.263 = 0.737$$

As p-value (0.737) > 0.05, we fail to reject H_0

3(b) To find the Pearson correlation coefficient:

$$S_{xy} = \frac{\sum_{i=1}^{n} \{ (X_i - \bar{X})(Y_i - \bar{Y}) \}}{\sqrt{\left(\sum_{i=1}^{n} (X_i - \bar{X})^2 \right) \left(\sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)}}$$

Dealer A : $\bar{A} = \sum_{i=1}^{n} X_i \times \frac{1}{n} = \frac{450}{10} = 45$

Dealer B : $\bar{B} = \sum_{i=1}^{n} X_i \times \frac{1}{n} = \frac{452}{10} = 45.2$

Dealer C : $\bar{C} = \sum_{i=1}^{n} X_i \times \frac{1}{n} = \frac{326}{10} = 32.6$

Day	A	$(A-\bar{A})$	$(A-\bar{A})^2$	B	$(B-\bar{B})$	$(B-\bar{B})^2$	C	$(C-\bar{C})$	$(C-\bar{C})^2$
1	48	3	9	54	8.8	77.44	19	-13.6	184.96
2	40	-5	25	48	2.8	7.84	40	7.4	54.76
3	58	13	169	51	5.8	33.64	35	2.4	5.76
4	53	8	64	47	1.8	3.24	41	8.4	70.56
5	65	20	400	62	16.8	282.24	38	5.4	29.16
6	25	-20	400	35	-10.2	104.24	32	-0.6	0.36
7	52	7	49	70	24.8	615.04	32	-0.6	0.36
8	34	-11	121	20	-25.2	635.04	37	4.4	19.36
9	30	-15	225	25	-20.2	408.04	37	4.4	19.36
10	45	0	0	40	-5.2	27.04	15	-17.6	309.76
Total	450		1462	452		2193.8	326		694.4

(i) Finding correlation coefficient between Dealer A and Dealer B:

$$S_{AB} = \frac{\sum_{i=1}^{n} \{ (A_i - \bar{A})(B_i - \bar{B}) \}}{\sqrt{\left(\sum_{i=1}^{n} (A_i - \bar{A})^2 \right) \left(\sum_{i=1}^{n} (B_i - \bar{B})^2 \right)}}$$

From the table;

$$\sum_{i=1}^{n} (A-\bar{A})(B-\bar{B}) = 1396 \quad \text{--- (1)}$$

$$\sum_{i=1}^{n} (B-\bar{B})(C-\bar{C}) = -96.20 \quad \text{--- (2)}$$

$$\sum_{i=1}^{n} (A-\bar{A})(C-\bar{C}) = 22.0 \quad \text{--- (3)}$$

$$\sqrt{\sum_{i=1}^{n} (A-\bar{A})^2} = 38.2361 \quad \text{--- (4)}$$

$$\sqrt{\sum_{i=1}^{n} (B-\bar{B})^2} = 46.836 \quad \text{--- (5)}$$

$$\sqrt{\sum_{i=1}^{n} (C-\bar{C})^2} = 26.35 \quad \text{--- (6)}$$

(i) $$\rho_{AB} = \frac{1396}{38.2361 \times 46.836} = 0.7795 \quad \text{--- (7)}$$ [Putting values from (1), (4) and (5)]

(ii) Finding correlation coeffecient between Dealer B and Dealer C:

$$\rho_{BC} = \frac{\sum_{i=1}^{n} \{(B_i-\bar{B})(C_i-\bar{C})\}}{\sqrt{\left(\sum_{i=1}^{n} (B_i-\bar{B})^2\right)\left(\sum_{i=1}^{n} (C_i-\bar{C})^2\right)}}$$

$$= \frac{-96.20}{46.836 \times 26.35} = -0.0779 \quad \text{--- (8)}$$ [Putting values from (2), (5) and (6)]

(iii) Finding correlation coeffecient between Dealer A and Dealer C:

$$\rho_{AC} = \frac{\sum_{i=1}^{n} \{(A_i - \bar{A})(C_i - \bar{C})\}}{\sqrt{\left(\sum_{i=1}^{n} (A_i - \bar{A})^2\right)\left(\sum_{i=1}^{n} (C_i - \bar{C})^2\right)}}$$

$$= \frac{22.0}{38.2361 \times 26.35} = 0.0218$$ [Putting values from (3), (4) and (6)]

(9)

From (7): $\rho_{A,B} = 0.7795\ (> 0.5) \rightarrow$ Positive linear correlation

From (8): $\rho_{BC} = |-0.0779|\ (< 0.5) \rightarrow$ No linear correlation

From (9): $\rho_{AC} = |0.0218|\ (< 0.5) \rightarrow$ No linear correlation

From the results above, we can conclude that Dealer A is positively correlated with B. However Dealer C is not correlated with Dealer A and Dealer B.

5. Type-1 and Type-2 error for one-sided unpaired T-test (Total 10 points)

Let $\{X_1, X_2, \dots, X_n\}$ be i.i.d. from Normal(μ_1, σ_1^2) and $\{Y_1, Y_2, \dots, Y_m\}$ be i.i.d. from Normal(μ_2, σ_2^2). Also suppose X's and Y's are independent, and $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ are unknown. Let S_x and S_y be the sample standard deviations of the two populations. Assume that n and m are large. Let H_0: $\mu_1 > \mu_2$ be the null hypothesis and H_1: $\mu_1 <= \mu_2$ be the alternate hypothesis. Consider the T statistic for the unpaired T test, as in class, with $\delta > 0$ being the critical value.

(a) For the above test, show that the probability of Type-1 and Type-2 errors are given by $\Phi(-\delta - \frac{\mu_1-\mu_2}{\sqrt{\frac{Sx^2}{n}+\frac{Sy^2}{m}}})$ and $1-\Phi(-\delta - \frac{\mu_1-\mu_2}{\sqrt{\frac{Sx^2}{n}+\frac{Sy^2}{m}}})$, respectively. (5 points)

(b) Show that the p-value is given by $\Phi\left(\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{Sx^2}{n}+\frac{Sy^2}{m}}}\right)$. (5 points)

Given:

$$X_1, X_2 \dots X_n \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$$

$$Y_1, Y_2 \dots Y_m \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$$

also $X's \perp Y's$.

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma_1^2}{n}\right)$$

$$\bar{Y} \sim N\left(\mu_2, \frac{\sigma_2^2}{m}\right)$$

$$\bar{X}-\bar{Y} \sim N\left(\mu_1-\mu_2, \frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}\right) \quad [\text{Since } X's \perp Y's]$$

$$T = \frac{\bar{X}-\bar{Y}}{\sqrt{\frac{Sx^2}{n}+\frac{Sy^2}{m}}}$$

(a) According to question:

$H_0: \mu_1 > \mu_2, \quad H_1: \mu_1 \le \mu_2$

Type 1 error: $Pr\,(\text{Test reject } H_0 \mid H_0 \text{ true})$

$$\Rightarrow P(T < -\delta \mid H_0 \text{ true})$$

$$= P\left(\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} < -\delta \;\middle|\; H_0 \text{ true} \right)$$

$$= P\left(\frac{\bar{X}-\bar{Y} - (\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} < -\delta - \frac{\mu_1-\mu_2}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \right)$$

[Subtracting $\frac{\mu_1-\mu_2}{\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}}$ from both sides]

Since n and m are very large, S_x is a consistent estimator of σ_x and S_y is a consistent estimator of σ_y.

$$\frac{\bar{X}-\bar{Y} - (\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \sim T \sim N(0,1)$$

From above: Type 1 error: $P\left(T < -\delta - \frac{(\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \right)$

$$= \Phi\left(-\delta - \frac{\mu_1-\mu_2}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \right) \text{ proved!}$$

Type 2 error: $Pr\,(\text{Test accept } H_0 \mid H_0 \text{ false})$

$$= P(T > -\delta \mid H_0 \text{ false})$$

$$= P\left(\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} > -\delta \;\middle|\; H_0 \text{ false} \right)$$

$$= 1 - P\left(\frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} < -\delta \right)$$

$$= 1 - P\left(\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} < -\delta - \frac{\mu_1 - \mu_2}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \right)$$

[Subtracting $\frac{\mu_1 - \mu_2}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}}$ from both sides]

Since n and m are very large, S_x is a consistent estimator of σ_x and S_y is a consistent estimator of σ_y.

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \sim T \sim N(0, 1)$$

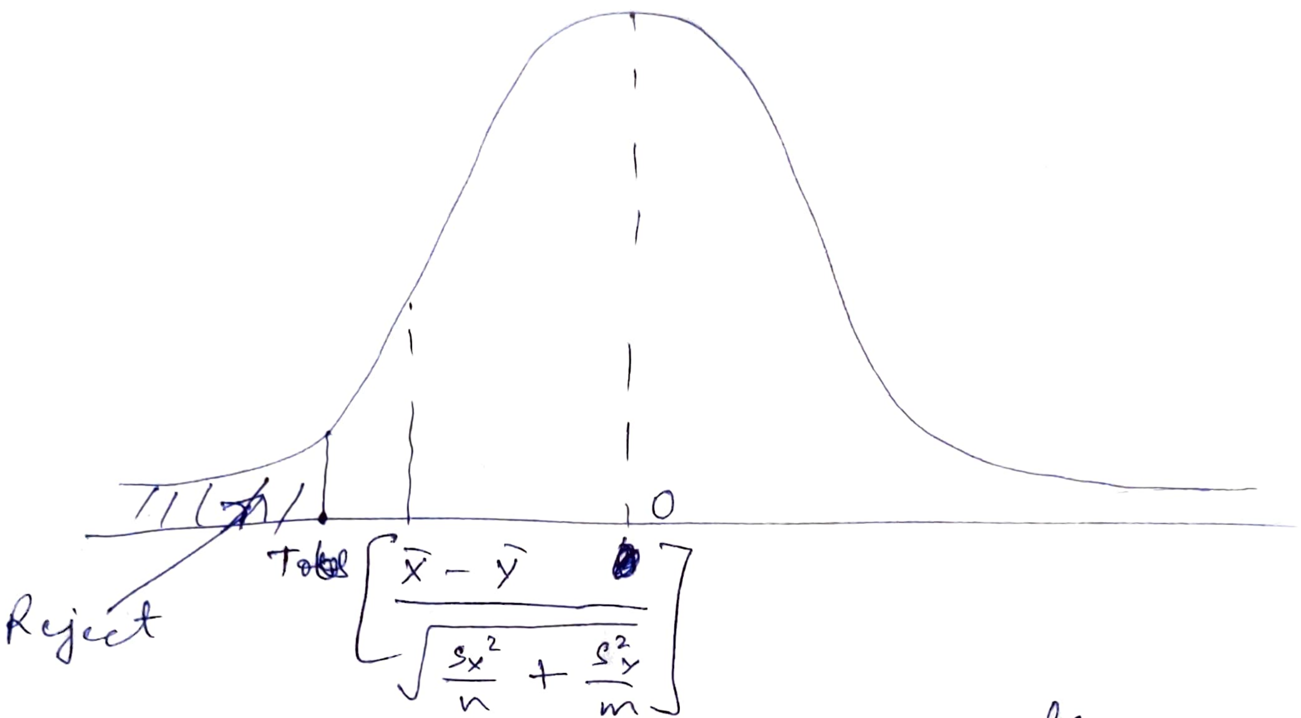
From above: Type 2 error $= 1 - P\left(T < -\delta - \frac{\mu_1 - \mu_2}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \right)$

$$= 1 - \Phi\left(-\delta - \frac{\mu_1 - \mu_2}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \right)$$

proved!

~~(b)~~

1(b) Ans. The Normal distribution graph is as follows:-



Ho is rejected if Tobs lies all the way to the left in the above graph.

p value = Area to the left of Tobs = ϕ (Tobs)

$$T_{obs} = \frac{\bar{x}-\bar{y}}{\sqrt{\frac{s_x^2}{n}+\frac{s_y^2}{m}}}$$ be the statistic that is observed.

$$p\text{-value} = P(T < t_{obs}) = P\left(\frac{\bar{x}-\bar{y}}{\sqrt{\frac{s_x^2}{n}+\frac{s_y^2}{m}}}\right)$$

On subtracting $(\mu_1 - \mu_2)$ from both side of the above inequality, we get:-

$$P\left(\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}} < \frac{t_{obs}-(\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}}\right)$$

$$= \Phi\left(t_{obs} - \frac{(\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}}\right)$$

$$= \Phi\left(\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{S_x^2}{n}+\frac{S_y^2}{m}}}\right)$$

{ Hence Proved }