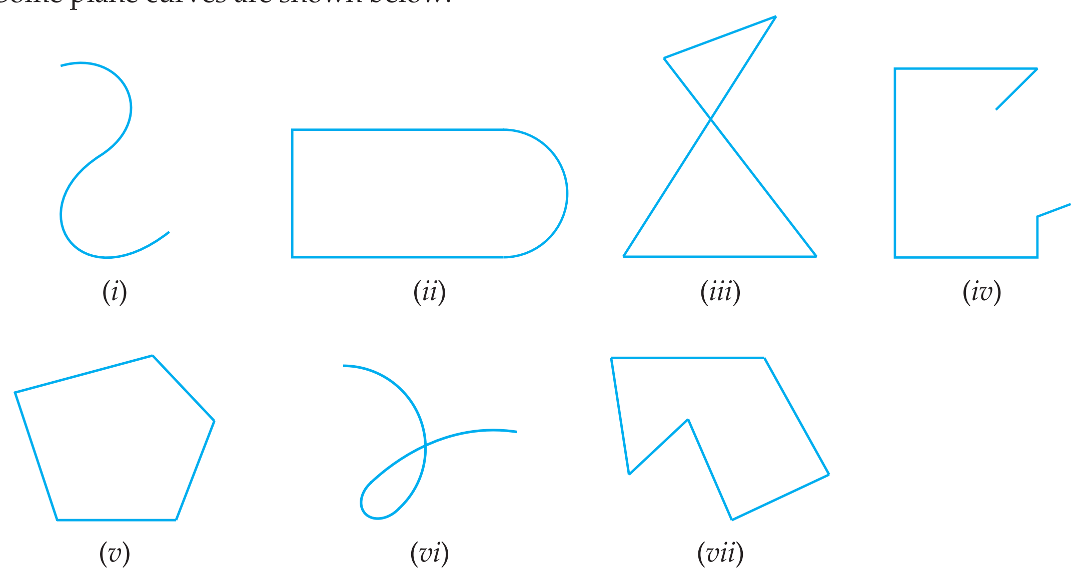
12 Rectilinear Figures

INTRODUCTION

If we put the sharp tip of a pencil on a sheet of paper and move from one point to the other, without lifting the pencil, then the shapes so formed are called **plane curves**.

Some plane curves are shown below:



The (plane) curves which have different beginning and end points are called **open curves** and the curves which have same beginning and end points are called **closed curves.** In the above figure, (*i*), (*iv*) and (*vi*) are open curves where as (*ii*), (*iii*), (*v*) and (*vii*) are closed curves.

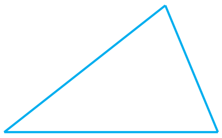
A curve which does not cross itself at any point is called a **simple curve**. In the above figure, (*i*), (*ii*), (*iv*), (*v*) and (*vii*) are all simple curves. Note that (*ii*), (*v*) and (*vii*) are **simple closed curves**. A simple closed plane curve made up entirely of line segments is called a **polygon**. In the above figure, (*v*) and (*vii*) are polygons.

In this chapter, we shall study about different kinds of polygons (parallelograms, rectangles, rhombuses, squares, kites) and their various properties. We shall also construct some quadrilaterals and regular hexagons by using ruler and compass.

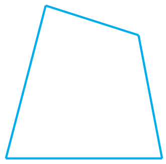
12.1 RECTILINEAR FIGURES

A plane figure made up entirely of line segments is called a rectilinear figure.

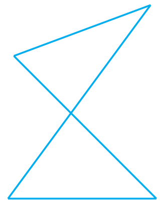
Look at the following plane figures:



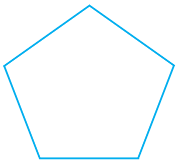
(*i*)



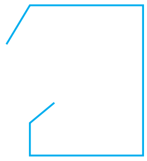
(*ii*)



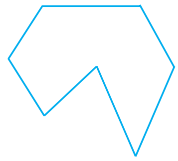
(*iii*)



(*iv*)



(*v*)



(*vi*)

All the six figures are made up entirely of line segments, so these are all rectilinear figures.

❐ Polygon

*A **polygon** is a simple closed rectilinear figure i.e. a **polygon** is a simple closed plane figure made up entirely of line segments.*

The figures (shown above) (*i*), (*ii*), (*iv*) and (*vi*) are polygons where as figures (*iii*) and (*v*) are not polygons since figure (*iii*) is not simple and figure (*v*) is not closed.

Thus, every polygon is a rectilinear figure but every rectilinear figure is not a polygon.

The line segments are called its **sides** and the points of intersection of consecutive sides are called its **vertices**. An angle formed by two consecutive sides of a polygon is called an **interior angle** or simply an **angle** of the polygon.

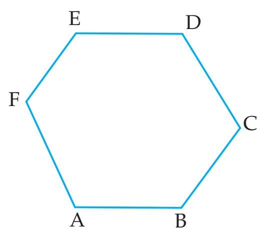
A polygon is named according to the number of sides it has.

No. of sides	3	4	5	6	7	8	10
Name	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon	Octagon	Decagon

Diagonal of a polygon. *Line segment joining any two non-consecutive vertices of a polygon is called its **diagonal**.*

Convex polygon. *If all the (interior) angles of a polygon are less than 180°, it is called a* **convex polygon.**

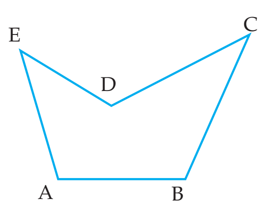
In the adjoining figure, ABCDEF is a convex polygon. In fact, it is a convex hexagon.



Convex polygon

Concave polygon. *If one or more of the (interior) angles of a polygon is greater than 180° i.e. reflex, it is called* **concave** (or **re-entrant**) polygon.

In the adjoining figure, ABCDEFG is a concave polygon. In fact, it is a concave pentagon.

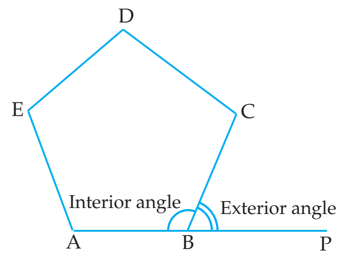


Concave polygon

However, we shall be dealing with convex polygons only.

❐ Exterior angle of convex polygon

If we produce a side of a polygon, the angle it makes with the next side is called an **exterior angle**.



In the adjoining figure, ABCDE is a pentagon. Its side AB has been produced to P, then $\angle$CBP is an exterior angle.

Note that corresponding to each interior angle, there is an exterior angle. Also, as an exterior angle and its adjacent interior angle make a line, so we have :

an exterior angle + adjacent interior angle = 180°.

Regular polygon. *A polygon is called* **regular polygon** *if all its sides have equal length and all its angles have equal size.*

Thus, in a regular polygon:

(*i*) *all sides are equal in length*

(*ii*) *all interior angles are equal in size*

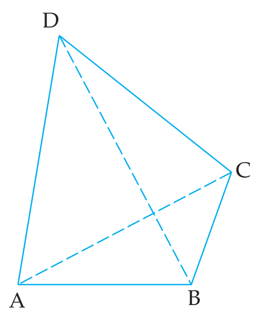
(*iii*) *all exterior angles are equal in size.*

All regular polygons are convex. All equilateral triangles and all squares are regular polygons.

12.2 QUADRILATERALS

A simple closed plane figure bounded by four line segments is called a ***quadrilateral****.*

In the adjoining figure, ABCD is a quadrilateral.



It has

four sides — AB, BC, CD and DA

four (interior) angles — $\angle$A, $\angle$B, $\angle$C and $\angle$D

four vertices — A, B, C and D

two diagonals — AC and BD.

In quadrilateral ABCD, sides AB, BC; BC, CD; CD, DA ; DA, AB are pairs of **adjacent sides**.

Sides AB, CD; BC, DA are pairs of **opposite sides**.

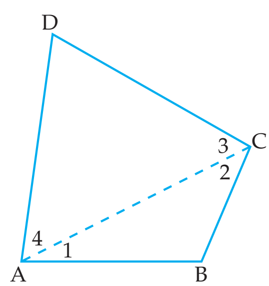
Angles $\angle$A, $\angle$B; $\angle$B, $\angle$C; $\angle$C, $\angle$D; $\angle$D, $\angle$A are pairs of **adjacent angles**.

Angles $\angle$A, $\angle$C; $\angle$B, $\angle$D are pairs of **opposite angles**.

❑ Angle sum property of a quadrilateral

Sum of (interior) angles of a quadrilateral is 360°.

In an adjoining figure, ABCD is *any* quadrilateral. Diagonal AC divides it inot two triangles. We know that the sum of angles of a triangle is 180°,



in ΔABC, $\angle 1 + \angle B + \angle 2 = 180°$ ...(*i*)

in ΔACD, $\angle 4 + \angle D + \angle 3 = 180°$ ...(*ii*)

On adding (i) and (ii), we get

$\angle 1 + \angle 4 + \angle B + \angle D + \angle 2 + \angle 3 = 360°$

$\angle A + \angle B + \angle D + \angle C = 360°$ (from figure)

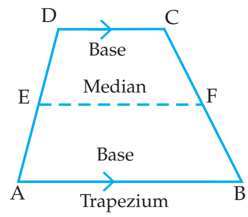
Hence, the sum of (interior) angles of a quadrilateral is 360°.

❑ Types of quadrilaterals

1. Trapezium

A quadrilateral in which one pair of opposite sides is parallel is called a ***trapezium*** *(abbreviated trap.)*

The parallel sides are called **bases** of the trapezium. The line segment joining mid-points of non-parallel sides is called its **median.**



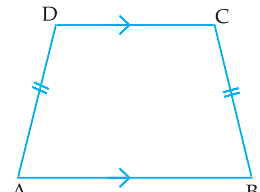
Trapezium

In the adjoining quadrilateral, AB || DC whereas AD and BC are non-parallel, so ABCD is a trapezium, AB and CD are its *bases*, and EF is its *median* where E, F are mid-points of the sides AD, BC respectively.

Isosceles trapezium

If non-parallel sides of a trapezium are equal, then it is called an ***isosceles trapezium.***

Here AB || DC, AD and BC are non-parallel and AD = BC, so ABCD is an isosceles trapezium.

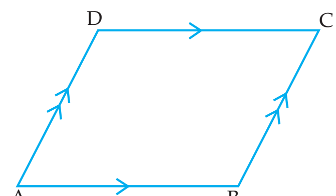


Isosceles trapezium

2. Parallelogram

A quadrilateral in which both pairs of opposite sides are parallel is called a ***parallelogram****. It is usually written as '||gm'.*

In the adjoining quadrilateral, AB || DC and AD || BC, so ABCD is a parallelogram.

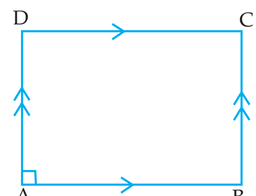


Parallelogram

3. Rectangle

If one of the angles of a parallelogram is a right angle, then it is called a ***rectangle****.*

In the adjoining parallelogram, $\angle A = 90°$, so ABCD is a rectangle. Of course, the remaining angles will also be right angles.



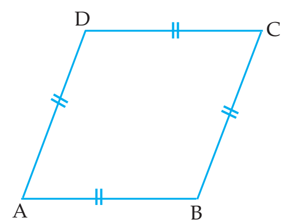
Rectangle

4. Rhombus

If all the sides of a quadrilateral are equal, then it is called a ***rhombus.***

In the adjoining quadrilateral, AB = BC = CD = DA, so ABCD is a rhombus.

[Every rhombus is a parallelogram, see corollary to theorem 12.3]

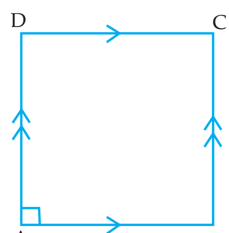


5. Square

If two adjacent sides of a rectangle are equal, then it is called a ***square****. Alternatively, if one angle of a rhombus is a right angle, it is called a* ***square****.*

In the adjoining rectangle, AB = AD, so ABCD is a square.

Of course, the remaining sides are also equal.

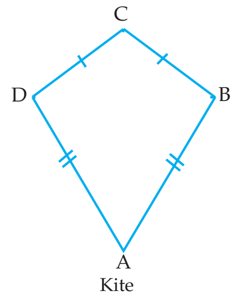


Square

6. Kite

A quadrilateral in which two pairs of adjacent sides are equal is called a ***kite*** *(or* ***diamond****).*

In the adjoining quadrilateral, AD = AB and DC = BC, so ABCD is a kite.



Kite

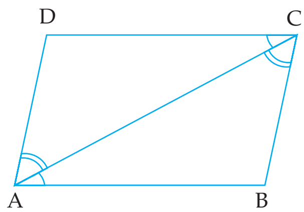
Remark

From the above definitions it follows that parallelograms include rectangles, squares and rhombi (plural of rhombus), therefore, any result which is true for a parallelogram is certainly true for all these figures.

12.2.1 Properties of parallelograms

Theorem 12.1

A diagonal of a parallelogram divides it into two congruent triangles.



Given. A parallelogram ABCD and diagonal AC divides it into two triangles ΔABC and ΔCDA.

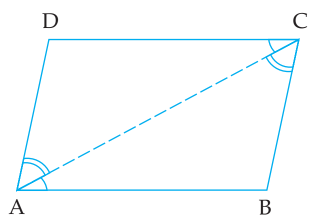
To prove. ΔABC ≅ ΔCDA.

Proof.

Statements	*Reasons*
In ΔABC and ΔCDA	
1. ∠BAC = ∠ACD	1. Alt. ∠s, AB || DC and AC is transversal.
2. ∠BCA = ∠CAD	2. Alt. ∠s, BC || AD and AC is transversal.
3. AC = CA	3. Common
4. ΔABC ≅ ΔCDA	4. ASA rule of congruency.

Theorem 12.2

In a parallelogram, opposite sides are equal.



Given. A parallelogram ABCD.

To prove. AB = DC and BC = AD.

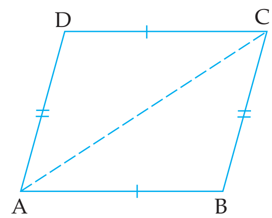
Construction. Join AC.

Proof.

Statements	*Reasons*
In ΔABC and ΔCDA	
1. ∠BAC = ∠ACD	1. Alt. ∠s, AB || DC and AC is transversal.
2. ∠BCA = ∠CAD	2. Alt. ∠s, BC || AD and AC is transversal.
3. AC = CA	3. Common
4. ΔABC ≅ ΔCDA	4. ASA rule of congruency.
5. AB = DC and BC = AD	5. c.p.c.t.

The converse of the above theorem is also true. In fact we have:

Theorem 12.3



If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.

Given. A quadrilateral ABCD in which AB = DC and BC = AD.

To prove. ABCD is a parallelogram.

Construction. Join AC.

Proof.

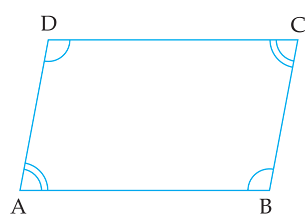
Statements	*Reasons*
In ΔABC and ΔCDA	
1. AB = DC	1. Given.
2. BC = AD	2. Given.
3. AC = AC	3. Common.
4. ΔABC ≅ ΔCDA	4. SSS rule of congruency
5. ∠BAC = ∠ACD ⇒ AB ∥ DC	5. c.p.c.t. alt. ∠s are equal formed by lines AB, DC and transversal AC.
6. ∠ACB = ∠CAD ⇒ BC ∥ AD	6. c.p.c.t. alt. ∠s are equal formed by lines AD, BC and transversal AC.

Hence, ABCD is a parallelogram.

Corollary. *Every rhombus is a parallelogram.*

[In a rhombus, all sides are equal, so opposite sides are equal. Therefore, every rhombus is a parallelogram.]

Theorem 12.4



In a parallelogram, opposite angles are equal.

Given. A parallelogram ABCD.

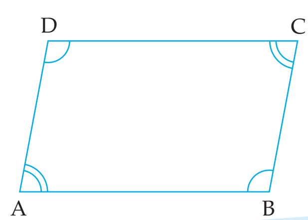
To prove. ∠A = ∠C and ∠B = ∠D.

Proof.

Statements	*Reasons*
1. ∠A + ∠B = 180°	1. AD ∥ BC and AB is a transversal, sum of co-interior angles = 180°
2. ∠B + ∠C = 180°	2. AB ∥ DC and BC is a transversal, sum of co-interior angles = 180°
3. ∠A + ∠B = ∠B + ∠C ⇒ ∠A = ∠C Similarly, ∠B = ∠D.	3. From 1 and 2

The converse of the above theorem is also true. In fact, we have:

Theorem 12.5



If each pair of opposite angles of a quadrilateral is equal, then it is a parallelogram.

Given. A quadrilateral ABCD in which ∠A = ∠C and ∠B = ∠D.

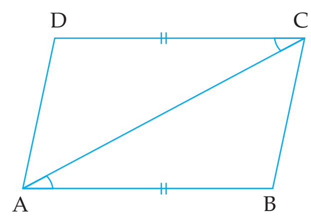
To prove. ABCD is a parallelogram.

Proof.

Statements	*Reasons*
1. $\angle A = \angle C$	1. Given
2. $\angle B = \angle D$	2. Given
3. $\angle A + \angle B = \angle C + \angle D$	3. Adding 1 and 2
4. $\angle A + \angle B + \angle C + \angle D = 360°$	4. Sum of angles of a quadrilateral.
5. $2(\angle A + \angle B) = 360°$ $\Rightarrow \angle A + \angle B = 180°$ $\Rightarrow BC \parallel AD$	5. Using 3 Sum of co-interior angles = 180°, formed by lines BC, AD and transversal AB.
Similarly, $AB \parallel DC$ Hence, ABCD is a parallelogram.	

Theorem 12.6

If a pair of opposite sides of a quadrilateral is equal and parallel, then it is a parallelogram.



Given. A quadrilateral ABCD in which $AB \parallel DC$ and $AB = DC$.

To prove. ABCD is a parallelogram.

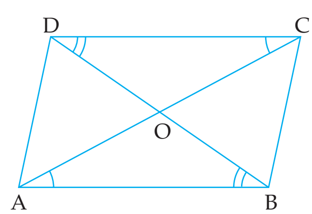
Construction. Join AC.

Proof.

Statements	*Reasons*
In ΔABC and ΔCDA	
1. $\angle BAC = \angle ACD$	1. Alt. $\angle$s, $AB \parallel DC$ and AC is a transversal.
2. $AB = DC$	2. Given
3. $AC = CA$	3. Common
4. $\Delta ABC \cong \Delta CDA$	4. SAS rule of congruency.
5. $\angle ACB = \angle CAD$ $\Rightarrow AD \parallel BC$ Hence, ABCD is a parallelogram.	5. c.p.c.t. Alt. $\angle$s are equal formed by lines AD, BC and transversal AC.

Theorem 12.7

The diagonals of a parallelogram bisect each other.



Given. A parallelogram ABCD whose diagonals AC and BD intersect at O.

To prove. $OA = OC$ and $OB = OD$.

Proof.

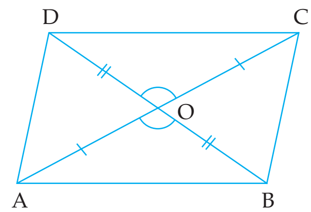
Statements	*Reasons*
In ΔOAB and ΔOCD	
1. $\angle BAC = \angle ACD$	1. Alt. $\angle$s, $AB \parallel DC$ and AC is a transversal.

2. $\angle ABD = \angle CDB$	2. Alt. $\angle$s, AB || DC and BD is a transversal.
3. AB = CD	3. Opp. sides of a parallelogram are equal.
4. $\Delta OAB \cong \Delta OCD$ $\therefore$ OA = OC and OB = OD	4. ASA rule of congruency c.p.c.t.

The converse of the above theorem is also true. In fact, we have:

Theorem 12.8

If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.



Given. A quadrilateral ABCD whose diagonals AC and BD intersect at O such that OA = OC and OB = OD.

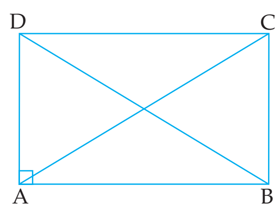
To prove. ABCD is a parallelogram.

Proof.

Statements	*Reasons*
In ΔOAB and ΔOCD	
1. OA = OC	1. Given
2. OB = OD	2. Given
3. $\angle AOB = \angle COD$	3. Vert. opp. $\angle s$
4. $\Delta OAB \cong \Delta OCD$	4. SAS rule of congruency
5. $\angle OAB = \angle OCD$ $\Rightarrow \angle CAB = \angle ACD$ $\Rightarrow$ AB || DC	5. c.p.c.t. Alt. $\angle$s are equal formed by lines AB, DC and transversal AC.
6. AB = CD	6. c.p.c.t.
7. ABCD is a parallelogram.	7. In quadrilateral ABCD, AB || DC and AB = DC (Theorem 12.6)

❑ Properties of a rectangle

Since every rectangle is a parallelogram, therefore, it has all the properties of a parallelogram. Additional properties of a rectangle are:



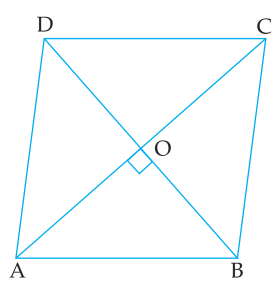
- *All the (interior) angles of a rectangle are right angles.*

  In the adjoining figure, $\angle A = \angle B = \angle C = \angle D = 90°$.
- *The diagonals of a rectangle are equal.*

  In the adjoining figure, AC = BD.

❑ Properties of a rhombus

Since every rhombus is a parallelogram, therefore, it has all the properties of a parallelogram. Additional properties of a rhombus are:



- *All the sides of a rhombus are equal.*

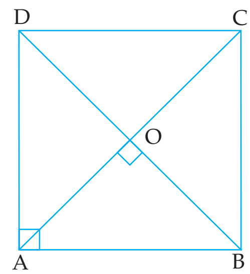
  In the adjoining figure, AB = BC = CD = DA.
- *The diagonals of a rhombus intersect at right angles.*

  In the adjoining figure, $AC \perp BD$.

- *The diagonals bisect the angles of a rhombus.*
  In the adjoining figure, diagonal AC bisects ∠A as well as ∠C and diagonal BD bisects ∠B as well as ∠D.

❑ Properties of a square

Since every square is a parallelogram, therefore, it has all the properties of a parallelogram. Additional properties of a square are:



- *All the interior angles of a square are right angles.*
  In the adjoining figure, ∠A = ∠B = ∠C = ∠D = 90°.
- *All the sides of a square are equal.*
  In the adjoining figure, AB = BC = CD = DA.
- *The diagonals of a square are equal.*
  In the adjoining figure, AC = BD.
- *The diagonals of a square intersect at right angles.*
  In the adjoining figure, AC ⊥ BD.
- *The diagonals bisect the angles of a square.*
  In the adjoining figure, diagonal AC bisects ∠A as well as ∠C and diagonal BD bisects ∠B as well as ∠D.

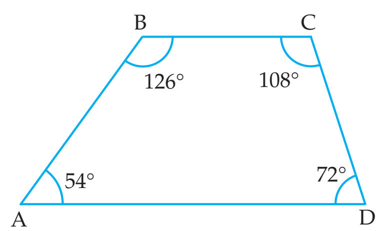
In fact, a square is a rectangle as well as a rhombus, so it has all the properties of a rectangle as well as that of a rhombus.

Illustrative Examples

Example 1. *If angles A, B, C and D of a quadrilateral ABCD, taken in order, are in the ratio 3 : 7 : 6 : 4 then ABCD is a trapezium. Is this statement true? Give reason for your answer.*

Solution. As the angles are in the ratio 3 : 7 : 6 : 4,

let these angles be $3x$, $7x$, $6x$ and $4x$.



Since sum of angles of a quadrilateral is 360°,

$$3x + 7x + 6x + 4x = 360°$$

$$\Rightarrow \quad 20x = 360° \Rightarrow x = 18°.$$

∴ The angles are: $\angle A = 3 \times 18° = 54°$,

$\angle B = 7 \times 18° = 126°$, $\angle C = 6 \times 18° = 108°$ and

$\angle D = 4 \times 18° = 72°$.

We note that $\angle A + \angle B = 54° + 126° = 180°$.

Thus, the sum of co-interior angles is 180° formed by lines AD, BC and transversal AB, therefore, AD || BC. So, ABCD is a trapezium. Hence, the given statement is true.

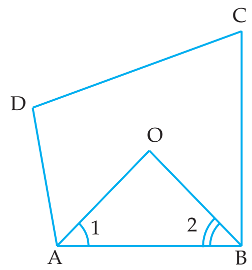
Example 2. *Three angles of a quadrilateral ABCD are equal. Is it a parallelogram? Why or why not?*

Solution. It need not be a parallelogram; because we may have $\angle A = \angle B = \angle C = 80°$, then $\angle D = 360° - 3 \times 80° = 120°$, so $\angle B \neq \angle D$ (opposite angles are not equal).

Example 3. *In a quadrilateral ABCD, AO and BO are the bisectors of ∠A and ∠B respectively. Prove that* $\angle AOB = \frac{1}{2}(\angle C + \angle D)$.

Solution. Given ABCD is a quadrilateral, OA and OB are the bisectors of ∠A and ∠B respectively. Mark the angles as shown in the figure given below.

As OA and OB are bisectors of $\angle A$ and $\angle B$ respectively,



$$\angle 1 = \frac{1}{2}\angle A \text{ and } \angle 2 = \frac{1}{2}\angle B \qquad \ldots(i)$$

$\angle AOB + \angle 1 + \angle 2 = 180°$

(sum of angles in ΔOAB)

$\Rightarrow \quad \angle AOB = 180° - (\angle 1 + \angle 2)$

$\Rightarrow \quad \angle AOB = 180° - \frac{1}{2}(\angle A + \angle B) \qquad$ (using (i))

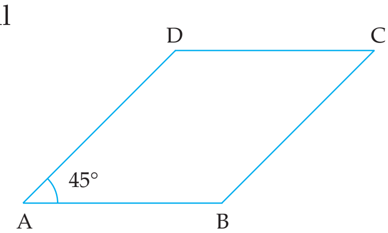
$\Rightarrow \quad \angle AOB = 180° - \frac{1}{2}(360° - (\angle C + \angle D))$

[$\because$ sum of angles in a quadrilateral is 360°, so $\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow \angle A + \angle B = 360° - (\angle C + \angle D)$]

$\Rightarrow \quad \angle AOB = \frac{1}{2}(\angle C + \angle D)$.

Example 4. *Diagonals of a quadrilateral ABCD bisect each other. If $\angle A = 45°$, determine $\angle B$.*



Solution. Since the diagonals AC and BD of quadrilateral ABCD bisect each other, ABCD is a parallelogram.

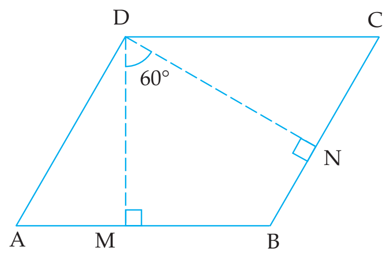
AD || BC and AB is a transversal,

$\angle A + \angle B = 180°$ (sum of co-interior angle = 180°)

$\Rightarrow \quad 45° + \angle B = 180°$

$\Rightarrow \quad \angle B = 135°$.

Example 5. *The angle between two altitudes of a parallelogram through the vertex of an obtuse angle of the parallelogram is 60°. Find the angles of the parallelogram.*



Solution. Let ABCD be parallelogram such that

$DM \perp AB$, $DN \perp BC$ and $\angle MDN = 60°$.

In quadrilateral DMBN,

$\angle MDN + \angle M + \angle N + \angle B = 360°$

$\Rightarrow \quad 60° + 90° + 90° + \angle B = 360°$

$\Rightarrow \quad \angle B = 120°$.

AD || BC and AB is a transversal,

$\angle A + \angle B = 180°$

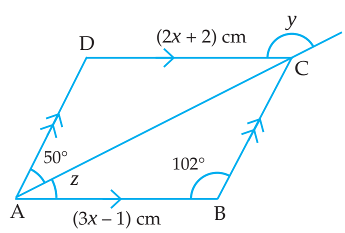
$\Rightarrow \quad \angle A + 120° = 180° \Rightarrow \angle A = 60°$.

$\angle C = \angle A$ and $\angle D = \angle B$ (opp. $\angle$s of a || gm are equal)

$\Rightarrow \quad \angle C = 60°$ and $\angle D = 120°$.

Hence, the angles of || gm ABCD are 60°, 120°, 60°, 120°.

Example 6. *In the adjoining figure, ABCD is parallelogram. Find the values of x, y and z.*



Solution. Given ABCD is a parallelogram,

$3x - 1 = 2x + 2$ (opp. sides are equal)

$\Rightarrow \quad x = 3$.

$\angle D = \angle B = 102°$ (opp. $\angle$s are equal)

For ΔACD, $y = 50° + \angle D$ ($\because$ ext. $\angle$ = sum of two int. opp. $\angle$s)

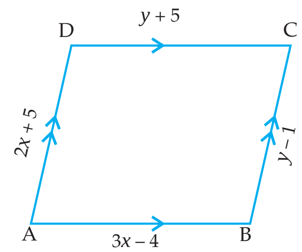
$\Rightarrow \quad y = 50° + 102° = 152°$

$\angle DAB + 102° = 180°$ (AD || BC, sum of co-int. $\angle$s = 180°)

$\Rightarrow \quad \angle DAB = 180° - 102° = 78°$.

From figure, $z = \angle DAB - \angle DAC = 78° - 50° = 28°$.

Example 7. *In the adjoining figure, ABCD is a parallelogram. Find the ratio of AB : BC. All measurements are in centimetres.*



Solution. Given ABCD is a parallelogram,

$3x - 4 = y + 5$ (opp. sides are equal)

$\Rightarrow$ $3x - y - 9 = 0$ ...(*i*)

and $2x + 5 = y - 1$ (opp. sides are equal)

$\Rightarrow$ $2x - y + 6 = 0$ ...(*ii*)

On subtracting (*ii*) from (*i*), we get

$x - 15 = 0 \Rightarrow x = 15.$

On substituting this value of x in (*i*), we get

$3 \times 15 - y - 9 = 0 \Rightarrow 36 - y = 0 \Rightarrow y = 36.$

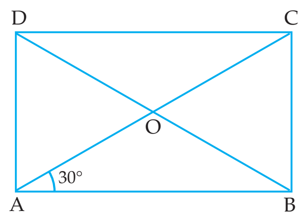
$\therefore$ $AB = 3x - 4 = 3 \times 15 - 4 = 41$

and $BC = y - 1 = 36 - 1 = 35.$

Hence, AB : BC = 41 : 35.

Example 8. *In a rectangle ABCD, diagonals intersect at O. If $\angle OAB = 30°$, find*

(i) $\angle ACB$ *(ii)* $\angle ABO$ *(iii)* $\angle COD$ *(iv)* $\angle BOC$.



Solution. (*i*) $\angle ABC = 90°$ (each angle of a rectangle = 90°)

$\angle ACB + 30° + 90° = 180°$ (sum of angles in ΔABC)

$\Rightarrow$ $\angle ACB = 180° - 30° - 90° = 60°.$

(*ii*) $AC = BD$ (diagonals are equal)

$\Rightarrow$ $2AO = 2BO$ (diagonals bisect each other)

$\Rightarrow$ $AO = BO$

$\Rightarrow$ $\angle ABO = \angle OAB$ (angles opp. equal sides in ΔOAB)

$\Rightarrow$ $\angle ABO = 30°$ ($\because \angle OAB = 30°$ given)

(*iii*) $\angle AOB + 30° + 30° = 180°$ (sum of angles in ΔOAB)

$\Rightarrow$ $\angle AOB = 180° - 30° - 30° = 120°.$

But $\angle COD = \angle AOB$ (vert. opp. $\angle$s)

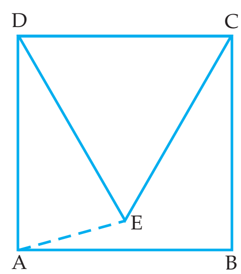
$\Rightarrow$ $\angle COD = 120°.$

(*iv*) $\angle BOC + 120° = 180°$ (linear pair)

$\Rightarrow$ $\angle BOC = 180° - 120° = 60°.$

Example 9. *In the adjoining figure, ABCD is a square and CDE is an equilateral triangle. Find*

(i) $\angle AED$ *(ii)* $\angle EAB$ *(iii) reflex* $\angle AEC$.



Solution. (*i*) From figure, $\angle ADE = 90° - 60°$

($\because$ each angle in a square = 90° and each angle in an equilateral triangle = 60°)

$\Rightarrow$ $\angle ADE = 30°.$

$ED = DC$ (sides of equilateral triangle)

$AD = DC$ (sides of square)

$\Rightarrow$ $ED = AD \Rightarrow \angle AED = \angle EAD$ (angles opp. equal sides in ΔAED)

But $\angle AED + \angle EAD + \angle ADE = 180°$ (sum of angles in ΔAED)

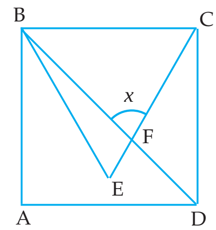
$\Rightarrow$ $2\angle AED + 30° = 180°$

$\Rightarrow$ $\angle AED = \dfrac{180° - 30°}{2} = 75°.$

(*ii*) $\angle EAB = 90° - 75° = 15°$ $\quad$ ($\because \angle EAD = \angle AED = 75°$)

(*iii*) Reflex $\angle AEC = 360° - 75° - 60° = 225°$.

Example 10. *BEC is an equilateral triangle in the square ABCD. Find the value of x in the figure.*



Solution. Since ABCD is a square and BD is a diagonal,

$\therefore \quad \angle DBC = 45°$.

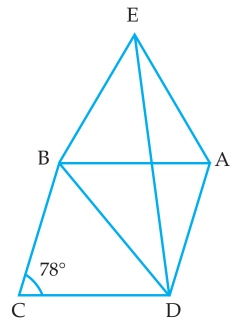
As BEC is an equilateral triangle,

$\angle BCE = 60°$.

In ΔBFC, $x + 45° + 60° = 180°$

$\Rightarrow \quad x = 180° - 45° - 60° = 75°$.

Example 11. *In the adjoining figure, ABCD is a rhombus and ABE is an equilateral triangle. E and D lie on opposite sides of AB. If $\angle BCD = 78°$, calculate $\angle ADE$ and $\angle BDE$.*



Solution. Since ABCD is a rhombus, $\angle DAB = \angle BCD = 78°$.

As ABE is an equilateral triangle, $\angle BAE = 60°$.

From figure,

$\angle DAE = \angle DAB + \angle BAE = 78° + 60° = 138°$.

Also BA = AE $\quad$ (Since ABE is equilateral triangle)

and BA = AD $\quad$ ($\because$ ABCD is a rhombus)

$\Rightarrow \quad AE = AD \Rightarrow \angle ADE = \angle AED$ $\quad$ ($\because$ angles opp. equal sides in ΔAED)

$\therefore \quad \angle ADE = \frac{1}{2}(180° - 138°) = \frac{1}{2} \times 42° = 21°$.

In ΔBCD, BC = CD $\quad$ ($\because$ ABCD is a rhombus)

$\Rightarrow \quad \angle CBD = \angle CDB$.

$\therefore \quad \angle CBD = \frac{1}{2}(180° - 78°) = \frac{1}{2}(102°) = 51°$.

But $\angle BDA = \angle CBD$ $\quad$ (BC || AD, alt. $\angle$s are equal)

$\Rightarrow \quad \angle BDA = 51°$.

From figure, $\angle BDE = \angle BDA - \angle EDA = 51° - 21° = 30°$.

Example 12. *In parallelogram ABCD, AB = 10 cm and AD = 6 cm. The bisector of $\angle A$ meets DC in E. AE and BC produced meet at F. Find the length of CF.*

Solution. Mark the angles as shown in the figure.

As AE is bisector of $\angle A$,

$\angle 1 = \angle 2$ $\quad$ ...(*i*)

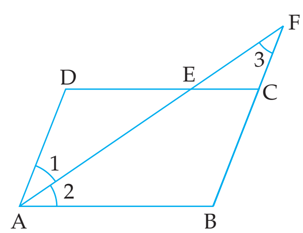
Since ABCD is a parallelogram, AD || BC *i.e.* AD || BF

$\angle 1 = \angle 3$ $\quad$ (alt. $\angle$s are equal)

$\Rightarrow \quad \angle 2 = \angle 3$ $\quad$ (using (*i*))

In ΔABF, $\quad \angle 2 = \angle 3$

$\Rightarrow \quad BF = AB$



(sides opp. equal angles are equal)

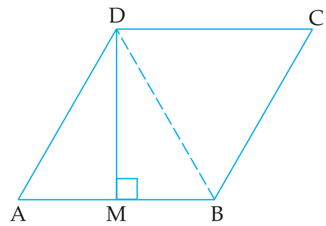
$\Rightarrow \quad BC + CF = 10$ cm

$\Rightarrow \quad AD + CF = 10$ cm $\quad$ (BC = AD, opp. sides of a || gm)

$\Rightarrow \quad 6$ cm + CF = 10 cm $\Rightarrow$ CF = 4 cm.

Example 13. *ABCD is a rhombus in which altitude from D to side AB bisects AB. Find the angles of the rhombus.*

Solution. ABCD is a rhombus in which $DM \perp AB$ such that M is mid-point of AB. Join BD.



In ΔDAM and ΔDBM,

$\angle AMD = \angle BMD$ (each = 90°)

$AM = BM$ (M is mid-point of AB)

$DM = DM$ (common)

$\therefore$ $\Delta DAM \cong \Delta DBM$ (SAS rule of congruency)

$\Rightarrow$ $AD = BD$

Also, $AD = AB$ ($\because$ ABCD is a rhombus)

$\Rightarrow$ $AD = BD = AB$

$\Rightarrow$ ΔABD is an equilateral triangle

$\Rightarrow$ $\angle A = 60°$.

Now, $AD \parallel BC$ and AB is a transversal,

$\therefore$ $\angle A + \angle B = 180°$ (sum of co-interior $\angle$s)

$\Rightarrow$ $60° + \angle B = 180° \Rightarrow \angle B = 120°$

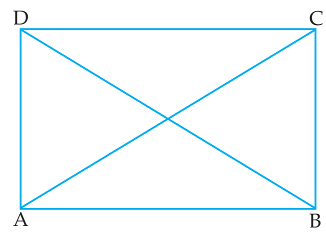
$\angle C = \angle A$ and $\angle D = \angle B$ (opp. $\angle$s in a $\parallel$ gm are equal)

$\Rightarrow$ $\angle C = 60°$ and $\angle D = 120°$.

Hence, the angles of the rhombus are 60°, 120°, 60°, 120°.

Example 14. *Prove that the diagonals of a rectangle are equal.*

Solution. Let ABCD be a rectangle. We need to prove that AC = BD.



In ΔABC and ΔBAD,

$BC = AD$ (opp. sides of a rectangle are equal)

$AB = BA$ (common)

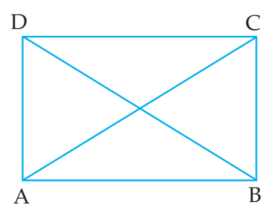
$\angle ABC = \angle BAD$ (each angle of a rectangle = 90°)

$\Delta ABC \cong \Delta BAD$ (SAS rule of congruency)

$\therefore$ $AC = BD$ (c.p.c.t.)

Example 15. *If the diagonals of a parallelogram are equal, then prove that it is a rectangle.*

Solution. Let ABCD be a parallelogram in which AC = BD. We need to prove that $\angle A = 90°$.



In ΔABC and ΔBAD,

$BC = AD$ (opp. sides of a $\parallel$ gm)

$AB = AB$ (common)

$AC = BD$ (given, diagonals are equal)

$\therefore$ $\Delta ABC \cong \Delta BAD$ (SSS rule of congruency)

$\therefore$ $\angle B = \angle A$ (c.p.c.t.)

As $AD \parallel BC$ and AB is a transversal,

$\angle A + \angle B = 180°$ (sum of co-int. $\angle$s)

$\Rightarrow$ $\angle A + \angle A = 180° \Rightarrow 2\angle A = 180° \Rightarrow \angle A = 90°$.

$\therefore$ ABCD is a rectangle.

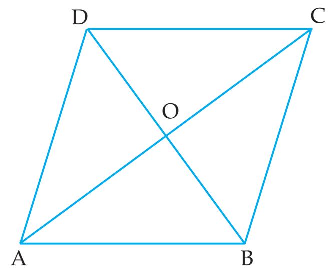
Example 16. *Show that the diagonals of a rhombus bisect each other at right angles.*

Solution. Let ABCD be a rhombus and let its diagonals AC and BD meet at O.

We need to prove that OA = OC, OB = OD and ∠AOB = 90°. As every rhombus is a || gm, therefore, the diagonals bisect each other *i.e.* OA = OC and OB = OD.

Thus, the diagonals of a rhombus bisect each other.

In ΔOAB and ΔOCB,



OA = OC (proved above)

OB = OB (common)

AB = BC (sides of a rhombus)

∴ ΔOAB ≅ ΔOCB (SSS rule of congruency)

∴ ∠AOB = ∠BOC (c.p.c.t.)

But ∠AOB + ∠BOC = 180° (linear pair)

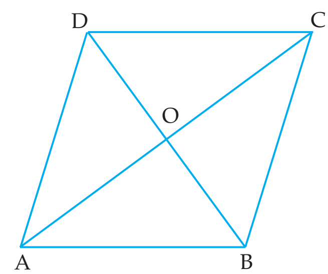
⇒ ∠AOB + ∠AOB = 180° ⇒ ∠AOB = 90°.

Hence, the diagonals of a rhombus bisect each other at right angles.

Example 17. *If the diagonals of a quadrilateral bisect each other at right angles, then prove that the quadrilateral is a rhombus.*

Solution. Let ABCD be a quadrilateral in which the diagonals AC and BD bisect each other at O and are at right angles *i.e.* OA = OC, OB = OD and AC ⊥ BD. We need to prove that ABCD is a rhombus.

As the diagonals of the quadrilateral ABCD bisect each other, ABCD is a parallelogram.



In ΔOAB and ΔOCB,

OA = OC (from given)

OB = OB (common)

∠AOB = ∠COB

(each = 90°, because AC ⊥ BD given)

∴ ΔOAB ≅ ΔOCB

(by SAS rule of congruency)

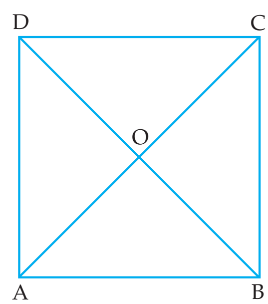
∴ AB = BC (c.p.c.t.)

Thus, ABCD is a parallelogram in which two adjacent sides are equal, therefore, ABCD is a rhombus.

Example 18. *Prove that the diagonals of a square are equal and bisect each other at right angles.*

Solution. Let ABCD is a square and let its diagonals AC and BD meet at O. We need to prove that OA = OC, OB = OD, ∠AOB = 90° and AC = BD.

As ABCD is a square, so it is a parallelogram. Therefore, its diagonals bisect each other



i.e. OA = OC and OB = OD.

Thus, the diagonals of a square bisect each other.

In ΔOAB and ΔOCB,

OA = OC (proved above)

AB = BC (sides of a square)

OB = OB (common)

∴ ΔOAB ≅ ΔOCB (by SSS rule of congruency)

∴ ∠AOB = ∠BOC (c.p.c.t.)

But ∠AOB + ∠BOC = 180° (linear pair)

⇒ ∠AOB + ∠AOB = 180° ⇒ ∠AOB = 90°

As ABCD is a square, so one of its angle is 90°. Let ∠A = 90°.

Now AD || BC and AB is a transversal

$\angle A + \angle B = 180°$ (sum of co-int. $\angle$s)

$\Rightarrow \quad 90° + \angle B = 180° \Rightarrow \angle B = 90°.$

In ΔABC and ΔBAD,

$BC = AD$ (sides of a square)

$AB = BA$ (common)

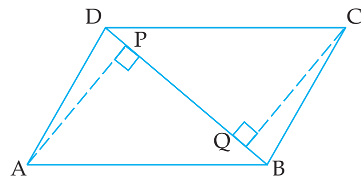
$\angle B = \angle A$ (each = 90°)

$\therefore \quad \Delta ABC \cong \Delta BAD$

$\therefore \quad AC = BD$ (c.p.c.t.)

Hence, the diagonals of a square are equal and bisect each other at right angles.

Example 19. *In the adjoining figure, ABCD is a parallelogram and AP, CQ are perpendiculars from the vertices A, C respectively on the diagonal BD. Show that*



(*i*) $\Delta APB \cong \Delta CQD$

(*ii*) $AP = CQ$.

Solution. (*i*) In ΔAPB and ΔCQD,

$AB = CD$ (opp. sides of a || gm)

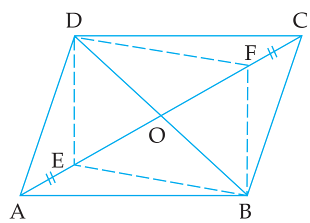
$\angle P = \angle Q$ (each = 90°)

$\angle ABP = \angle CDQ$ (alt. $\angle$s, AB || DC and BD is a transversal)

$\therefore \quad \Delta APB \cong \Delta CQD$ (by AAS rule of congruency)

(*ii*) $AP = CQ$ (c.p.c.t.)

Example 20. *If E and F are points on diagonal AC of a parallelogram ABCD such that AE = CF, then show that BFDE is a parallelogram.*



Solution. As ABCD is a parallelogram, its diagonals bisect each other *i.e.* OA = OC and OB = OD.

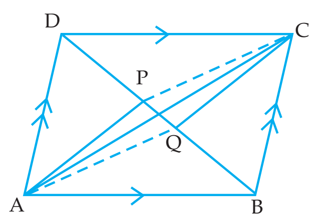
Given $AE = CF$

$\therefore \quad OA - AE = OC - CF$

$\Rightarrow \quad OE = OF$

Thus, in quadrilateral BFDE, OE = OF and OB = OD *i.e.* its diagonals EF and BD bisect each other, therefore, BFDE is a parallelogram.

Example 21. *ABCD is a parallelogram. If the bisectors of ∠A and ∠C meet the diagonal BD at P and Q respectively, prove that the quadrilateral PCQA is a parallelogram.*



Solution. Given. ABCD is a ||gm, AP bisects ∠A and CQ bisects ∠C.

To prove. AP || QC and PC || AQ.

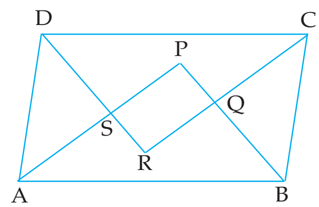
Construction. Join AC.

Proof.

Statements	*Reasons*
1. $\angle BAP = \frac{1}{2}\angle A$	1. AP is bisector of ∠A.
2. $\angle DCQ = \frac{1}{2}\angle C$	2. CQ is bisector of ∠C.
3. $\angle BAP = \angle DCQ$	3. $\angle A = \angle C$, since ABCD is a ||gm.

4. $\angle BAC = \angle DCA$	4. Alt. $\angle$s, since AB ∥ DC.
5. $\angle BAP - \angle BAC = \angle DCQ - \angle DCA$	5. Subtracting 4 from 3.
6. $\angle CAP = \angle ACQ$	6. From figure.
7. AP ∥ QC Similarly, PC ∥ AQ. Hence, PCQA is a parallelogram. **Q.E.D**	7. Alt. $\angle$s are equal.

Example 22. *Show that the bisectors of the angles of a parallelogram form a rectangle.*



Solution. Let ABCD be a parallelogram and let P, Q, R and S be the points of intersection of the bisectors of $\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$ respectively. We need to show that PQRS is a rectangle.

As ABCD is a ∥ gm, AD ∥ BC and AB is a transversal.

$\therefore \quad \angle A + \angle B = 180°$ (sum of co-int. $\angle$s = 180°)

$\Rightarrow \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90°$

$\Rightarrow \quad \angle PAB + \angle PBA = 90°$ ($\because$ AP is bisector of $\angle A$ and BP is bisector of $\angle B$)

In ΔPAB, $\angle APB + \angle PAB + \angle PBA = 180°$ (sum of angles in a Δ)

$\Rightarrow \quad \angle APB + 90° = 180° \Rightarrow \angle APB = 90° \Rightarrow \angle SPQ = 90°$.

Similarly, $\quad \angle PQR = 90°$, $\angle QRS = 90°$ and $\angle RSP = 90°$.

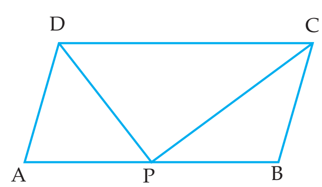
So, PQRS is a quadrilateral in which each angle is 90°.

Now, $\quad \angle SPQ = \angle QRS$ (each = 90°)

and $\quad \angle PQR = \angle RSP$ (each = 90°)

Thus, PQRS is a quadrilateral in which both pairs of opposite angles are equal, therefore, PQRS is a parallelogram. Also, in this parallelogram one angle (in fact all angles) is 90°. Therefore, PQRS is a rectangle.

Example 23. *In the adjoining figure, ABCD is a parallelogram. If AB = 2AD and P is mid-point of AB, prove that $\angle DPC = 90°$.*



Solution. Given P is mid-point of AB

$\Rightarrow \quad AP = PB = \frac{1}{2}AB.$

Also AB = 2AD $\Rightarrow$ AD = $\frac{1}{2}$AB.

$\therefore \quad AP = AD.$

In ΔAPD, AP = AD

$\Rightarrow \quad \angle APD = \angle ADP$ (angles opp. equal sides)

But $\angle A + \angle APD + \angle ADP = 180°$ (sum of angles in a Δ = 180°)

$\Rightarrow \quad \angle A + \angle APD + \angle APD = 180°$ ($\because \angle ADP = \angle APD$)

$\Rightarrow \quad 2\angle APD = 180° - \angle A \Rightarrow \angle APD = \frac{180° - \angle A}{2}$ …(*i*)

As PB = AP and BC = AD (opp. sides of ∥ gm ABCD)

$\Rightarrow \quad PB = BC.$

In ΔBPC, $PB = BC \Rightarrow \angle CPB = \angle BCP$.

But $\angle B + \angle CPB + \angle BCP = 180°$

$\Rightarrow \quad \angle B + \angle CPB + \angle CPB = 180°$

$\Rightarrow \quad 2\angle CPB = 180° - \angle B \Rightarrow \angle CPB = \dfrac{180° - \angle B}{2}$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$\angle APD + \angle CPB = 180° - \frac{1}{2}(\angle A + \angle B)$$

$$= 180° - \frac{1}{2}(180°) \quad (\because \text{ ABCD is a } \| \text{gm, AD} \parallel \text{BC, so } \angle A + \angle B = 180°)$$

$\Rightarrow \quad \angle APD + \angle CPB = 90°$ ...(*iii*)

But $\angle APD + \angle DPC + \angle CPB = 180°$ ($\because$ APB is a line)

$\Rightarrow \quad (\angle APD + \angle CPB) + \angle DPC = 180°$

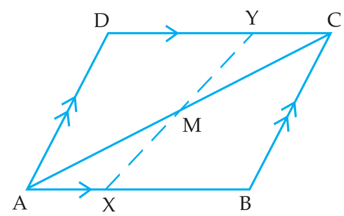
$\Rightarrow \quad 90° + \angle DPC = 180°$

$\Rightarrow \quad \angle DPC = 90°$ (using (*iii*))

Example 24. *In the parallelogram ABCD, M is mid-point of AC, and X, Y are points on AB and DC respectively such that AX = CY. Prove that*

(i) triangle AXM is congruent to triangle CYM.

(ii) XMY is a straight line.



Given. ABCD is a ∥gm, M is mid-point of AC, X, Y are points on AB, CD such that AX = CY.

To prove. (*i*) $\Delta AXM \cong \Delta CYM$

(*ii*) XMY is a straight line.

Construction. Join XM and MY.

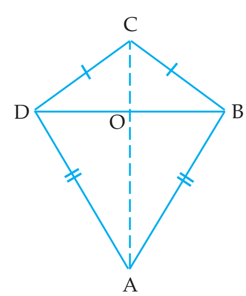
Proof.

Statements	*Reasons*
In Δs AXM and CYM	
1. AX = CY	1. Given.
2. AM = MC	2. M is mid-point of AC.
3. $\angle XAM = \angle MCY$ $\therefore$ (*i*) $\Delta AXM \cong \Delta CYM$	3. Alt. $\angle$s, since AB ∥ DC. SAS rule of congruency.
4. $\angle CMY = \angle AMX$	4. 'c.p.c.t.'
5. $\angle XMC = \angle XAM + \angle AXM$	5. Ext. $\angle$ = sum of two int. opp. $\angle$s.
6. $\angle CMY + \angle XMC$ $= \angle AMX + \angle XAM + \angle AXM$	6. Adding 4 and 5.
7. $\angle CMY + \angle XMC = 180°$ (*ii*) XMY is a straight line **Q.E.D.**	7. Sum of $\angle$s of a Δ = 180°. Sum of adj. $\angle$s = 180°.

Example 25. *In the adjoining figure, ABCD is a kite in which AB = AD and BC = CD. Prove that:*

(i) AC is a bisector of $\angle A$ and of $\angle C$.

(ii) AC is perpendicular bisector of BD.



Solution. (*i*) In ΔABC and ΔADC,

AB = AD (given)

$BC = CD$ (given)

$CA = CA$ (common)

$\therefore \quad \Delta ABC \cong \Delta ADC$ (SSS rule of congruency)

$\angle BAC = \angle CAD$ and $\angle BCA = \angle ACD$. (c.p.c.t.)

Hence, AC is bisector of $\angle A$ and of $\angle C$.

(*ii*) In ΔOBC and ΔODC,

$BC = CD$ (given)

$\angle BCO = \angle OCD$ (proved above)

$OC = OC$ (common)

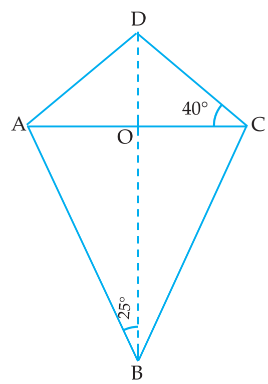
$\therefore \quad \Delta OBC \cong \Delta ODC$ (SAS rule of congruency)

$\therefore \quad OB = OD$ and $\angle BOC = \angle COD$ (c.p.c.t.)

But $\angle BOC + \angle COD = 180°$ (linear pair)

$\Rightarrow \quad 2\angle BOC = 180° \Rightarrow \angle BOC = 90°$.

Hence, AC is perpendicular bisector of BD.



Example 26. *In the adjoining kite, diagonals intersect at O. If $\angle ABO = 25°$ and $\angle OCD = 40°$, find*

(i) $\angle ABC$ (ii) $\angle ADC$ (iii) $\angle BAD$.

Solution. (*i*) Since the diagonal BD bisects $\angle ABC$, $\angle ABC = 2\angle ABO = 2 \times 25° = 50°$.

(*ii*) $\angle DOC = 90°$ (diagonals intersect at right angles)

$\angle ODC + 40° + 90° = 180°$

(sum of angles in ΔOCD)

$\Rightarrow \quad \angle ODC = 180° - 40° - 90° = 50°$.

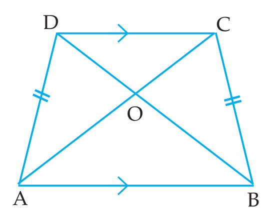
Since the diagonal BD bisects $\angle ADC$,

$\angle ADC = 2\angle ODC = 2 \times 50° = 100°$.

(*iii*) Since the diagonal BD bisects $\angle ADC$, $\angle ADB = \angle ODC = 50°$.

$\angle BAD + 50° + 25° = 180°$ (sum of angles in ΔABD)

$\Rightarrow \quad \angle BAD = 180° - 50° - 25° = 105°$.



Example 27. *In the adjoining figure, ABCD is an isosceles trapezium and its diagonals meet at O. Prove that:*

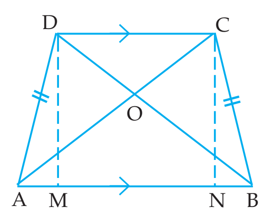
(i) $\angle A = \angle B$ and $\angle C = \angle D$.

(ii) $AC = BD$.

(iii) $OA = OB$ and $OC = OD$.

Solution. (*i*) From C and D, draw perpendiculars CN and DM on AB respectively.

In ΔAMD and ΔBNC,



$AD = BC$ (given)

$\angle AMD = \angle CNB$

($\because$ $DM \perp AB$ and $CN \perp AB$, by construction)

$MD = CN$ (distance between parallel lines)

$\therefore \quad \Delta AMD \cong \Delta BNC$ (RHS rule of congruency)

$\therefore \quad \angle A = \angle B$ (c.p.c.t.)

Also $\angle A + \angle D = 180°$ and $\angle B + \angle C = 180°$) ($\because$ AB || DC, sum of co-int. $\angle$s = 180°)

$\Rightarrow \quad \angle A + \angle D = \angle B + \angle C$

$\Rightarrow$ $\angle D = \angle C$ ($\because$ $\angle A = \angle B$, proved above)

(*ii*) In ΔABD and ΔBAC,

$\angle A = \angle B$ (proved above)

$AD = BC$ (given)

$AB = AB$ (common)

$\therefore$ $\Delta ABD \cong \Delta BAC$ (SAS rule of congruency)

$\therefore$ $AC = BD$ (c.p.c.t.)

(*iii*) In ΔOAD and ΔOBC,

$AD = BC$ (given)

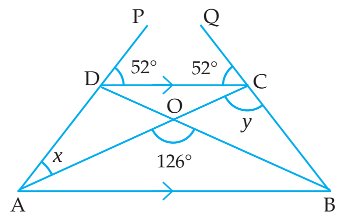
$\angle AOD = \angle BOC$ (vert. opp. $\angle$s)

$\angle ADO = \angle BCO$ ($\because$ $\Delta ABD \cong \Delta BAC$, so $\angle ADB = \angle ACB$)

$\therefore$ $\Delta OAD \cong \Delta OBC$ (AAS rule of congruency)

$\therefore$ $OA = OB$ and $OC = OD$. (c.p.c.t.)

Example 28. *In the adjoining figure, ABCD is a trapezium. If $\angle AOB = 126°$ and $\angle PDC = \angle QCD = 52°$, find the values of x and y.*



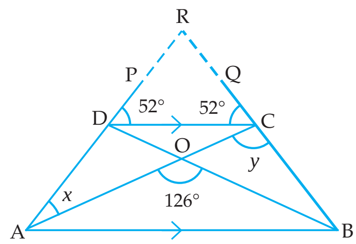
Solution. Produce AP and BQ to meet at R.

In ΔRDC, $\angle RDC = \angle RCD$ (each angle = 52°)

$\therefore$ $DR = CR$ (sides opp. equal angles are equal)

$\angle RAB = \angle RDC$ (corres. $\angle$s, $AB \parallel DC$).

Similarly, $\angle RBA = \angle RCD$.



$\therefore$ $\angle RAB = \angle RBA$

$\Rightarrow$ $AR = RB$.

$\therefore$ $AD = AR - DR = RB - CR = BC$

$\Rightarrow$ ABCD is an isosceles trapezium.

$\therefore$ $OA = OB$ (Example 27)

$\Rightarrow$ $\angle OAB = \angle OBA$.

$\therefore$ $\angle OAB = \dfrac{180° - 126°}{2} = 27°$.

$\angle DAC = \angle DAB - \angle OAB = 52° - 27° = 25°$

$\therefore$ $x = 25°$.

$\angle ACB + \angle CAB + \angle ABC = 180°$ (sum of angle in ΔABC)

$\Rightarrow$ $y + 27° + 52° = 180° \Rightarrow y = 180° - 27° - 52°$

$\Rightarrow$ $y = 101°$.

Exercise 12.1

1. If two angles of a quadrilateral are 40° and 110° and the other two are in the ratio 3 : 4, find these angles.
2. If the angles of a quadrilateral, taken in order, are in the ratio 1 : 2 : 3 : 4, prove that it is a trapezium.
3. If an angle of a parallelogram is two-thirds of its adjacent angle, find the angles of the parallelogram.
4. (*a*) In figure (1) given below, ABCD is a parallelogram in which $\angle DAB = 70°$, $\angle DBC = 80°$. Calculate angles CDB and ADB.