Łukasiewicz $[0,1]$ $0_K = 0$, $1_K = 1$

$a \ominus b = a \leq b \,?\, 0 : a$

$a \oplus b = \max(a, b)$

$a \otimes b = \max(a + b - 1, 0)$

Idempotent : $\max(a, a) = a$

Absorptive : $\max(1, a) = 1$

Commutative

A13

$$a \otimes (b \ominus c) \stackrel{?}{=} \underbrace{a \otimes b}_{\alpha} \ominus \underbrace{a \otimes c}_{\beta}$$

$\alpha = \max(a + (b \leq c \,?\, 0 : b) - 1, 0)$

Si $b \leq c$ alors $a \otimes b \leq a \otimes c$

$\alpha = \max(a - 1, 0) = 0$

$\beta = 0$

Si $b > c$, $a + b > 1$

$\alpha = a \otimes b > 0$

$\beta = a \otimes b$

Si $b > c$, $a + b < 1$

$\alpha = 0$

$\beta = 0 \ominus a \otimes c = 0$

A14 $\underbrace{(a+b)}_{\alpha} - c = \underbrace{(a - c) + (b - c)}_{\beta}$?

Si $\max(a, b) \leq c$

$\alpha = 0$, $\beta = 0$

Si $a \leq c$, $\max(a, b) > c$

$\alpha = b$

$\beta = b \leq c \,?\, 0 : b = b$

!

A14 doesn't hold

Si $b \leq c$, $a > c$

$\alpha = a$, $\beta = a$

Si $b > c$, $a > c$

$\alpha = \max(a, b)$

Semantics equivalence, two var?

$$\left.t_1(1-t_2)+t_2(1-t_1)\right)\ \alpha$$
$$\stackrel{?}{=} \left.t_1+t_2-t_1t_2\right)\ \beta$$

$t_1 = 1 \quad t_2 = 1/2$

$\alpha = \cancel{0}\ \underline{1}$

$$\beta = \max\Big(\underbrace{(t_1+t_2)}_{1} \leq \underbrace{\max(t_1+t_2-1,\ 0)}_{\frac{1}{2}}\ ?\ 0 : \underbrace{\max(t_1,t_2)}_{1}\Big)$$

$\mathrm{PosBool}[\{x_1, x_2\}]$

Pos → absorptive, Bool → $\cancel{AB}$

$a - b = \cancel{\operatorname{argmin}}\ \min\{c \mid a \leq b + c\}$

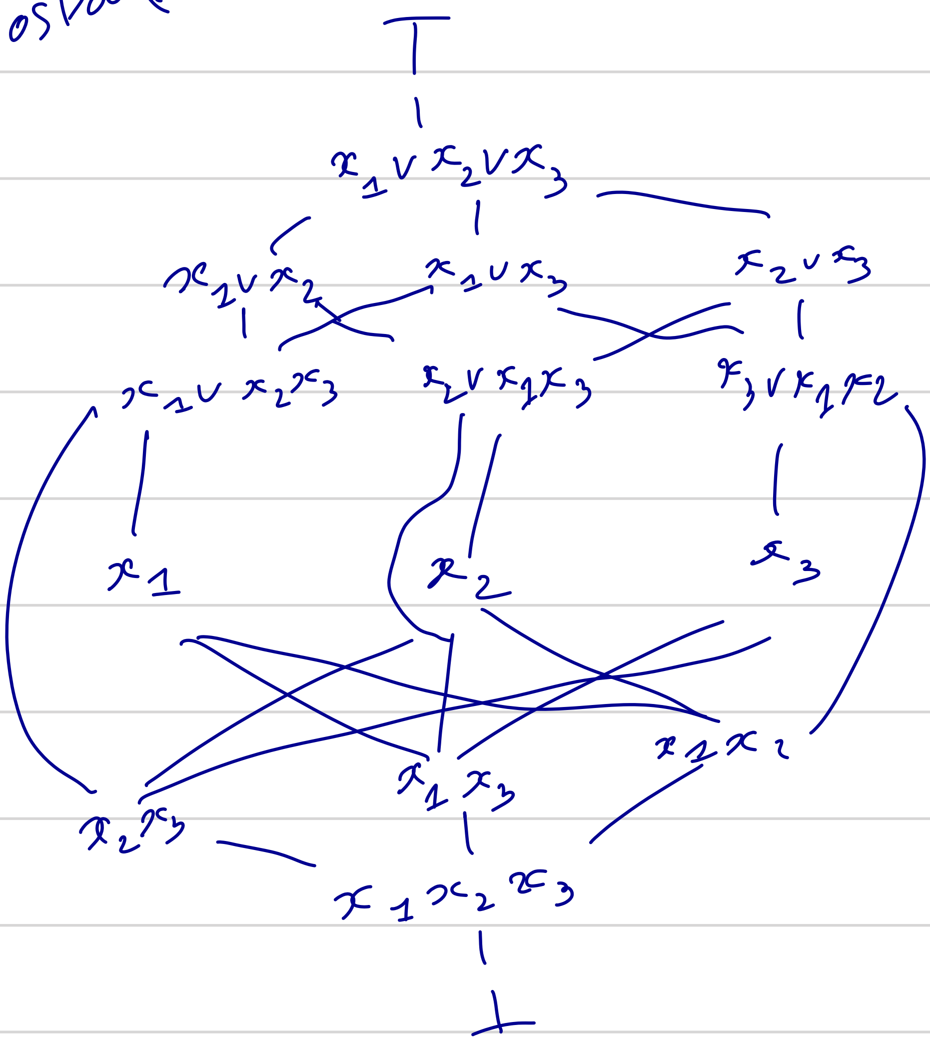
$$\begin{tikzcd}[column sep=small]
 & \top \arrow[d, dash] & \\
 & x_1 \vee x_2 \arrow[dl, dash] \arrow[dr, dash] & \\
x_1 \arrow[dr, dash] & & x_2 \arrow[dl, dash] \\
 & x_1 \wedge x_2 \arrow[d, dash] & \\
 & \bot &
\end{tikzcd}$$

Diamond

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}$$

$$\begin{cases} (x_1 + x_2) - x_1 = x_2 \\ (x_1 - x_1) + (x_2 - x_1) = 0 + x_2 = x_2 \end{cases}$$

PosBool$\{x_1, x_2, x_3\}$



$$(x_1 + x_2) - x_1 = x_2$$

$$x_1 - x_1 + x_2 - x_2 = x_2$$

Idempotent $\Rightarrow$ (A14).

Assume $+$ idempotent then $+$ is the join of a semi-lattice for the natural order!

* If $a \leq b \Rightarrow a+b=b$ (thanks to idempotence)

$\exists x,\ a+x=b$

$a+b = a+(a+x) = a+x = b$

* $a+b=b \Rightarrow a \leq b$ (always)

$\boxed{a \leq b \Leftrightarrow a+b=b}$

$a \leq b+c$

$\Leftrightarrow a-b \leq c$

If u, $a \leq u$ and $b \leq u$

$a+u=u$ $\quad$ $a+b+u = a+u = u$ $\quad$ so $a+b \leq u$

$b+u=u$

But $\quad a \leq a+b, \quad b \leq a+b$

$a+b$ is the least u s.t. $a \leq u \wedge b \leq u$.

$(a-c)+(b-c) \leq (a+b)-c$?

Show that $(a-c) \leq (a+b)-c$ and $(b-c) \leq (a+b)-c$

$a \leq a+b \leq c+\underbrace{[(a+b)-c]}$ def of $\ominus$

So $\quad a-c \leq (a+b)-c \quad$ (by $x \leq y+z \Leftrightarrow x-y \leq z$)

Same proof $(b-c) \leq (a+b)-c$

$(a+b)-c \leq (a-c)+(b-c)$?

Show that $(a+b) \leq c + ((a-c)+(b-c))$

We have $a \leq c + (a-c)$

$b \leq c + (b-c)$

$a+b \leq \underbrace{c+c}_{\substack{=c \\ \text{by idempotence}}} + (a-c)+(b-c)$

$\blacksquare$

Right-distributivity of minus over plus

$$(a+b)-c = (a-c)+(b-c)$$

Left-distributivity: always.

$$a-(b+c) = (a-b)-c$$

Result

In an m-semiring:

Idempotence $\iff$ Right-distr. of $\ominus$.

$\Leftarrow$ $(a+a)-a = \underbrace{(a-a)}_{0} + \underbrace{(a-a)}_{0} = 0$

so $a+a \leq a$

We always have $a \leq a+a$

So $a = a+a$

	A13	Idemp. (=A14)	Abs.
Bool, Bool[X]	✓	✓	✓
Why[X]	✓	✓	✓
Temporal[X]	✓	✓	✓
Tropical	✓	✓	✓
TVL/MinMax/S	✗	✓	✓
Counting, ℕ[X]	✓	✗	✗
Which	✓	✓	✗
Łukasiewicz	✓	✓	✓
PosBool[X]	✗	✓	✓