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Math exercise : Calculus

1. Prove that $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$, $y = f(g(x))$

$$\frac{dy}{dx} = \lim_{h \to 0} \left(\frac{f(g(x+h)) - f(g(x))}{h} \right) \quad \text{[def}^n \text{ off derivative]}$$

$$= \lim_{h \to 0} \left(\frac{f(g(x+h)) - f(g(x))}{h} \right) * \frac{g(x+h) - g(x)}{g(x+h) - g(x)}$$

$$= \lim_{h \to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \underbrace{\lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)}_{\text{[def}^n \text{ of derivative]}}$$

$$= \lim_{h \to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \cdot g'(x)$$

Let $k = g(x+h) - g(x)$

$\therefore$ as $h \to 0$, $k \to 0$

$g(x+h) = g(x) + h$ —— (1)

Substitute (1) in above eq^n,

$$= \lim_{k \to 0} \left(\frac{f(g(x) + k) - f(g(x))}{k} \right) g'(x)$$

$$= f'(g(x)) \cdot g'(x) \quad \text{[From def}^n \text{ of derivative]}$$

hence, $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

2. write chain rule for

$$u = f(x, y, z) \quad \text{and}$$

$$x = x(a, b), \; y = y(a, b), \; z = z(a, b)$$

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial a}$$

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial b} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial b} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial b}$$

3. Find the gradients $\left(\nabla_x f(x) = \left[\frac{\partial}{\partial x_i}, \ldots\right]^T\right)$ of

$$y = f(x) = 3x_1^2 + 5e^{x_2}$$

$$\frac{\partial y}{\partial x_1} = 6x_1$$

$$\frac{\partial y}{\partial x_2} = 5e^{x_2}$$

$$\nabla_x f(x) = \left[6x_1, 5e^{x_2}\right]^T$$

4. write the gradient $\nabla f(x)$ of following fun.

$$f(x) = \|x\|_2$$

The norm of a vector X, $\|X\|_2 = \sqrt{(x_1^2 + x_2^2 + \ldots x_n^2)}$

where $X = [x_1, x_2 \ldots x_n]^T$

$$\nabla_x \|X\|_2 = \left[\frac{\partial}{\partial x_1}\|X\|_2, \frac{\partial}{\partial x_2}\|X\|_2, \ldots \frac{\partial}{\partial x_n}\|x\|_2\right]^T$$

for x_i elemement

$$\frac{\partial}{\partial x_i}\|X\| = \frac{\partial}{\partial x_i}\sqrt{(x_1^2 + x_2^2 + \ldots x_n^2)}$$

$$\frac{\partial}{\partial x_i}\|x\|_2 = \frac{1}{2}\,\frac{2x_i}{(x_1^2+x_2^2+\dots x_n^2)^{1/2}}$$

$$= \frac{x_i}{\sqrt{x_1^2+x_2^2+\dots x_n^2}}$$

Putting all partial derivaties together,

$$\boxed{\nabla_x \|x\|_2 = \frac{x}{\|x\|_2}}$$

5. Find the gradient $\nabla f(x)$ of the following fun and then solve the eqn to find the minimu value of f.

$$f(x) = (y-x)^T(y-x)$$
$$= (Y^T - X^T)\,Y - X$$
$$= Y^TY - Y^TX - X^TY + X^TX$$

$$\nabla_x f(x) = -Y - Y + 2x \qquad [X^TY = X.Y]$$
$$= 2(X-Y)$$

now for minimum value of f,

$$\nabla_x f(x) = 0$$
$$2(X-Y) = 0$$
$$\boxed{X = Y}$$

$\boxed{f(x) \text{ is } 0 \text{ when } X = Y.}$