Lecture 4: Scribe

1 Continuous Random Variables

1.1 Definition: Continuous Random Variable

A random variable X is continuous if there exists a nonnegative function f_X, defined for all $x \in \mathbb{R}$, having the property that for every subset B of the real line,

$$P(X \in B) = \int_B f_X(x)\,dx$$

The function f_X is called the probability density function (pdf) of the random variable X.

1.2 Properties of Probability Density Function

For any pdf f_X:

1. $f_X(x) \geq 0$ for all $x \in \mathbb{R}$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
3. For any interval $[a, b]$:
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$
4. $P(X = a) = \int_a^a f_X(x)\,dx = 0$ for any $a \in \mathbb{R}$

1.3 Cumulative Distribution Function (CDF)

1.3.1 Definition: CDF for Continuous Random Variable

The cumulative distribution function F_X of a continuous random variable X with pdf f_X is defined by:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt, \quad -\infty < x < \infty$$

1.3.2 Relationship between PDF and CDF

If X is a continuous random variable with pdf f_X and cdf F_X, then at all points x at which the derivative $F'_X(x)$ exists:

$$f_X(x) = \frac{d}{dx}F_X(x) = F'_X(x)$$

1.4 Expectation of Continuous Random Variable

1.4.1 Definition: Expected Value

The expected value or expectation or mean of a continuous random variable X with pdf f_X is defined by:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

provided that the integral exists (i.e., $\int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty$).

1.4.2 Expected Value of a Function of X

If X is a continuous random variable with pdf f_X and g is a real-valued function, then:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

provided the integral exists.

1.5 Variance and Standard Deviation

1.5.1 Definition: Variance

The variance of a continuous random variable X is defined by:

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

1.5.2 Definition: Standard Deviation

The standard deviation of X is defined by:

$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$

1.6 Properties of Expectation and Variance

For a continuous random variable X and constants $a, b \in \mathbb{R}$:

1. $E[aX + b] = aE[X] + b$
2. $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$

2 Uniform Distribution

2.1 Definition: Uniform Distribution

A random variable X is said to be uniformly distributed over the interval $[a, b]$, denoted $X \sim U(a, b)$, if its pdf is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

2.2 CDF of Uniform Distribution

The cumulative distribution function of $X \sim U(a, b)$ is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

2.3 Expectation and Variance of Uniform Distribution

For $X \sim U(a, b)$:

$$E[X] = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

2.3.1 Derivation of $E[X]$

$$\begin{aligned} E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx &= \int_a^b x \cdot \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \int_a^b x\, dx \\ &= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b \\ &= \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2} \right) \\ &= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} \\ &= \frac{1}{b-a} \cdot \frac{(b-a)(b+a)}{2} \\ &= \frac{a+b}{2} \end{aligned}$$

2.3.2 Derivation of Var(X)

First, calculate $E[X^2]$:

$$\begin{aligned} E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx &= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \int_a^b x^2\, dx \\ &= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b \\ &= \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3} \right) \end{aligned}$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{b^2 + ab + a^2}{3}$$

Then:

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - (E[X])^2 \\
&= \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2)}{12} \\
&= \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

2.4 Example 1: Uniform Distribution

Let $X \sim U(0,1)$. Find $P(X > 0.3)$.

Solution:

$$\begin{aligned}
P(X > 0.3) &= \int_{0.3}^{1} f_X(x)\,dx \\
&= \int_{0.3}^{1} 1\,dx \\
&= [x]_{0.3}^{1} \\
&= 1 - 0.3 \\
&= 0.7
\end{aligned}$$

Alternatively, using the CDF:

$$P(X > 0.3) = 1 - P(X \leq 0.3)$$

$$\begin{aligned}
&= 1 - F_X(0.3) \\
&= 1 - \frac{0.3 - 0}{1 - 0} \\
&= 1 - 0.3 \\
&= 0.7
\end{aligned}$$

2.5 Example 2: Uniform Distribution Application

Suppose buses arrive at a bus stop according to a uniform distribution over a 30-minute interval. If you arrive at the bus stop at a random time, what is the probability you will wait more than 10 minutes?

Solution:

Let X be the time until the next bus arrives. Then $X \sim U(0, 30)$.

We need to find $P(X > 10)$.

$$P(X > 10) = \int_{10}^{30} \frac{1}{30}\, dx$$

$$= \frac{1}{30}[x]_{10}^{30}$$

$$= \frac{1}{30}(30 - 10)$$

$$= \frac{20}{30}$$

$$= \frac{2}{3}$$

3 Exponential Distribution

3.1 Definition: Exponential Distribution

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$, denoted $X \sim \text{Exp}(\lambda)$, if its pdf is given by:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

3.2 CDF of Exponential Distribution

The cumulative distribution function of $X \sim \text{Exp}(\lambda)$ is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

3.2.1 Derivation

For $x \geq 0$:

$$F_X(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

Let $u = -\lambda t$, then $du = -\lambda dt$, so $dt = -du/\lambda$.

When $t = 0$, $u = 0$; when $t = x$, $u = -\lambda x$.

$$F_X(x) = \int_0^{-\lambda x} \lambda e^u \left(-\frac{du}{\lambda}\right)$$

$$
\begin{aligned}
&= -\int_0^{-\lambda x} e^u \, du \\
&= -[e^u]_0^{-\lambda x} \\
&= -(e^{-\lambda x} - 1) \\
&= 1 - e^{-\lambda x}
\end{aligned}
$$

3.3 Expectation and Variance of Exponential Distribution

For $X \sim \text{Exp}(\lambda)$:

$$
\begin{aligned}
E[X] &= \frac{1}{\lambda} \\
\text{Var}(X) &= \frac{1}{\lambda^2}
\end{aligned}
$$

3.3.1 Derivation of $E[X]$

$$E[X] = \int_0^{\infty} x\lambda e^{-\lambda x} \, dx$$

Using integration by parts with $u = x$, $dv = \lambda e^{-\lambda x} dx$:

$$du = dx, \quad v = -e^{-\lambda x}$$

$$
\begin{aligned}
E[X] &= [x(-e^{-\lambda x})]_0^{\infty} - \int_0^{\infty} (-e^{-\lambda x}) \, dx \\
&= 0 + \int_0^{\infty} e^{-\lambda x} \, dx \\
&= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^{\infty} \\
&= -\frac{1}{\lambda}(0 - 1) \\
&= \frac{1}{\lambda}
\end{aligned}
$$

3.3.2 Derivation of Var(X)

First, calculate $E[X^2]$:

$$E[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x} \, dx$$

Using integration by parts: Let $u = x^2$, $dv = \lambda e^{-\lambda x} dx$:

$$du = 2x \, dx, \quad v = -e^{-\lambda x}$$

$$
\begin{aligned}
E[X^2] &= [x^2(-e^{-\lambda x})]_0^{\infty} - \int_0^{\infty} 2x(-e^{-\lambda x}) \, dx \\
&= 0 + 2\int_0^{\infty} x e^{-\lambda x} \, dx
\end{aligned}
$$

$$= 2 \cdot \frac{1}{\lambda} \int_0^\infty x\lambda e^{-\lambda x}\, dx$$
$$= \frac{2}{\lambda} E[X]$$
$$= \frac{2}{\lambda} \cdot \frac{1}{\lambda}$$
$$= \frac{2}{\lambda^2}$$

Then:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$
$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$
$$= \frac{1}{\lambda^2}$$

3.4 Memoryless Property of Exponential Distribution

3.4.1 Theorem: Memoryless Property

The exponential distribution has the memoryless property: for all $s, t \geq 0$,

$$P(X > s + t \mid X > s) = P(X > t)$$

3.4.2 Proof

$$P(X > s + t \mid X > s) = \frac{P(X > s + t \text{ AND } X > s)}{P(X > s)}$$

Since $X > s + t$ implies $X > s$:

$$= \frac{P(X > s + t)}{P(X > s)}$$

For $X \sim \mathrm{Exp}(\lambda)$:

$$P(X > x) = 1 - F_X(x) = e^{-\lambda x}$$

Therefore:

$$P(X > s + t \mid X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$
$$= e^{-\lambda t}$$
$$= P(X > t)$$

This completes the proof.

3.5 Example 3: Exponential Distribution

The time (in hours) required to repair a machine is an exponential random variable with parameter $\lambda = \frac{1}{2}$. What is the probability that a repair takes at least 3 hours?

Solution:

Let $X \sim \text{Exp}\left(\frac{1}{2}\right)$.

We need $P(X \geq 3)$.

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) \\ &= 1 - F_X(3) \\ &= 1 - \left(1 - e^{-(1/2)\cdot 3}\right) \\ &= e^{-3/2} \\ &\approx 0.2231 \end{aligned}$$

Alternatively:

$$\begin{aligned} P(X \geq 3) &= \int_3^\infty \frac{1}{2} e^{-(1/2)x}\, dx \\ &= [-e^{-(1/2)x}]_3^\infty \\ &= e^{-3/2} \end{aligned}$$

3.6 Example 4: Memoryless Property Application

Suppose the lifetime of a component (in years) is exponentially distributed with mean 5 years. If the component has already lasted 3 years, what is the probability it will last at least 2 more years?

Solution:

Since $E[X] = 5$ and $E[X] = 1/\lambda$, we have $\lambda = 1/5$.

So $X \sim \text{Exp}(1/5)$.

We need:

$$P(X \geq 5 \mid X \geq 3) = P(X \geq 3 + 2 \mid X \geq 3)$$

By memoryless property:

$$\begin{aligned} P(X \geq 3 + 2 \mid X \geq 3) &= P(X \geq 2) \\ &= 1 - F_X(2) \\ &= 1 - \left(1 - e^{-(1/5)\cdot 2}\right) \\ &= e^{-2/5} \\ &\approx 0.6703 \end{aligned}$$

4 Normal (Gaussian) Distribution

4.1 Definition: Normal Distribution

A continuous random variable X is said to have a normal (or Gaussian) distribution with parameters μ and σ^2 (where $\mu \in \mathbb{R}$ and $\sigma > 0$), denoted $X \sim N(\mu, \sigma^2)$, if its pdf is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty$$

4.2 Parameters of Normal Distribution

- μ: mean (location parameter)
- σ^2: variance (scale parameter)
- σ: standard deviation

4.3 Expectation and Variance of Normal Distribution

For $X \sim N(\mu, \sigma^2)$:

$$E[X] = \mu$$
$$\text{Var}(X) = \sigma^2$$

4.4 Standard Normal Distribution

4.4.1 Definition: Standard Normal Distribution

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a standard normal random variable, denoted $Z \sim N(0, 1)$.

The pdf of Z is:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

The cdf of Z is denoted by:

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

4.5 Properties of Standard Normal CDF

1. $\Phi(-z) = 1 - \Phi(z)$ for all $z \in \mathbb{R}$
2. $P(Z > z) = 1 - \Phi(z)$
3. $P(a < Z \leq b) = \Phi(b) - \Phi(a)$

4.6 Standardization

4.6.1 Theorem: Standardization

If $X \sim N(\mu, \sigma^2)$, then the random variable

$$Z = \frac{X - \mu}{\sigma}$$

has a standard normal distribution, i.e., $Z \sim N(0, 1)$.

4.7 Computing Probabilities for Normal Random Variables

For $X \sim N(\mu, \sigma^2)$:

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P\left(Z \leq \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

4.8 Example 5: Normal Distribution

Let $X \sim N(10, 4)$. Find $P(X \leq 12)$.

Solution:

Here $\mu = 10$ and $\sigma^2 = 4$, so $\sigma = 2$.

$$\begin{aligned} P(X \leq 12) &= P\left(\frac{X-10}{2} \leq \frac{12-10}{2}\right) \\ &= P(Z \leq 1) = \Phi(1) \approx 0.8413 \end{aligned}$$

4.9 Example 6: Normal Distribution Probability

Let $X \sim N(50, 25)$. Find $P(45 < X < 60)$.

Solution:

Here $\mu = 50$ and $\sigma^2 = 25$, so $\sigma = 5$.

$$\begin{aligned} P(45 < X < 60) &= P\left(\frac{45-50}{5} < \frac{X-50}{5} < \frac{60-50}{5}\right) \\ &= P(-1 < Z < 2) \\ &= \Phi(2) - \Phi(-1) \\ &= \Phi(2) - (1 - \Phi(1)) \\ &= \Phi(2) + \Phi(1) - 1 \\ &= 0.9772 + 0.8413 - 1 \\ &= 0.8185 \end{aligned}$$

4.10 Example 7: Finding Values from Probabilities

Let $X \sim N(100, 100)$. Find the value c such that $P(X \leq c) = 0.95$.

Solution:

Here $\mu = 100$ and $\sigma^2 = 100$, so $\sigma = 10$.

$$\begin{aligned} P\left(\frac{X-100}{10} \leq \frac{c-100}{10}\right) &= 0.95 \\ P\left(Z \leq \frac{c-100}{10}\right) &= 0.95 \end{aligned}$$

From standard normal tables, $\Phi(1.645) \approx 0.95$.

$$\begin{aligned} \frac{c-100}{10} &= 1.645 \\ c &= 116.45 \end{aligned}$$

4.11 Linear Transformation of Normal Random Variables

4.11.1 Theorem: Linear Transformation

If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ where $a \neq 0$ and b are constants, then:

$$Y \sim N(a\mu + b, a^2\sigma^2)$$

4.11.2 Proof Outline

Using properties of expectation and variance:

$$E[Y] = E[aX + b] = aE[X] + b = a\mu + b$$

$$\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$$

4.12 Example 8: Linear Transformation

Let $X \sim N(5, 9)$. Define $Y = 2X - 3$. Find the distribution of Y.

Solution:

Here $X \sim N(5, 9)$, so $\mu = 5$ and $\sigma^2 = 9$.

For $Y = 2X - 3$, we have $a = 2$ and $b = -3$.

$$\mu_Y = a\mu + b = 2(5) - 3 = 7$$
$$\sigma_Y^2 = a^2\sigma^2 = 2^2(9) = 36$$

Therefore:

$$Y \sim N(7, 36)$$

5 Gamma Distribution

5.1 Gamma Function

5.1.1 Definition: Gamma Function

The gamma function $\Gamma(\alpha)$ is defined for $\alpha > 0$ by:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

5.1.2 Properties of Gamma Function

1. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$

   Derivation:

   $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

   Using integration by parts with $u = x^{\alpha-1}$, $dv = e^{-x}dx$:

   $$du = (\alpha - 1)x^{\alpha-2}dx, \quad v = -e^{-x}$$

   $$\begin{aligned}\Gamma(\alpha) &= \left[x^{\alpha-1}(-e^{-x})\right]_0^\infty - \int_0^\infty (\alpha - 1)x^{\alpha-2}(-e^{-x})\,dx \\ &= (\alpha - 1)\int_0^\infty x^{\alpha-2}e^{-x}\,dx \\ &= (\alpha - 1)\Gamma(\alpha - 1)\end{aligned}$$

2. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$

   Derivation:

   From property 1:

   $$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots$$

   We need $\Gamma(1)$:

   $$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = 1$$

   Therefore:

   $$\Gamma(n) = (n-1)!$$

3. $\Gamma(1/2) = \sqrt{\pi}$

5.2 Definition: Gamma Distribution

A continuous random variable X is said to have a gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\lambda > 0$, denoted $X \sim \text{Gamma}(\alpha, \lambda)$, if its pdf is given by:

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

5.3 Expectation and Variance of Gamma Distribution

For $X \sim \text{Gamma}(\alpha, \lambda)$:

$$E[X] = \frac{\alpha}{\lambda}$$

$$\text{Var}(X) = \frac{\alpha}{\lambda^2}$$

5.3.1 Derivation of $E[X]$

$$E[X] = \int_0^\infty x \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x}\,dx$$

Let $u = \lambda x$, then $x = u/\lambda$ and $dx = du/\lambda$:

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\lambda}\right)^\alpha e^{-u} \frac{du}{\lambda}$$

$$= \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty u^\alpha e^{-u}\,du$$

$$= \frac{1}{\lambda\Gamma(\alpha)} \Gamma(\alpha+1)$$

Using $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$:

$$E[X] = \frac{\alpha}{\lambda}$$

5.4 Relationship Between Exponential and Gamma Distributions

The exponential distribution is a special case of the gamma distribution:

$$\text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$$

5.4.1 Verification

For $\text{Gamma}(1, \lambda)$:

$$f_X(x) = \frac{\lambda^1}{\Gamma(1)} x^{1-1} e^{-\lambda x} = \lambda e^{-\lambda x}, \quad x > 0$$

which matches the exponential pdf.

5.5 Example 9: Gamma Distribution

Let $X \sim \text{Gamma}(3, 2)$. Find $E[X]$ and $\text{Var}(X)$.

Solution:

Here $\alpha = 3$ and $\lambda = 2$.

$$E[X] = \frac{\alpha}{\lambda} = \frac{3}{2} = 1.5$$

$$\text{Var}(X) = \frac{\alpha}{\lambda^2} = \frac{3}{4} = 0.75$$

5.6 Example 10: Gamma Distribution Application

The time (in hours) to complete a task follows a $\text{Gamma}(4, 1/2)$ distribution. What is the expected time to complete the task?

Solution:

Here $\alpha = 4$ and $\lambda = 1/2$.

$$E[X] = \frac{\alpha}{\lambda} = \frac{4}{1/2} = 8$$

The expected time is 8 hours.