Lecture Scribe: CSE400 – Fundamentals of Probability in Computing
Lecture 9: Uniform, Exponential, Laplace, and Gamma Random Variables
(Prepared strictly from lecture slides content)

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1 Introduction

This lecture discusses types of continuous random variables, focusing on:

- Uniform Random Variable
- Exponential Random Variable
- Laplace Random Variable
- Gamma Random Variable

The lecture includes definitions, probability density functions (PDFs), cumulative distribution functions (CDFs), graphical interpretations, applications, and worked examples.

2 Uniform Random Variable

2.1 Definition

A continuous random variable X is said to be uniformly distributed over the interval $[a, b]$ if its probability density function (PDF) is given by:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x < b \\ 0, & \text{elsewhere} \end{cases}$$

2.2 Cumulative Distribution Function (CDF)

The cumulative distribution function corresponding to the uniform distribution is:

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

2.3 Graphical Representation

The PDF of a uniform random variable is constant over the interval $[a, b]$. The CDF increases linearly from 0 to 1 over the interval.

3 Example 1: Uniform Random Variable

3.1 Problem Statement

The phase of a sinusoid Θ is uniformly distributed over the interval $[0, 2\pi]$. The PDF is given by:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

3.2 Part (a): $\Pr(\Theta > 3\pi/4)$

For a uniform random variable over $[0, 2\pi]$:

$$\Pr(a < \Theta < b) = \frac{b-a}{2\pi}$$

Thus,

$$\Pr(\Theta > 3\pi/4) = \frac{2\pi - 3\pi/4}{2\pi} = \frac{5}{8}$$

3.3 Part (b): $\Pr(\Theta < \pi \mid \Theta > 3\pi/4)$

Using conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Let $A = \{\Theta < \pi\}$ and $B = \{\Theta > 3\pi/4\}$.

$$\Pr(3\pi/4 < \Theta < \pi) = \frac{\pi - 3\pi/4}{2\pi} = \frac{1}{8}$$

Since $\Pr(B) = \frac{5}{8}$,

$$\Pr(\Theta < \pi \mid \Theta > 3\pi/4) = \frac{1/8}{5/8} = \frac{1}{5}$$

3.4 Part (c): $\Pr(\cos\Theta < 1/2)$

The values of Θ satisfying $\cos\Theta = 1/2$ are:

$$\Theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Thus,

$$\cos\Theta < \frac{1}{2} \quad \text{for} \quad \frac{\pi}{3} < \Theta < \frac{5\pi}{3}$$

Therefore,

$$\Pr(\cos\Theta < 1/2) = \frac{5\pi/3 - \pi/3}{2\pi} = \frac{2}{3}$$

4 Applications of Uniform Random Variable

- Phase of a sinusoidal signal when angles between 0 and 2π are equally likely
- Random number generation between 0 and 1 for simulations
- Arrival time of a user within a known time window assuming no preference

5 Exponential Random Variable

5.1 Definition

The exponential random variable has PDF and CDF defined for any $b > 0$.

5.2 Probability Density Function

$$f_X(x) = \frac{1}{b} e^{-x/b} u(x)$$

where $u(x)$ denotes the unit step function.

5.3 Cumulative Distribution Function

$$F_X(x) = [1 - e^{-x/b}] u(x)$$

5.4 Graphical Interpretation

The PDF decreases exponentially as x increases. The CDF increases monotonically and approaches 1 asymptotically.

6 Example 2: Exponential Random Variable

6.1 Problem Statement

Let X be an exponential random variable with PDF:

$$f_X(x) = e^{-x}u(x)$$

6.2 Part (a): $\Pr(3X < 5)$

Since $3X < 5 \Rightarrow X < 5/3$,

$$\Pr(3X < 5) = \Pr(X < 5/3) = 1 - e^{-5/3}$$

6.3 Part (b): Generalization

For arbitrary constant y:

$$\Pr(3X < y) = \Pr(X < y/3) = 1 - e^{-y/3}$$

7 Remaining Topics in Lecture Outline

The lecture outline lists the following additional topics:

- Laplace Random Variable
- Gamma Random Variable
- Graph and Special Cases
- Example
- Homework Problem
- Problem Solving and In-class Activity

Detailed derivations of these topics are not provided within the given slides.

8 Logical Progression of Concepts

1. Introduction to uniform distribution and its properties
2. Application-based problem solving using uniform distribution

3. Introduction to exponential distribution
4. Application-based problems using exponential distribution
5. Preview of additional continuous distributions

9 Summary of Key Mathematical Results

Uniform Random Variable

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x < b \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

Exponential Random Variable

$$f_X(x) = \frac{1}{b} e^{-x/b} u(x)$$

$$F_X(x) = [1 - e^{-x/b}] u(x)$$