Lecture Scribe — CSE400

Lecture 4: Joint Probability and Conditional Probability
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1. Introduction to Probability Theory

1.1 Experiments, Sample Space, and Events

Definitions

- **Experiment (E):** A procedure we perform that produces some result.
  Example: Tossing a coin five times.
- **Outcome (ω):** A possible result of an experiment.
  Example: One outcome of five coin tosses: `HHTHT`.
- **Event (A, B, C, . . .):** A set of outcomes of an experiment.
  Example: Event (C): all outcomes consisting of an even number of heads.
- **Sample Space (S):** The set of all possible distinct outcomes of an experiment.

Outcomes in (S) are:

- **Mutually Exclusive:** Only one can occur at a time.
- **Collectively Exhaustive:** No other outcome is possible.

Sample space can be:

- Discrete
- Countably infinite
- Continuous

Examples

- Flipping a fair coin once
- Rolling a cubical die
- Rolling two dice
- Flipping a coin until tails occurs
- Random number generator on ([0,1))

2. Axioms of Probability

Probability is a function that assigns a numerical measure to events.

Axioms

1. For any event (A):
$$0 \leq Pr(A) \leq 1$$
2. If (S) is the sample space:
$$Pr(S) = 1$$
3. If (A $\cap B = \varnothing$) : $Pr(A \cup B) = Pr(A) + Pr(B)$

   For infinite mutually exclusive events (A_i):
$$Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} Pr(A_i)$$

3. Corollary from Axioms

Corollary 2.1 (Finite case)

For a finite number of mutually exclusive events (A_i):
$$Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} Pr(A_i)$$

4. Propositions from Axioms

- **Proposition 2.1**
$$Pr(A^c) = 1 - Pr(A)$$
- **Proposition 2.2**

  If (A $\subset$ B), then
$$Pr(A) \leq Pr(B)$$
- **Proposition 2.3**
$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$
- **Proposition 2.4 (Inclusion–Exclusion Principle)**
$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n)$$
  expressed using alternating sums of intersections.

5. Assigning Probabilities

5.1 Classical Approach

- Coin flip: (Pr(H) = Pr(T))
- Die roll: (Pr(even))
- Two dice: (Pr(sum = 5))

5.2 Relative Frequency Approach

$$Pr(A) = \lim_{n \to \infty} \frac{\text{Number of times A occurs}}{n}$$

Limitations:

- Requires infinite repetitions
- Many phenomena are not repeatable

6. Joint Probability

6.1 Motivation

Events are not always mutually exclusive. We are interested in:

$$Pr(A \cap B)$$

6.2 Notation

$$Pr(A, B) \quad \text{or} \quad Pr(A \cap B)$$

For multiple events:

$$Pr(A_1, A_2, \ldots, A_n)$$

6.3 Calculation Approaches

- **Classical:** Express events using atomic outcomes and find common outcomes.
- **Relative Frequency:**

$$Pr(A, B) = \frac{n_{A,B}}{n}$$

Example 1: Card Deck Example

52 cards, 13 per suit.

Let:

- (A): Red card
- (B): Number card (Ace included)
- (C): Heart card

Individual Probabilities

$$Pr(A) = \frac{26}{52} = \frac{1}{2}$$

$$Pr(B) = \frac{40}{52} = \frac{5}{26}$$

$$Pr(C) = \frac{13}{52} = \frac{1}{4}$$

Joint Probabilities

$$Pr(A, B) = \frac{20}{52}$$

$$Pr(A, C) = \frac{13}{52}$$

$$Pr(B, C) = \frac{10}{52}$$

Example 2: Costume Party

Tops: 3 T-shirts, 1 cape → (Pr(Cape) = $\frac{1}{4}$)

Bottoms: 2 pants, 4 boxers → (Pr(Boxers) = $\frac{4}{6}$)

Joint probability:

$$Pr(\text{Cape and Boxers}) = \frac{1}{4} \times \frac{4}{6} = \frac{1}{6}$$

Correct option: C

7. Conditional Probability

7.1 Motivation

Occurrence of one event may depend on another.

7.2 Definition

$$Pr(A|B) = \frac{Pr(A, B)}{Pr(B)}, \quad Pr(B) > 0$$

7.3 Product Rule

$$Pr(A, B) = Pr(A|B)Pr(B) = Pr(B|A)Pr(A)$$

7.4 Three Events

$$Pr(A, B, C) = Pr(C|A, B)Pr(B|A)Pr(A)$$

7.5 Chain Rule

$$Pr(A_1, A_2, \ldots, A_n) = Pr(A_n|A_1, \ldots, A_{n-1}) \ldots Pr(A_2|A_1)Pr(A_1)$$

Example 3: Cards Without Replacement

Two cards drawn, no replacement.

- (A): First card is Spade
- (B): Second card is Spade

After one spade removed:

- Remaining cards = 51
- Remaining spades = 12

$$Pr(B|A) = \frac{12}{51}$$

Example 4: Game of Poker — Flush

Flush in Spades

$$Pr = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48}$$

Any Flush

$$4 \times Pr(\text{Spade Flush})$$

Example 5: The Missing Key

Given:

$$P(L) = 0.4, \quad P(R) = 0.4, \quad P(K) = 0.8$$

Find:

$$P(R|L^c)$$

Since key cannot be in both pockets:

$$P(R|L^c) = \frac{P(R)}{P(L^c)} = \frac{0.4}{0.6} = \frac{2}{3}$$

Correct option: C

End of Lecture