Lecture 9: Uniform, Exponential, Laplace, and Gamma Random Variables

Course: CSE400 – Fundamentals of Probability in Computing
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1 Uniform Random Variable

1.1 Mathematical Definition

A continuous random variable X is uniform over the interval $[a, b]$ if its probability density function (PDF) and cumulative distribution function (CDF) are given by:

Probability Density Function (PDF):

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x < b \\ 0, & \text{elsewhere} \end{cases}$$

Cumulative Distribution Function (CDF):

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$$

1.2 Application Examples

The phase of a sinusoidal signal, where all angles between 0 and 2π are equally likely.

A random number generated by a computer between 0 and 1 for simulations.

The arrival time of a user within a known time window, assuming no time preference.

1.3 Worked Example: Phase of a Sinusoid

Problem: The phase of a sinusoid, Θ, is uniformly distributed over $[0, 2\pi)$.

Find $\Pr(\Theta > \frac{4\pi}{3})$:

$$\Pr(\Theta > \tfrac{4\pi}{3}) = \frac{2\pi - \frac{4\pi}{3}}{2\pi} = \frac{5}{8}$$

Find $\Pr(\Theta < \pi \mid \Theta > \frac{4\pi}{3})$:
Using $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$:

$$\Pr\big(\tfrac{4\pi}{3} < \Theta < \pi\big) = \frac{\pi/4}{2\pi} = \frac{1}{8}$$

$$\Pr(\Theta < \pi \mid \Theta > \tfrac{4\pi}{3}) = \frac{1/8}{5/8} = \frac{1}{5}$$

Find $\Pr(\cos(\Theta) < \frac{1}{2})$:

$$\cos(\Theta) = \tfrac{1}{2} \Rightarrow \Theta = \tfrac{\pi}{3}, \tfrac{5\pi}{3}$$

$$\Pr(\cos(\Theta) < \tfrac{1}{2}) = \frac{\frac{5\pi}{3} - \frac{\pi}{3}}{2\pi} = \frac{2}{3}$$

2 Exponential Random Variable

2.1 Mathematical Definition (for $b > 0$)

PDF:

$$f_X(x) = \frac{1}{b} \exp\Big(-\frac{x}{b}\Big)\, u(x)$$

CDF:

$$F_X(x) = \Big[1 - \exp\Big(-\frac{x}{b}\Big)\Big]\, u(x)$$

2.2 Worked Example: Transformation of RV

Problem: Let X be an exponential RV with $f_X(x) = e^{-x}u(x)$. Define $Y = 3X$.

$$F_Y(y) = F_X\Big(\frac{y}{3}\Big) = \Big[1 - e^{-y/3}\Big]\, u(y)$$

$$f_Y(y) = \frac{1}{3} e^{-y/3} u(y)$$

3 Laplace Random Variable

3.1 Mathematical Definition

$$f_X(x) = \frac{1}{2b} \exp\Bigg(-\frac{|x|}{b}\Bigg)$$

$$F_X(x) = \begin{cases} \frac{1}{2} \exp\big(\frac{x}{b}\big), & x < 0 \\ 1 - \frac{1}{2} \exp\big(-\frac{x}{b}\big), & x \geq 0 \end{cases}$$

4 Gamma Random Variable

4.1 Mathematical Definition

$$f_X(x) = \frac{1}{b\Gamma(c)} \left(\frac{x}{b}\right)^{c-1} \exp(-x/b)\, u(x)$$

$$F_X(x) = \frac{\gamma(c, x/b)}{\Gamma(c)}\, u(x)$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

$$\gamma(\alpha, \beta) = \int_0^\beta e^{-t} t^{\alpha-1} dt$$

5 Problem Solving Exercises

Given: $F_X(x) = x^2$ for $0 \leq x \leq 1$.

$$f_X(x) = 2x$$

$$\mu_X = \int_0^1 x(2x)\, dx = \frac{2}{3}$$

$$\sigma_X^2 = \int_0^1 x^2(2x)\, dx - \mu_X^2 = \frac{1}{18}$$