Lecture Notes: Probability Distributions

L9_S1_A (1)

P PPTX

Generate a lecture scribe intended to serve as exam reference material for a reading-based, closed-notes examination.

Use only the provided context, which consists of the lecture slides/PDF and the relevant textbook content.

The scribe must faithfully reconstruct what was taught in the lecture so that a student can rely on it later for revision.

Include definitions, notation, assumptions, theorem or result statements, proofs or derivations, and worked examples only if they appear in the provided context.

Do not introduce new examples, explanations, intuition, or material that is not present in the context.

Maintain strict faithfulness to the lecture's structure, scope, and logical flow.

Organize the scribe clearly with appropriate sections and headings so it can be used directly for exam preparation. make sure this is the only prompt no imporvisation would be done after this

Lecture 9: Uniform, Exponential, Laplace, and Gamma Random Variables Course: CSE400 – Fundamentals of Probability in Computing **Instructor:** Dhaval Patel, PhD **Date:** February 2, 2026

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1. Uniform Random Variable

1.1 Mathematical Definition

A continuous random variable X is uniform over the interval $[a, b]$ if its probability density function (PDF) and cumulative distribution function (CDF) are given by:

- **Probability Density Function (PDF):**

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & \text{elsewhere} \end{cases}$$

- **Cumulative Distribution Function (CDF):**

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

1.2 Application Examples

- The **phase of a sinusoidal signal**, where all angles between 0 and 2π are equally likely.
- A **random number generated** by a computer between 0 and 1 for simulations.
- The **arrival time of a user** within a known time window, assuming no time preference.

1.3 Worked Example: Phase of a Sinusoid

Problem: The phase of a sinusoid, Θ, is uniformly distributed over $[0, 2\pi)$. **Solutions:**

- **Find $Pr(\Theta > \frac{3\pi}{4})$:**

$$Pr(\Theta > \frac{3\pi}{4}) = \frac{2\pi - \frac{3\pi}{4}}{2\pi} = \frac{5}{8}$$

- **Find $Pr(\Theta < \pi \mid \Theta > \frac{3\pi}{4})$:** Using $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$: $Pr(\frac{3\pi}{4} < \Theta < \pi) = \frac{\pi/4}{2\pi} = \frac{1}{8}$. $Pr(\Theta < \pi \mid \Theta > \frac{3\pi}{4}) = \frac{1/8}{5/8} = \frac{1}{5}$.
- **Find $Pr(\cos(\Theta) < \frac{1}{2})$:** $\cos(\Theta) = \frac{1}{2} \implies \Theta = \frac{\pi}{3}, \frac{5\pi}{3}$. $\cos(\Theta) < \frac{1}{2}$ for $\frac{\pi}{3} < \Theta < \frac{5\pi}{3}$. $Pr(\cos(\Theta) < \frac{1}{2}) = \frac{\frac{5\pi}{3} - \frac{\pi}{3}}{2\pi} = \frac{4\pi/3}{2\pi} = \frac{2}{3}$.

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2. Exponential Random Variable

2.1 Mathematical Definition (for $b > 0$)

- **PDF:** $f_X(x) = \frac{1}{b}\exp(-\frac{x}{b})u(x)$
- **CDF:** $F_X(x) = [1 - \exp(-\frac{x}{b})]u(x)$ *(Note: $u(x)$ is the unit step function, indicating the function is 0 for $x < 0$).*

2.2 Application Examples

- **Time until a radioactive particle decays**, assuming a constant decay rate.
- **Duration of an idle period** of a wireless communication channel before it becomes busy.
- **Time until the next earthquake** in a simplified seismic activity model.

2.3 Worked Example: Transformation of RV

Problem: Let X be an exponential RV with $f_X(x) = e^{-x}u(x)$. Define $Y = 3X$. **Solution for $f_Y(y)$:**

1. **CDF of X:** $F_X(x) = [1 - e^{-x}]u(x)$.
2. **CDF of Y:** $Pr(3X < y) = Pr(X < \frac{y}{3}) = F_X(\frac{y}{3}) = [1 - e^{-y/3}]u(y)$.
3. **PDF of Y:** Differentiating $F_Y(y)$ gives $f_Y(y) = \frac{1}{3}e^{-y/3}u(y)$. **Conclusion:** Y is an exponential random variable with parameter $\lambda = \frac{1}{3}$.

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3. Laplace Random Variable

3.1 Mathematical Definition

- **PDF:** $f_X(x) = \frac{1}{2b}\exp(-\frac{|x|}{b})$
- **CDF:**

$$F_X(x) = \begin{cases} \frac{1}{2}\exp(\frac{x}{b}), & x < 0 \\ 1 - \frac{1}{2}\exp(-\frac{x}{b}), & x \geq 0 \end{cases}$$

3.2 Application Examples

- Modeling the **amplitude of a speech (voice) signal**.
- Modeling **impulsive noise** in communication systems where large noise spikes occur frequently.
- **Error distribution in L1 regression** (minimizing absolute error).
- **Differential privacy mechanisms**, where Laplace noise protects sensitive database information.

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4. Gamma Random Variable

4.1 Mathematical Definition

- **PDF:** $f_X(x) = \frac{(x/b)^{c-1}\exp(-x/b)}{b\Gamma(c)}u(x)$
- **CDF:** $F_X(x) = \frac{\gamma(c,x/b)}{\Gamma(c)}u(x)$
- **Gamma Function:** $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$
- **Incomplete Gamma Function:** $\gamma(\alpha, \beta) = \int_0^\beta e^{-t}t^{\alpha-1}dt$

4.2 Special Cases

- **Chi-square RV:** Results when $b = 2$ and $c = 0.5$.
- **Exponential RV:** Results when $c = 1$. *(Note: parameters are often denoted as $k = c$ and $\theta = b$).*

4.3 Statistical Properties and Applications

- **Mean:** $E[X] = \frac{\alpha}{\lambda}$
- **Variance:** $Var(X) = \frac{\alpha}{\lambda^2}$
- **Applications:**
  - **Total service time** for completing multiple independent tasks with exponentially distributed durations.
  - **Received signal energy** in detection systems (sum of multiple independent signal samples).
  - **Aggregate waiting time** until the k-th event in a Poisson process.

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5. Problem Solving Exercises

5.1 Optimization: Movie Theater Departure

Scenario: Fixed showtime T. Early arrival cost $c \cdot s$; late arrival cost $k \cdot s$. Travel time T_r is a random variable with PDF $f(t)$ and CDF $F(t)$. **Goal:** Minimize expected cost $E[C(d)]$. **Optimality Condition:** $F(T - d^*) = \frac{k}{c+k}$. **Interpretation:** Depart such that the probability of arriving early equals $\frac{k}{c+k}$.

5.2 CDF Analysis

Given: $F_X(x) = x^2$ for $0 \leq x \leq 1$.

- **PDF:** $f_X(x) = \frac{d}{dx}F_X(x) = 2x$.
- **Mean (μ_X):** $\int_0^1 x(2x)dx = \frac{2}{3}$.
- **Variance (σ_X^2):** $\int_0^1 x^2(2x)dx - \mu_X^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.
- **Skewness (c_s):** $-\frac{2\ \ 2}{5}$ (**Left-skewed**).
- **Kurtosis (c_k):** $\frac{12}{5}$ (**Platykurtic**; lighter tails than Gaussian).

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6. Gaussian Modeling and Density Estimation

6.1 Modeling Uncertainty

- **Reality:** Sensors do not give "flat" lines due to thermal noise and interference; repeated measurements form a "cloud" of uncertainty.
- **Simulation:** Noise is modeled using the Gaussian RV: $X \sim \mathcal{N}(\mu, \sigma^2)$.
- **Generative Formula:** $X = \sigma Z + \mu$.

6.2 Density Estimation

We observe raw noisy samples (a histogram) and use **Sample Statistics** ($\hat{\mu}, \hat{\sigma}$) to estimate the distribution (PDF).

- **Key Concept:** Estimating probability distributions "smooths" noise into a predictive tool.
- **Network Application:** Modeling **Packet Delay** (latency analysis) to understand typical delay ($\hat{\mu}$) and jitter/variation ($\hat{\sigma}$).

Would you like me to generate a set of practice problems based on the derivations for Mean and Variance of the Gamma distribution shown in Slides 38 and 39?