CSE400: Fundamentals of Probability in Computing
Lecture 9 – Uniform, Exponential, Laplace and Gamma Random Variables

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Outline

- Types of Continuous Random Variables
  - Uniform Random Variable: Example
  - Exponential Random Variable: Example
  - Laplace Random Variable: Example
  - Gamma Random Variable
- Graphs and Special Cases
- Problem Solving
- Applications

1 Uniform Random Variable

Definition (PDF)

Let X be a uniform random variable on $[a, b)$. The probability density function (PDF) is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x < b, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition (CDF)

The cumulative distribution function (CDF) is

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x < b, \\ 1, & x \ge b. \end{cases}$$

2 Example 1: Uniform Random Variable

Problem

The phase of a sinusoid, Θ, is uniformly distributed over $[0, 2\pi)$ with PDF

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

(a) $\Pr(\Theta > \frac{3\pi}{4})$

For a uniform random variable on $[0, 2\pi)$:

$$\Pr(a < \Theta < b) = \frac{b-a}{2\pi}.$$

Thus,

$$\Pr\left(\Theta > \frac{3\pi}{4}\right) = \frac{2\pi - \frac{3\pi}{4}}{2\pi} = \frac{5}{8}.$$

(b) $\Pr(\Theta < \pi \mid \Theta > \frac{3\pi}{4})$

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

$$\Pr\left(\frac{3\pi}{4} < \Theta < \pi\right) = \frac{\pi - \frac{3\pi}{4}}{2\pi} = \frac{1}{8},$$

$$\Pr\left(\Theta > \frac{3\pi}{4}\right) = \frac{5}{8}.$$

Therefore,

$$\Pr\left(\Theta < \pi \mid \Theta > \frac{3\pi}{4}\right) = \frac{1/8}{5/8} = \frac{1}{5}.$$

(c) $\Pr(\cos(\Theta) < \frac{1}{2})$

$$\cos(\Theta) = \frac{1}{2} \Rightarrow \Theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

$$\cos(\Theta) < \frac{1}{2} \quad \text{for} \quad \frac{\pi}{3} < \Theta < \frac{5\pi}{3}.$$

$$\Pr\left(\cos(\Theta) < \frac{1}{2}\right) = \frac{\frac{5\pi}{3} - \frac{\pi}{3}}{2\pi} = \frac{2}{3}.$$

3 Exponential Random Variable

Definition

The exponential random variable has PDF and CDF (for $b > 0$):

$$f_X(x) = \frac{1}{b} e^{-x/b} u(x),$$

$$F_X(x) = \left[1 - e^{-x/b}\right] u(x).$$

4 Example 2: Exponential Random Variable

Problem

Let X be an exponential random variable with PDF $f_X(x) = e^{-x} u(x)$.

(a) $\Pr(3X < 5)$

$$\Pr(3X < 5) = \Pr\left(X < \frac{5}{3}\right) = F_X\left(\frac{5}{3}\right) = 1 - e^{-5/3}.$$

(b) $\Pr(3X < y)$

$$\Pr(3X < y) = \Pr\left(X < \frac{y}{3}\right) = F_X\left(\frac{y}{3}\right) = \left[1 - e^{-y/3}\right] u(y).$$

(c) Let $Y = 3X$. Find $f_Y(y)$.

From part (b),

$$F_Y(y) = \left[1 - e^{-y/3}\right] u(y).$$

Differentiating,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{3} e^{-y/3} u(y).$$

5 Laplace Random Variable

Definition

$$f_X(x) = \frac{1}{2b} \exp\left(-\frac{|x|}{b}\right).$$

$$F_X(x) = \begin{cases} \frac{1}{2} e^{x/b}, & x < 0, \\ 1 - \frac{1}{2} e^{-x/b}, & x \geq 0. \end{cases}$$

6 Example 3: Laplace Random Variable

Let W have PDF $f_W(w) = ce^{-2|w|}$.

(a) Find c

$$\int_{-\infty}^{\infty} ce^{-2|w|} dw = 2c \int_0^{\infty} e^{-2w} dw = c = 1.$$

(b) $\Pr(-1 < W < 2)$

$$\Pr(-1 < W < 2) = \frac{1}{2}(1 - e^{-2}) + \frac{1}{2}(1 - e^{-4}) = 1 - \frac{1}{2}(e^{-2} + e^{-4}).$$

(c) $\Pr(W > 0 \mid -1 < W < 2)$

$$\Pr(W > 0 \mid -1 < W < 2) = \frac{\frac{1}{2}(1 - e^{-4})}{1 - \frac{1}{2}(e^{-2} + e^{-4})}.$$

7 Gamma Random Variable

Definition

$$f_X(x) = \frac{x^{\alpha-1} e^{-x/b}}{b^\alpha \Gamma(\alpha)} u(x).$$

$$F_X(x) = \frac{\gamma(\alpha, x/b)}{\Gamma(\alpha)} u(x).$$

Gamma Functions

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

$$\gamma(\alpha, \beta) = \int_0^\beta t^{\alpha-1} e^{-t} dt.$$

8 Example 4: Gamma Random Variable

Let $X \sim \text{Gamma}(\alpha, \lambda)$.

Mean

$$E[X] = \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}.$$

Variance

$$E[X^2] = \frac{\alpha(\alpha+1)}{\lambda^2}, \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}.$$

9 Example 5: Properties of Gamma Function

(a) $\Gamma(n) = (n-1)!$, $n = 1, 2, 3, \ldots$

(b) $\Gamma(x+1) = x\Gamma(x)$

(c) $\Gamma(1/2) = \sqrt{\pi}$

10 Problem Solving 1: Movie Theater

Arrival time:

$$A = d + T_r.$$

Expected cost:

$$E[C(d)] = \int_0^{T-d} c(T-d-t) f(t) dt + \int_{T-d}^\infty k(t+d-T) f(t) dt.$$

Optimality condition:

$$F(T-d^*) = \frac{k}{c+k}, \quad d^* = T - F^{-1}\left(\frac{k}{c+k}\right).$$

11 Problem Solving 2: CDF Analysis

Given

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

$$f_X(x) = 2x, \ 0 \le x \le 1.$$

$$\mu = \frac{2}{3}, \quad \sigma^2 = \frac{1}{18}, \quad c_s = -\frac{2\sqrt{2}}{5}, \quad c_k = \frac{12}{5}.$$

Density Estimation: Learning from Data and Applications

Density estimation is the problem of estimating the probability density function (PDF) of a continuous random variable using observed data. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed samples drawn from an unknown distribution with density $f_X(x)$. The objective is to construct an estimator $\hat{f}_X(x)$ such that $\hat{f}_X(x) \approx f_X(x)$.

Histogram-Based Density Estimation

The histogram method partitions the range of data into disjoint intervals (bins) of equal width h. Let N_k denote the number of observations that fall into the k-th bin. The histogram density estimator is defined as

$$\hat{f}_X(x) = \frac{N_k}{nh}, \quad x \text{ belongs to bin } k.$$

This estimator satisfies the normalization condition

$$\int_{-\infty}^{\infty} \hat{f}_X(x)\, dx = 1.$$

Kernel Density Estimation

Kernel density estimation provides a smooth estimate of the density. Let $K(\cdot)$ be a kernel function such that

$$\int_{-\infty}^{\infty} K(u)\, du = 1.$$

For a bandwidth parameter $h > 0$, the kernel density estimator is given by

$$\hat{f}_X(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right).$$

Learning from Data

Density estimation is a fundamental task in learning from data. It enables probabilistic modeling of unknown continuous distributions using finite samples. Once a density is estimated, probabilities, expectations, and likelihoods can be computed directly from data.

Applications

Density estimation is widely used in pattern recognition, anomaly detection, signal processing, and statistical inference. It also forms the basis for clustering, classification, and probabilistic decision-making systems.