Date: / /

Solution 1:

To prove: $\sum_x |f(x)|^2 = \sum_\omega |F(\omega)|^2$

Given: $F(\omega) = \frac{1}{\sqrt{N}} \sum_x f(x) e^{-i\omega x}$

$$f(x) = \frac{1}{\sqrt{N}} \sum_\omega F(\omega) e^{i\omega x}$$

LHS $= \sum_x |f(x)|^2$

Since $f(x) = \frac{1}{\sqrt{N}} \sum_\omega F(\omega) e^{i\omega x}$, (given)

Complex conjugate: $f^*(x) = \frac{1}{\sqrt{N}} \sum_{\omega'} F^*(\omega') e^{-i\omega' x}$

$\therefore |f(x)|^2 = f(x) . f^*(x)$

$$= \left\{\frac{1}{\sqrt{N}} \sum_\omega F(\omega) e^{i\omega x}\right\} . \left\{\frac{1}{\sqrt{N}} \sum_{\omega'} F^*(\omega') e^{-i\omega' x}\right\}$$

$$= \frac{1}{\sqrt{N}} \sum_\omega \sum_{\omega'} F(\omega) F^*(\omega') e^{i(\omega-\omega')x}$$

$\therefore \sum_x |f(x)|^2 = \frac{1}{N} . \sum_\omega \sum_{\omega'} F(\omega) F^*(\omega') \left(\sum_x e^{i(\omega-\omega')x}\right).$

--- by summing over x.

Here,

$$\sum_x \cancel{t} e^{i(\omega-\omega')x} = \begin{cases} N & \text{if } \omega = \omega' \\ 0 & \text{if } \omega \neq \omega' \end{cases}$$

Date: / /

Because: if $\omega = \omega'$: $e^{i(\omega-\omega')x} = e^0 = 1 \Rightarrow$ Sum $= N$

if $\omega \neq \omega'$, we get geometric series with $r = e^{i(\omega-\omega')}$

for $\omega = 2\pi k/N$; $r^N = e^{i(\omega-\omega')N}$ (from above)

$\therefore r^N = \cancel{e^{i(\omega-\omega')}}\ e^{i\left(\frac{2\pi k}{N} - \frac{2\pi k'}{N}\right)N}$

$\therefore r^N = e^{i2\pi(k-k')} = \cos(2\pi(k-k')) + i\sin(2\pi(k-k'))$

Since $k-k'$ is an integer, $r^N = 1$ — ①

$$\sum_x e^{i(\omega-\omega')x} = \sum_x r^x = r^0 + r^1 + r^2 + r^3 + \dots + r^{N-1}$$

$$= \frac{r^N - 1}{r-1} \quad \text{---- } r \neq 1 \text{ because } \omega \neq \omega'.$$

But we have $r^N = 1$ from ①.

$$\therefore \frac{r^N - 1}{r-1} = 0$$

$\therefore$ we can write relation using Kronecker delta.

$$\sum_x e^{i(\omega-\omega')x} = N \cdot \delta_{\omega\omega'} \qquad \left\{ \delta_{\omega\omega'} = \begin{cases} 1 & \text{if } \omega = \omega' \\ 0 & \text{if } \omega \neq \omega' \end{cases} \right.$$

By substituting back

$$\sum_x |f(x)|^2 = \frac{1}{\cancel{N}} \sum_\omega \sum_{\omega'} F(\omega) F^*(\omega') \left[\cancel{N} \cdot \delta_{\omega\omega'}\right]$$

$$= \sum_\omega \sum_{\omega'} F(\omega) F^*(\omega') \cdot \delta_{\omega\omega'}$$

while double summing, $\delta_{\omega\omega'}$ zeros out all terms except where $\omega = \omega'$

Date: / /

$$= \sum_{\omega} F(\omega) F^*(\omega)$$

$$= \sum_{\omega} |F(\omega)|^2$$ --- by property of complex conjugate

$= RHS.$

Thus, we have proved Parseval's Theorem