Date: / /

(a) Distributive Property (Discrete time) - Proof.

To prove: $f[n] * (g[n] + h[n]) = (f[n] * g[n]) + (f[n] * h[n])$

By the definition of Convolution, we know:

$$f[n] * g[n] = \sum_k f[k] \cdot g[n-k].$$

Similarly:

$$LHS = \sum_k f[k] \cdot (g+h)[n-k]$$

$$= \sum_k f[k] \cdot (g[n-k] + h[n-k])$$

$$= \sum_k (f[k] \cdot g[n-k] + f[k] \cdot h[n-k])$$

Splitting the summation across the addition, we get.

$$LHS = \sum_k f[k] \cdot g[n-k] + \sum_k f[k] \cdot h[n-k]$$

Now, consider RHS.

$$RHS = (f[n] * g[n]) + (f[n] * h[n])$$

By definition of Convolution:

$$RHS = \sum_k f[k] \cdot g[n-k] + \sum_k f[k] \cdot h[n-k]$$

Thus, because LHS = RHS, we have proved the Distributive Property (Discrete time) of the Convolution operator.

Date: / /

Ans 1. (b) Associative Property (Continuous-Time) - Proof.

To prove: $(f(t) * g(t)) * h(t) = f(t) * (g(t) * h(t))$

Consider LHS:

By definition of Convolution.

$$f(t) * g(t) = \int f(\tau_1) \cdot g(t - \tau_1)\, d\tau_1 \quad --- ①$$

Now: $(f(t) * g(t)) * h(t)$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau_1) \cdot g(\tau_2 - \tau_1)\, d\tau_1 \right] h(t - \tau_2)\, d\tau_2$$

--- Reasoning: [Consider $u(t) = f(t) * g(t)$ ---- ②

Then Convolute $u(t)$ & $h(t)$

This will give:

$$\int u(\tau_2) \cdot h(t - \tau_2)\, d\tau_2.$$

Then substitute u from ① & ②.]

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau_1) \cdot g(\tau_2 - \tau_1) \cdot h(t - \tau_2)\, d\tau_1\, d\tau_2.$$

Date: / /

Consider RHS :

$$u(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau_1) \cdot h(t-\tau_1)\, d\tau_1 \quad \text{—③}$$

$$\therefore f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau_2) \cdot u(t-\tau_2)\, d\tau_2 .$$

Substituting from ①, the value of $u(t-\tau_2)$.

$$RHS = \int_{-\infty}^{\infty} f(\tau_2) \left[\int_{-\infty}^{\infty} g(\tau_1) \cdot h((t-\tau_2)-\tau_1)\, d\tau_1 \right] d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau_2) \cdot g(\tau_1) \cdot h(t-\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

—④.

Consider LHS again :

$$LHS = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau_1) \cdot g(\tau_2-\tau_1) \cdot h(t-\tau_2)\, d\tau_1\, d\tau_2 .$$

Let $u = \tau_2 - \tau_1$ & $v = \tau_1$

$\therefore \tau_1 = v$

& $\tau_2 = u + v$

$$\frac{\partial(\tau_1, \tau_2)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial \tau_1}{\partial u} & \frac{\partial \tau_1}{\partial v} \\ \frac{\partial \tau_2}{\partial u} & \frac{\partial \tau_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$|J| = 1 .$

Date: / /

$\therefore$ LHS = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v) \cdot g(u) \cdot h(t-u-v)\, du \cdot dv$.

Rename: $v = \tau_2$, $u = \tau_1$.

we get LHS = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau_2) \cdot g(\tau_1) \cdot h(t-\tau_1-\tau_2)\, d\tau_1 d\tau_2$

$\therefore$ LHS = RHS.

Thus we have proved associative property of convolution in continous time.