# Project 1: Hunter Drone Net Stabilization

Pratik Prajapati

## Contents

**1 Single Quadcopter with MRS Net Mechanism First Rigid Link and Connected** **2**
1.1 Some Preliminaries . . . 2
1.2 Derivation of EOM for Single Quadcopter with Constrained Link . . . 2
1.3 Dynamical Model for Single Quadcopter with MRS Net Mechanism First Rigid Link with Free Ends . . . 13
1.3.1 Special Case 1: . . . 55

**A Useful Derivations** **57**
A.1 Useful Properties . . . 57
A.2 Useful Derivations on $\mathbb{S}^2$ . . . 58
A.3 Useful Derivations on $\mathbb{SO}(3)$ . . . 60
A.4 Useful Derivations used for Lagrangian . . . 61

# 1 Single Quadcopter with MRS Net Mechanism First Rigid Link and Connected

## 1.1 Some Preliminaries

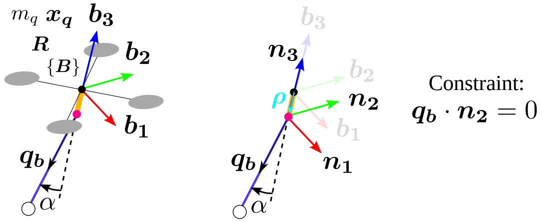


Figure 1: Line Diagram of Quadcopter with Single Link with Constrained Link Motion

As shown in Fig. 1, consider the line diagram of a quadcopter with a constrained link. This link is constraint to oscillate along $\boldsymbol{n_2}$ axis only. Consider the angular position of the link from $-\boldsymbol{n_3}$ axis is $\alpha$. Hence, the expression for the unit vector which represents the direction of the link is given as follows.

$$\boldsymbol{q_b} = \begin{bmatrix} -sin\alpha & 0 & -cos\alpha \end{bmatrix}^T \tag{1}$$

$$\begin{aligned} \boldsymbol{x_p} &= \boldsymbol{x_q} + \boldsymbol{R}\boldsymbol{\rho} + l\boldsymbol{R}\boldsymbol{q_b} \\ \boldsymbol{x_p} &= \boldsymbol{x_q} + \boldsymbol{R}(\boldsymbol{\rho} + l\boldsymbol{q_b}) \end{aligned} \tag{2}$$

Overall, the configuration space of the system is $\mathbb{R}^3 \times \mathbb{SO}(3) \times \mathbb{S}^1$ with a total of seven degrees of freedom (DOF) - six DOF of the quadcopter and one DOF for the payload attitude, as there is inherent constraint $\boldsymbol{q_b} \cdot \boldsymbol{n_2} = 0$. Expressing the angular velocity of the quadcopter with respect to frame $\{B\}$ as $\boldsymbol{\Omega} \in \mathbb{R}^3$ respectively, the kinematic relations for the quadcopter's attitude are as follows:

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\Omega}} \tag{3}$$

Here, the *hat map* $\widehat{\cdot}: \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined as $\widehat{\boldsymbol{x}}\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, where $\mathfrak{so}(3)$ is the skew-symmetric matrix. Using the Lagrange-d'Alembert principle on a manifold [1], the system's equation of motion (EOM) is derived as follows.

## 1.2 Derivation of EOM for Single Quadcopter with Constrained Link

First we will derive the expressions of total kinetic and potential energy of the system. The total kinetic energy ($\mathcal{T}$) of the system is given by the summation of the total kinetic energy of the quadcopter and the total kinetic energy of the payload. The total kinetic energy of the quadcopter is written as given in Eq. (5). Time derivative of position of the payload is

$$\dot{\boldsymbol{x}}_{\boldsymbol{p}} = \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \boldsymbol{R}(l\dot{\boldsymbol{q}}_{\boldsymbol{b}}) \tag{4}$$

Hence, the kinetic energy of the payload is written as given in Eq. (6). Using Eqs. (5 & 6), the total kinetic energy of the system is derived as given in Eq. (7).

$$\mathcal{T}_q = \frac{1}{2}m_q\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) \tag{5}$$

$$\begin{aligned}
\mathcal{T}_p &= \frac{1}{2}m_p\|\dot{\boldsymbol{x}}_{\boldsymbol{p}}\|^2 \\
&= \frac{1}{2}m_p\|\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) + \boldsymbol{R}(l\dot{\boldsymbol{q}}_{\boldsymbol{b}})\|^2 \\
&= \frac{1}{2}m_p\|\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) + l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
\mathcal{T}_p &= \frac{1}{2}m_p\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}m_p\|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})\|^2 + \frac{1}{2}m_p\|l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) + m_p(\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) \cdot l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}})
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathcal{T} &= \mathcal{T}_q + \mathcal{T}_p \\
&= \frac{1}{2}m_q\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}m_p\|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})\|^2 + \frac{1}{2}m_p\|l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) + m_p(\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) \cdot l\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) \\
&= \frac{1}{2}(m_q + m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})\|^2 + \frac{1}{2}m_p l^2\|\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) + m_p l(\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) \cdot \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}})
\end{aligned} \tag{7}$$

The total potential energy of the system is the sum of the total potential energy of the quadcopter and the total potential energy of the payload and its expression is given in Eq. (8).

$$\begin{aligned}
\mathcal{V} &= m_q g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{q}} + m_p g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{p}} \\
&= m_q g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{q}} + m_p g\boldsymbol{e}_{\boldsymbol{3}} \cdot (\boldsymbol{x}_{\boldsymbol{q}} + \boldsymbol{R}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) \\
\mathcal{V} &= (m_q + m_p)g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{q}} + m_p g\boldsymbol{e}_{\boldsymbol{3}} \cdot (\boldsymbol{R}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) \\
\mathcal{V} &= (m_q + m_p)g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{q}} + m_p g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{R}\boldsymbol{\rho} + m_p l g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}
\end{aligned} \tag{8}$$

Using Eqs. (7 & 8), the Lagrangian ($\mathcal{L}$) of the system is written as given in Eq. (10).

$$\begin{aligned}
\mathcal{L} &= \mathcal{T} - \mathcal{V} \\
\mathcal{L} &= \frac{1}{2}(m_q + m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})\|^2 + \frac{1}{2}m_p l^2\|\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) + m_p l(\dot{\boldsymbol{R}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) \cdot \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) \\
&\quad - (m_q + m_p)g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{q}} - m_p g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p l g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}
\end{aligned} \tag{9}$$

Using the relation, $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\Omega}}$, the above equation is updated as follows

$$\begin{aligned}
\mathcal{L} &= \mathcal{T} - \mathcal{V} \\
\mathcal{L} &= \frac{1}{2}(m_q + m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})\|^2 + \frac{1}{2}m_p l^2\|\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q}_{\boldsymbol{b}}) \cdot \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) \\
&\quad - (m_q + m_p)g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{x}_{\boldsymbol{q}} - m_p g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p l g\boldsymbol{e}_{\boldsymbol{3}} \cdot \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}
\end{aligned} \tag{10}$$

According to Lagrange-d'Alembert principle [2], the infinitesimal variation in the action integral ($\mathcal{I}$) is equal to the negative of the infinitesimal variation in the work done ($\mathcal{W}$) by the system, i.e.,

$$\delta\mathcal{I} = -\delta\mathcal{W} \tag{11}$$

The variations [2] in the cable attitude is defined as $\delta \boldsymbol{q} = \boldsymbol{\xi} \times \boldsymbol{q}$, where, differential curves $\boldsymbol{\xi}, \boldsymbol{\xi} : [t_o, t_f] \to \mathbb{R}^3$, satisfying $\boldsymbol{\xi(t_0) = \xi(t_f) = 0}$. Similarly, the variations in quadcopter's attitude and angular velocities are defined as $\delta \boldsymbol{R} = \boldsymbol{R}\hat{\boldsymbol{\eta}}$ and $\delta \boldsymbol{\Omega} = \dot{\boldsymbol{\eta}} + \hat{\boldsymbol{\Omega}}\boldsymbol{\eta}$, $\boldsymbol{\eta} \in \mathbb{R}^{\boldsymbol{3}}$ respectively. Hence, the infinitesimal workdone by the system is given in Eq. (12).

$$
\begin{aligned}
\delta \mathcal{W} &= \int_{t_0}^{t_f} \Big(\boldsymbol{u} \cdot \delta \boldsymbol{x_q} + \boldsymbol{\tau} \cdot \boldsymbol{\eta}\Big) dt \\
\delta \mathcal{W} &= \int_{t_0}^{t_f} \boldsymbol{u} \cdot \delta \boldsymbol{x_q} dt + \int_{t_0}^{t_f} \boldsymbol{\tau} \cdot \boldsymbol{\eta} dt
\end{aligned} \tag{12}
$$

The infinitesimal variation in the action integral is given as follows,

$$
\begin{aligned}
\delta \mathcal{I} &= \int_{t_0}^{t_f} \delta \mathcal{L} \; dt \\
&= \int_{t_0}^{t_f} \left[ \left( \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \cdot \delta \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} \cdot \delta \boldsymbol{x_q} \right) + \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \cdot \delta \dot{\alpha} + \frac{\partial \mathcal{L}}{\partial \alpha} \cdot \delta \alpha \right) + \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \delta \boldsymbol{\Omega} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} \right) \right] dt
\end{aligned} \tag{13}
$$

Finally, using Eqs. (11, 12, and 13), the following equations are derived.

$$
\int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \cdot \delta \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} \cdot \delta \boldsymbol{x_q} + \boldsymbol{u} \cdot \delta \boldsymbol{x_q} \right) dt = 0 \tag{14}
$$

$$
\int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \cdot \delta \dot{\alpha} + \frac{\partial \mathcal{L}}{\partial \alpha} \cdot \delta \alpha \right) dt = 0 \tag{15}
$$

$$
\int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \delta \boldsymbol{\Omega} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt = 0 \tag{16}
$$

Solve Eq. (14) using integral by parts and rearrange the terms,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \cdot \int_{t_0}^{t_f} \delta \dot{\boldsymbol{x}}_{\boldsymbol{q}} dt - \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \right) \cdot \delta \boldsymbol{x_q} dt + \int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} + \boldsymbol{u} \right) \cdot \delta \boldsymbol{x_q} dt &= 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \cdot [\cancelto{0}{\delta \boldsymbol{x_q}(t_f)} - \cancelto{0}{\delta \boldsymbol{x_q}(t_0)}] - \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \right) \cdot \delta \boldsymbol{x_q} dt + \int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} + \boldsymbol{u} \right) \cdot \delta \boldsymbol{x_q} dt &= 0 \\
\int_{t_0}^{t_f} \left( \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} - \boldsymbol{u} \right) \cdot \delta \boldsymbol{x_q} dt &= 0 \\
\frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} = \boldsymbol{u}
\end{aligned} \tag{17}
$$

Solve Eq. (15) using integral by parts and rearrange the terms,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \cdot \int_{t_0}^{t_f} \delta\dot{\alpha} dt - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) \cdot \delta\alpha dt + \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \alpha}\right) \cdot \delta\alpha dt &= 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \cdot [\cancelto{0}{\delta\alpha(t_f)} - \cancelto{0}{\delta\alpha(t_0)}] - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) \cdot \delta\alpha dt + \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \alpha}\right) \cdot \delta\alpha dt &= 0 \\
\int_{t_0}^{t_f} \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha}\right) \cdot \delta\alpha dt &= 0 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} &= 0 \qquad (18)
\end{aligned}
$$

Using relation $\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\eta}} + \boldsymbol{\Omega} \times \boldsymbol{\eta}$, Eq. (16) is resolved as follows.

$$
\int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \delta\boldsymbol{\Omega} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta\boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta}\right) dt = 0 \qquad (19)
$$

$$
\begin{aligned}
\int_{t_0}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\Omega}+\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt &= 0\\
\int_{t_0}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot(\dot{\boldsymbol{\eta}}+\boldsymbol{\Omega}\times\boldsymbol{\eta})+\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt &= 0\\
\int_{t_0}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\dot{\boldsymbol{\eta}}+\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\eta})+\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt &= 0\\
\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot[\cancelto{0}{\boldsymbol{\eta}(t_f)}-\cancelto{0}{\boldsymbol{\eta}(t_0)}]-\int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)\cdot\boldsymbol{\eta}dt+\int_{t_0}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\eta})+\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt &= 0\\
-\int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)\cdot\boldsymbol{\eta}dt+\int_{t_0}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\eta})+\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt &= 0\\
-\int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)\cdot\boldsymbol{\eta}dt+\int_{t_0}^{t_f}\left(\widehat{\boldsymbol{\Omega}}^T\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\boldsymbol{\eta}+d_{\boldsymbol{R}}\cdot\boldsymbol{\eta}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt &= 0\\
\left(\because\ \frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot(\boldsymbol{\Omega}\times\boldsymbol{\eta})=\widehat{\boldsymbol{\Omega}}^T\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\boldsymbol{\eta},\ \text{Dev.(4), and consider}\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}=d_{\boldsymbol{R}}\cdot\boldsymbol{\eta}\right)&\\
-\int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)\cdot\boldsymbol{\eta}dt+\int_{t_0}^{t_f}\left(\widehat{\boldsymbol{\Omega}}^T\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}+d_{\boldsymbol{R}}+\boldsymbol{\tau}\right)\cdot\boldsymbol{\eta}dt &= 0\\
-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)+\widehat{\boldsymbol{\Omega}}^T\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}+d_{\boldsymbol{R}}+\boldsymbol{\tau} &= 0\\
-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)-\widehat{\boldsymbol{\Omega}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}+d_{\boldsymbol{R}}+\boldsymbol{\tau} &= 0\\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\right)+\widehat{\boldsymbol{\Omega}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}-d_{\boldsymbol{R}} &= \boldsymbol{\tau} \qquad (20)
\end{aligned}
$$

From Eq. (1), we know $\boldsymbol{q_b}=\boldsymbol{q_b}=\begin{bmatrix}-sin\alpha & 0 & -cos\alpha\end{bmatrix}^T$. The time derivative of $\boldsymbol{q_b}$ is

$$
\begin{aligned}
\dot{\boldsymbol{q}}_{\boldsymbol{b}} &= \begin{bmatrix}-cos\alpha & 0 & sin\alpha\end{bmatrix}^T\dot{\alpha}\\
\dot{\boldsymbol{q}}_{\boldsymbol{b}} &= \boldsymbol{q_t}\dot{\alpha}
\end{aligned}
\qquad (21)
$$

Where,

$$
\boldsymbol{q_t}=\begin{bmatrix}-cos\alpha & 0 & sin\alpha\end{bmatrix}^T \qquad (22)
$$

The derivative of $\boldsymbol{q_b}$ w.r.t. $\alpha$ is

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\boldsymbol{q_b} &= \begin{bmatrix}-cos\alpha & 0 & sin\alpha\end{bmatrix}^T\\
&= \boldsymbol{q_t}
\end{aligned}
\qquad (23)
$$

$$\begin{aligned}\frac{d}{dt}\boldsymbol{q_t} &= \begin{bmatrix} sin\alpha & 0 & cos\alpha \end{bmatrix}^T \dot{\alpha} \\ &= -\boldsymbol{q_b}\dot{\alpha}\end{aligned} \tag{24}$$

Put expression $\dot{\boldsymbol{q}}_{\boldsymbol{b}} = \boldsymbol{q_t}\dot{\alpha}$ in Lagrangian as follows.

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}m_p l^2\|\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}m_p l^2\|\boldsymbol{Rq_t}\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}m_p l^2\|\boldsymbol{Rq_t}\|^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}m_p l^2(\boldsymbol{Rq_t})^T\boldsymbol{Rq_t}\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}m_p l^2\boldsymbol{q_t}^T\boldsymbol{R}^T\boldsymbol{Rq_t}\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\|\boldsymbol{R}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\boldsymbol{\Omega}\|^2 + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}). + \frac{1}{2}m_p(\boldsymbol{R}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\boldsymbol{\Omega})^T\boldsymbol{R}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_p\boldsymbol{\Omega}^T(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^T\boldsymbol{R}^T\boldsymbol{R}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) - \frac{1}{2}m_p\boldsymbol{\Omega}^T(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^2\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b} \\
\mathcal{L} &= \frac{1}{2}(m_q+m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{Rq_t}\dot{\alpha}) + m_p l(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+l\boldsymbol{q_b})\cdot\boldsymbol{Rq_t}\dot{\alpha}) \\
&\quad -(m_q+m_p)g\boldsymbol{e_3}\cdot\boldsymbol{x_q} - m_p g\boldsymbol{e_3}\cdot\boldsymbol{R\rho} - m_p l g\boldsymbol{e_3}\cdot\boldsymbol{Rq_b}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(m_q + m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\dot{\alpha}) + m_p l\dot{\alpha}(\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b}))^T\boldsymbol{R}\boldsymbol{q_t} \\
&\quad - (m_q + m_p)g\boldsymbol{e_3} \cdot \boldsymbol{x_q} - m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p l g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_b} \\
\mathcal{L} &= \frac{1}{2}(m_q + m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\dot{\alpha}) + m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{q_t} \\
&\quad - (m_q + m_p)g\boldsymbol{e_3} \cdot \boldsymbol{x_q} - m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p l g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_b} \\
\mathcal{L} &= \frac{1}{2}(m_q + m_p)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{1}{2}m_p l^2\|\dot{\alpha}\|^2 + m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\dot{\alpha}) - m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} \\
&\quad - (m_q + m_p)g\boldsymbol{e_3} \cdot \boldsymbol{x_q} - m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p l g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_b}
\end{aligned} \tag{25}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\dot{\boldsymbol{x}}_{\boldsymbol{q}}$***

$$
\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} = (m_q + m_p)\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p l\boldsymbol{R}\boldsymbol{q_t}\dot{\alpha} \tag{26}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{x_q}$***

$$
\frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} = -(m_q + m_p)g\boldsymbol{e_3} \tag{27}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\dot{\alpha}$***

$$
\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = m_p l^2\dot{\alpha} + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}) - m_p l(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} \tag{28}
$$

$$
\tag{29}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\alpha$***

$$
\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{1}{2}\boldsymbol{\Omega}^T(-2m_p l(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}})\boldsymbol{\Omega} + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}) - m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}\dot{\alpha}) - m_p l^2\dot{\alpha}\boldsymbol{q_t}^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} + m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_b} - m_p l g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_t} \tag{30}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{R}$***

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \partial \boldsymbol{R} &= \frac{\partial}{\partial \boldsymbol{R}}\Big(m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\dot{\alpha}) - m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_b}\Big) \cdot \boldsymbol{R}\widehat{\eta} \\
&= \frac{\partial}{\partial \boldsymbol{R}}\Big(m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\dot{\alpha}) - m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{\rho} - m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_b}\Big) \cdot \boldsymbol{R}\widehat{\eta} \\
&\quad \text{(To solve this derivative, just simply replace } \boldsymbol{R} \text{ with } \boldsymbol{R}\widehat{\boldsymbol{\eta}} \text{ as done in [3])} \\
&= m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\eta}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\eta}\boldsymbol{q_t}\dot{\alpha}) - m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\widehat{\eta}\boldsymbol{\rho} - m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\widehat{\eta}\boldsymbol{q_b} \\
&= -m_p\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\eta - m_p l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{q_t}}\boldsymbol{\eta} + m_p g\boldsymbol{e_3} \cdot \boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\eta} + m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\widehat{\boldsymbol{q_b}}\boldsymbol{\eta} \\
&= -m_p(\boldsymbol{R}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\eta)^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}(\boldsymbol{R}\widehat{\boldsymbol{q_t}}\boldsymbol{\eta})^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p g(\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\eta})^T\boldsymbol{e_3} + m_p lg(\boldsymbol{R}\widehat{\boldsymbol{q_b}}\boldsymbol{\eta})^T\boldsymbol{e_3} \\
&= -m_p(\boldsymbol{R}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\eta)^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}(\boldsymbol{R}\widehat{\boldsymbol{q_t}}\boldsymbol{\eta})^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p g(\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\eta})^T\boldsymbol{e_3} + m_p lg(\boldsymbol{R}\widehat{\boldsymbol{q_b}}\boldsymbol{\eta})^T\boldsymbol{e_3} \\
&= -m_p\eta^T[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\boldsymbol{\eta}^T\widehat{\boldsymbol{q_t}}^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p g\boldsymbol{\eta}^T\widehat{\boldsymbol{\rho}}^T\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\boldsymbol{\eta}^T\widehat{\boldsymbol{q_b}}^T\boldsymbol{R}^T\boldsymbol{e_3} \\
&= m_p\eta \cdot [\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}\boldsymbol{\eta} \cdot \widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p g\boldsymbol{\eta} \cdot \widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - m_p lg\boldsymbol{\eta} \cdot \widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} \\
&= \Big(m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3}\Big) \cdot \boldsymbol{\eta}
\end{aligned} \tag{31}
$$

Hence, the expression of $\boldsymbol{d_R}$ is

$$
\boldsymbol{d_R} = m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} \tag{32}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{\Omega}$***

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} &= (\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{\partial}{\partial \boldsymbol{\Omega}}\Big(m_p(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})) - m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\Big) \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} &= (\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{\partial}{\partial \boldsymbol{\Omega}}\Big(-m_p(\boldsymbol{R}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{\Omega})^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega}\Big) \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} &= (\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{\partial}{\partial \boldsymbol{\Omega}}\Big(-m_p\boldsymbol{\Omega}^T(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}(\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega})^T(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big) \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} &= (\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{\partial}{\partial \boldsymbol{\Omega}}\Big(m_p\boldsymbol{\Omega} \cdot (\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}\boldsymbol{\Omega} \cdot \widehat{\boldsymbol{q_t}}^T(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big) \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} &= (\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + \frac{\partial}{\partial \boldsymbol{\Omega}}\Big(m_p\boldsymbol{\Omega} \cdot (\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\boldsymbol{\Omega} \cdot \widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big) \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} &= (\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b})
\end{aligned} \tag{33}
$$

From Eqs. (17, 26, & 27), the first dynamical equation of the system is derived as follows,

$$(m_q + m_p)\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p \dot{\boldsymbol{R}}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p \boldsymbol{R}\widehat{\dot{\boldsymbol{\Omega}}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p l \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} + m_p l \dot{\boldsymbol{R}}\boldsymbol{q_t}\dot{\alpha} - m_p l \boldsymbol{R}\boldsymbol{q_b}\dot{\alpha}^2 + m_p l \boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + (m_q + m_p)g\boldsymbol{e_3} = \boldsymbol{u}$$

$$(m_q + m_p)\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p \boldsymbol{R}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\dot{\boldsymbol{\Omega}} + m_p l \boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + m_p \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + l\boldsymbol{q_b}) + 2m_p l \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - m_p l \boldsymbol{R}\boldsymbol{q_b}\dot{\alpha}^2 + (m_q + m_p)g\boldsymbol{e_3} = \boldsymbol{u} \tag{34}$$

From Eqs. (18, 28, & 30), the second dynamical equation of the system is derived as follows,

$$\begin{aligned}
&m_p l^2\ddot{\alpha} + m_p l(\ddot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}) + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}\boldsymbol{q_t}) - m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_d}\dot{\alpha}) - m_p l^2\dot{\alpha}\boldsymbol{q_t}^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - m_p l(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\dot{\boldsymbol{\Omega}}}\boldsymbol{q_t} + m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_b}\\
&\quad - \Big(\boldsymbol{\Omega}^T(-m_p l(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}})\boldsymbol{\Omega} + m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}) - m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}\dot{\alpha}) - m_p l^2\dot{\alpha}\boldsymbol{q_t}^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} + m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_b} - m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_t}\Big) = 0\\
&m_p l^2\ddot{\alpha} + m_p l(\ddot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}) + \cancelto{0}{m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}\boldsymbol{q_t})} - \cancelto{1}{m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_d}\dot{\alpha})} - \cancelto{2}{m_p l^2\dot{\alpha}\boldsymbol{q_t}^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}} - m_p l(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\dot{\boldsymbol{\Omega}}}\boldsymbol{q_t} + \cancelto{3}{m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_b}}\\
&\quad - \Big(\boldsymbol{\Omega}^T(-m_p l(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}})\boldsymbol{\Omega} + \cancelto{0}{m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t})} - \cancelto{1}{m_p l(\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}\dot{\alpha})} - \cancelto{2}{m_p l^2\dot{\alpha}\boldsymbol{q_t}^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}} + \cancelto{3}{m_p l\dot{\alpha}(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{\Omega}}\boldsymbol{q_b}} - m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_t}\Big) = 0\\
&m_p l^2\ddot{\alpha} + m_p l(\ddot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}) - m_p l(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\dot{\boldsymbol{\Omega}}}\boldsymbol{q_t} + m_p l\boldsymbol{\Omega}^T(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_t} = 0
\end{aligned}$$

$$m_p l^2\ddot{\alpha} + m_p l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l(\boldsymbol{\rho} + l\boldsymbol{q_b})^T\widehat{\boldsymbol{q_t}}\dot{\boldsymbol{\Omega}} + m_p l\boldsymbol{\Omega}^T(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p lg\boldsymbol{e_3} \cdot \boldsymbol{R}\boldsymbol{q_t} = 0 \tag{35}$$

From Eqs. (20, 33), the third dynamical equation of the system is derived as follows,

$$\begin{aligned}
&\frac{d}{dt}\Big((\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big) + \widehat{\boldsymbol{\Omega}}\Big((\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big)\\
&\qquad - \Big(m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3}\Big) = \boldsymbol{\tau}\\
&(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} - m_p l(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{\Omega} + m_p l\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\dot{\boldsymbol{R}}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) - m_p l\dot{\alpha}^2(-\widehat{\boldsymbol{q_b}})(\boldsymbol{\rho} + l\boldsymbol{q_b}) - \cancel{m_p l\dot{\alpha}^2(\widehat{\boldsymbol{q_t}})\boldsymbol{q_t}}\\
&\qquad + \widehat{\boldsymbol{\Omega}}\Big((\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big) - \Big(m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3}\Big) = \boldsymbol{\tau}\\
&(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}}^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})\\
&\qquad + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b})\Big) - m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned} \tag{36}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$+\widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) - m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$


$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$+\widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} + m_p\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) - m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$


$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + \cancelto{0}{m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$+m_p\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) - m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cancelto{0}{- m_p l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}} + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$+m_p\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$+m_p(\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}}) - (\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}} - [\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}])\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

As we know $\widehat{\boldsymbol{a}}\widehat{\boldsymbol{b}} - \widehat{\boldsymbol{b}}\widehat{\boldsymbol{a}} = \widehat{\widehat{\boldsymbol{a}}\boldsymbol{b}}$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$\cancelto{0}{+m_p(\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}}) - (\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{\Omega}} - [\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + l\boldsymbol{q_b})}])\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}(\boldsymbol{\rho} + l\boldsymbol{q_b})$$

$$-m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{\rho} + \cancelto{0}{m_p l^2\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{q_b}})$$

$$-m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{\rho} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau} \tag{37}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{\rho} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + m_p lg\widehat{\boldsymbol{q_b}}\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}} + m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - m_p l\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + \widehat{\boldsymbol{\Omega}}(\boldsymbol{J} - m_p(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})^2)\boldsymbol{\Omega} - m_p l\dot{\alpha}(\widehat{\boldsymbol{\rho} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + m_p l\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{\rho} - m_p l\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho} + l\boldsymbol{q_b}) + m_p g(\widehat{\boldsymbol{\rho}} + l\widehat{\boldsymbol{q_b}})\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}$$

$$(\boldsymbol{J}-m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^2)\dot{\boldsymbol{\Omega}}+m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}}-m_pl\ddot{\alpha}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho}+l\boldsymbol{q_b})+\widehat{\boldsymbol{\Omega}}(\boldsymbol{J}-m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^2)\boldsymbol{\Omega}-m_pl\dot{\alpha}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega}+m_pl\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{\rho}-m_pl\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho}+l\boldsymbol{q_b})+m_pg(\widehat{\boldsymbol{\rho}}+l\widehat{\boldsymbol{q_b}})\boldsymbol{R}^T\boldsymbol{e_3}=\boldsymbol{\tau} \quad (38)$$

Hence, this exercise is enough to go ahead and derive the dynamics for the quadcopter with net where the first link rigid and ends of the links are the connected.

$$\begin{bmatrix} (m_q+m_p) & -m_p\boldsymbol{R}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}}) & m_pl\boldsymbol{R}\boldsymbol{q_t} \\ m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\boldsymbol{R}^T & (\boldsymbol{J}-m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^2) & -m_pl\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho}+l\boldsymbol{q_b}) \\ m_pl\boldsymbol{q_t}^T\boldsymbol{R}^T & m_pl(\boldsymbol{\rho}+l\boldsymbol{q_b})^T\widehat{\boldsymbol{q_t}} & m_pl^2 \end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} \\ \dot{\boldsymbol{\Omega}} \\ \ddot{\alpha}\end{bmatrix}+\begin{bmatrix} m_p\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+l\boldsymbol{q_b})+2m_pl\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha}-m_pl\boldsymbol{R}\boldsymbol{q_b}\dot{\alpha}^2+(m_q+m_p)g\boldsymbol{e_3} \\ \widehat{\boldsymbol{\Omega}}(\boldsymbol{J}-m_p(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})^2)\boldsymbol{\Omega}-m_pl\dot{\alpha}(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega}+m_pl\dot{\alpha}^2\widehat{\boldsymbol{q_b}}\boldsymbol{\rho}-m_pl\dot{\alpha}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{q_t}}(\boldsymbol{\rho}+l\boldsymbol{q_b})+m_pg(\widehat{\boldsymbol{\rho}}+l\widehat{\boldsymbol{q_b}})\boldsymbol{R}^T\boldsymbol{e_3} \\ m_pl\boldsymbol{\Omega}^T(\widehat{\boldsymbol{\rho}+l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega}+m_plg\boldsymbol{e_3}\cdot\boldsymbol{R}\boldsymbol{q_t}\end{bmatrix}=\begin{bmatrix}\boldsymbol{u} \\ \boldsymbol{\tau} \\ \boldsymbol{0}\end{bmatrix} \quad (39)$$

### 1.3 Dynamical Model for Single Quadcopter with MRS Net Mechanism First Rigid Link with Free Ends

As shown in Fig. 2, consider the quadcopter with a net mechanism where the $m$ numbers of ropes are attached with a massless rigid link. After that each rope is modeled as $n$ numbers of rigid links with point masses. Considering the $(\cdot)_i$ as $i^{th}$ cable attachment point on the rod and $(\cdot)_j$ as $j^{th}$ link, the attitude of each link is represented as a unit vector, $\boldsymbol{q_{ij}} \in \mathbb{S}^2$, $\mathbb{S}^2 \triangleq \{\boldsymbol{q_i} \in \mathbb{R}^3 |\ \|\boldsymbol{q_{ij}}\|_2{=}1\}$ in frame $\{\boldsymbol{E}\}$, pointing towards each point mass. $\|\cdot\|_2$ represents the second norm. The bottom rod is again considered a rigid rod which is pin jointed with the top rod. The bottom link is constraint to oscillate along $\boldsymbol{n_2}$ axis only. Consider the angular position of the bottom link from $-\boldsymbol{n_3}$ axis is $\alpha$. Hence, the expression for the unit vector which represents the direction of the first link is given as follows.

$$\boldsymbol{q_b}=\begin{bmatrix}-sin\alpha & 0 & -cos\alpha\end{bmatrix}^T \quad (40)$$

**Assumption 1.** *The rigid links are massless.*

The position of $ij^{th}$ point mass in frame $\{\boldsymbol{E}\}$ is computed as follows:

$$\boldsymbol{x_{ij}}=\boldsymbol{x_q}+\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})+\sum_{a=1}^{j}l_{ia}\boldsymbol{q_{ia}} \quad (41)$$

The position of the rod which is net mechanism is given as

$$\boldsymbol{x_r}=\boldsymbol{x_q}+\boldsymbol{R}\boldsymbol{\rho} \quad (42)$$

Overall, the configuration space of the system is $\mathbb{R}^3\times\mathbb{SO}(3)\times(\mathbb{S}^2)^{(n\times m)}\times(\mathbb{S}^1)$ with a total of $(6+2(n\times m)+1)$ degrees of freedom (DOF) - 6 DOF of the quadcopter, $2(n\times m)$ DOF for $2(n\times m)$ links and 1 DOF for bottom rigid rod. Expressing the angular velocity of the quadcopter with respect to frame $\{B\}$ and angular velocity of the cable with respect to frame $\{\boldsymbol{E}\}$ as $\boldsymbol{\Omega}, \boldsymbol{\omega_{ij}} \in \mathbb{R}^3$ respectively, the kinematic relations for the quadcopter's attitude and link's attitude are as follows:

$$\dot{\boldsymbol{R}}=\boldsymbol{R}\widehat{\boldsymbol{\Omega}}, \quad \dot{\boldsymbol{q}}_{\boldsymbol{ij}}=\boldsymbol{\omega_{ij}}\times\boldsymbol{q_{ij}} \quad (43)$$

Here, the *hat map* $\widehat{\cdot}:\mathbb{R}^3\rightarrow\mathfrak{so}(3)$ is defined as $\widehat{\boldsymbol{x}}\boldsymbol{y}=\boldsymbol{x}\times\boldsymbol{y}$ for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$, where $\mathfrak{so}(3)$ is the skew-symmetric matrix. Using the Lagrange-d'Alembert principle on a manifold [1], the system's equation of motion (EOM) is derived. First we will derive the expressions of total kinetic and potential energy of the system. The total kinetic energy ($\mathcal{T}$) of the system is given by the summation of the total kinetic energy of the quadcopter, the total kinetic energy of the rigid rod which is net mechanism, and the total kinetic energy of the point masses.

***Total kinetic energy of quadcopter:*** The total kinetic energy of the quadcopter is written as given in Eq. (44).

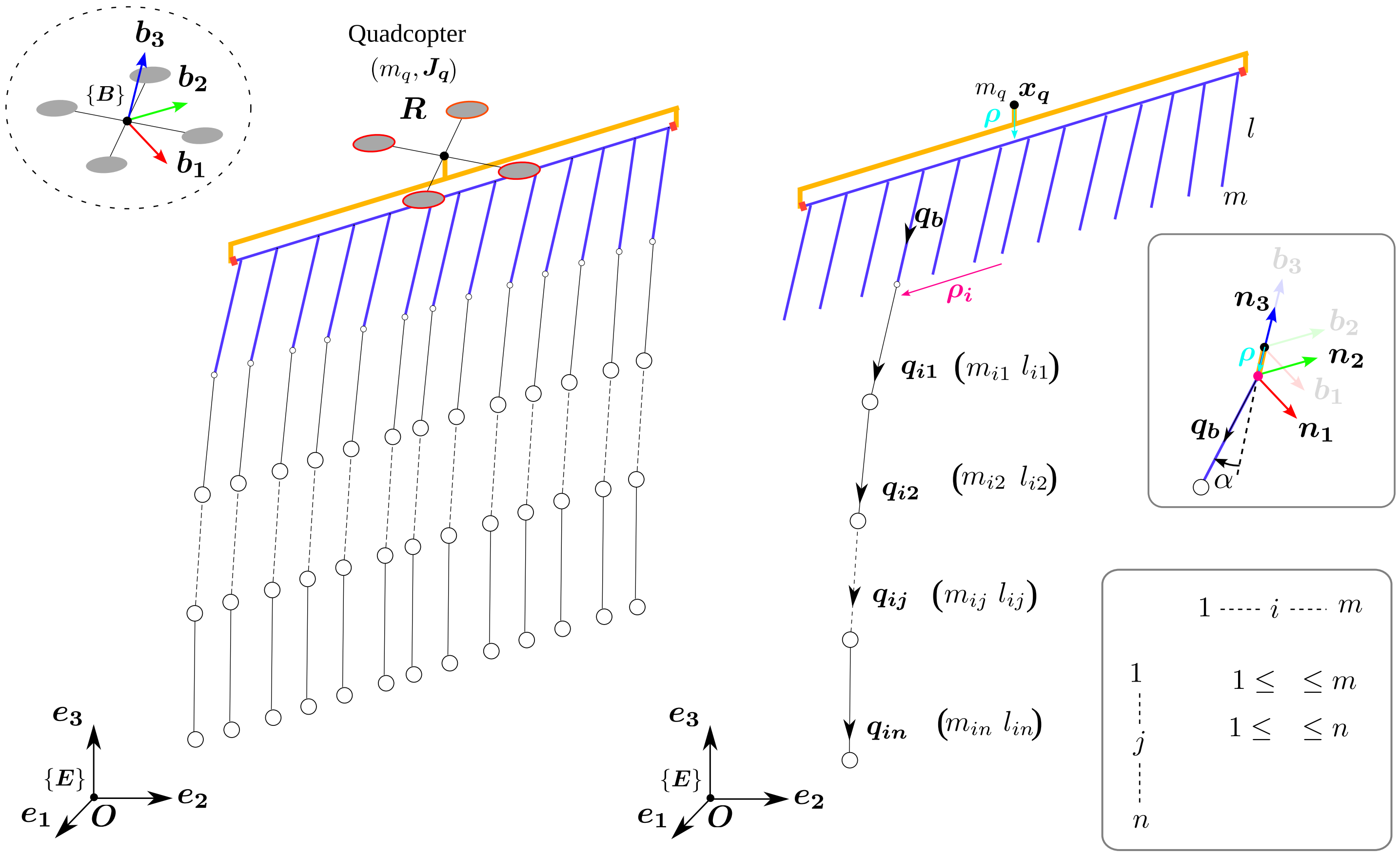


Figure 2: Line diagram of a quadcopter with a flexible ropes modeled as point masses with rigid links. The first links are modeled as a connected rigid single link. The inertial frame of reference and body-fixed frame of reference of the quadcopter are represented as frame $\{\boldsymbol{E}\}$ and frame $\{\boldsymbol{B}\}$ respectively. $\boldsymbol{x_q}$ is the position vectors of the quadcopter in frame $\{\boldsymbol{E}\}$.

$$\mathcal{T}_q = \frac{1}{2} m_q \|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2} \boldsymbol{\Omega}^T (\boldsymbol{J\Omega}) \tag{44}$$

***Total kinetic energy of rod:***

Consider a infinitesimal small element of length $dr$ of the rod of the net mechanism as shown in Fig. 3. The position of this infinitesimal small element is taken as $\boldsymbol{r} \in \mathbb{R}$ along the $\boldsymbol{r_2}$ axis. The position of this infinitesimal small element in frame $\{\boldsymbol{E}\}$ is given as follows.

$$\begin{aligned}
\boldsymbol{x}_{\boldsymbol{r}_{element}} &= \boldsymbol{x_q} + \boldsymbol{R\rho} + r\boldsymbol{Re_2} \\
\boldsymbol{x}_{\boldsymbol{r}_{element}} &= \boldsymbol{x_q} + \boldsymbol{R}(\boldsymbol{\rho} + r\boldsymbol{e_2})
\end{aligned} \tag{45}$$

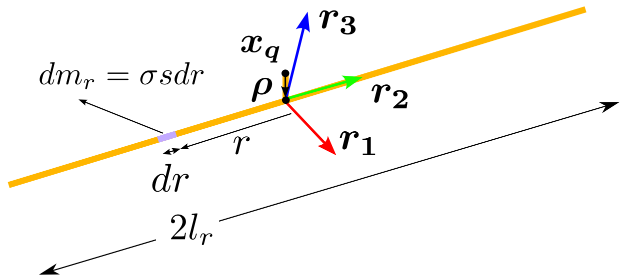


Figure 3: Line diagram representing the infinitesimal element of the net mechanism rod

Consider the net mechanism rod has uniform density, $\sigma \in \mathbb{R}$ and the uniform cross-sectional area denoted as $\boldsymbol{s} \in \mathbb{R}$, the mass the of the infinitesimal element is written as

$$dm_r = \sigma s dr \tag{46}$$

The total kinetic energy of the rod is derived as follows.

$$\begin{aligned}
\mathcal{T}_r &= \frac{1}{2}\int_{r=-l_r}^{r=l_r} ||\dot{\boldsymbol{x}}_{\boldsymbol{r}_{element}}||^2 dm_r \\
&= \frac{1}{2}\int_{r=-l_r}^{r=l_r} ||\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + r\boldsymbol{e_2})||^2 \sigma s dr \\
&= \frac{1}{2}\int_{r=-l_r}^{r=l_r} \Big(||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 + ||\dot{\boldsymbol{R}}(\boldsymbol{\rho} + r\boldsymbol{e_2})||^2 + 2\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + r\boldsymbol{e_2})\Big)\sigma s dr \\
&= \frac{1}{2}\int_{r=-l_r}^{r=l_r} ||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 \sigma s dr + \frac{1}{2}\int_{r=-l_r}^{r=l_r} ||\dot{\boldsymbol{R}}(\boldsymbol{\rho} + r\boldsymbol{e_2})||^2 \sigma s dr + 2\frac{1}{2}\int_{r=-l_r}^{r=l_r} \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + r\boldsymbol{e_2})\sigma s dr \\
&= \frac{1}{2}||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 \sigma s [r]_{r=-l_r}^{r=l_r} + \frac{1}{2}\int_{r=-l_r}^{r=l_r} ||\dot{\boldsymbol{R}}||^2 (||\boldsymbol{\rho}||^2 + r^2||\boldsymbol{e_2}||^2 + 2\boldsymbol{\rho} \cdot \boldsymbol{e_2} r)\sigma s dr + \int_{r=-l_r}^{r=l_r} \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + r\boldsymbol{e_2})\sigma s dr \\
&= \frac{1}{2}||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 \sigma s 2l_r + \frac{1}{2}\Bigg(||\dot{\boldsymbol{R}}||^2\Bigg(||\boldsymbol{\rho}||^2 [r]_{r=-l_r}^{r=l_r} + \left[\frac{r^3}{3}\right]_{r=-l_r}^{r=l_r} ||\boldsymbol{e_2}||^2 + 2\boldsymbol{\rho} \cdot \boldsymbol{e_2}\left[\frac{r^2}{2}\right]_{r=-l_r}^{r=l_r}\Bigg)\Bigg)\sigma s + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}\Bigg(\boldsymbol{\rho}[r]_{r=-l_r}^{r=l_r} + \left[\frac{r^2}{2}\right]_{r=-l_r}^{r=l_r} \boldsymbol{e_2}\Bigg)\sigma s \\
&= \frac{1}{2}m_r||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 + \frac{1}{2}||\dot{\boldsymbol{R}}||^2||\boldsymbol{\rho}||^2 2l_r\sigma s + \frac{1}{2}||\dot{\boldsymbol{R}}||^2\frac{2}{3}l_r^3||\boldsymbol{e_2}||^2\sigma s + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}\boldsymbol{\rho} 2l_r \sigma s \\
&= \frac{1}{2}m_r||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 + \frac{1}{2}m_r||\dot{\boldsymbol{R}}\boldsymbol{\rho}||^2 + \frac{1}{2}||\dot{\boldsymbol{R}}\boldsymbol{e_2}||^2\frac{1}{3}l_r^2(2l_r)\sigma s + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}\boldsymbol{\rho} \\
\mathcal{T}_r &= \frac{1}{2}m_r||\dot{\boldsymbol{x}}_{\boldsymbol{q}}||^2 + \frac{1}{2}m_r||\dot{\boldsymbol{R}}\boldsymbol{\rho}||^2 + \frac{1}{6}m_r l_r^2||\dot{\boldsymbol{R}}\boldsymbol{e_2}||^2 + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}\boldsymbol{\rho}
\end{aligned} \tag{47}$$

***Total kinetic energy of point masses:***

As the position of each point mass is defined as $\boldsymbol{x_{ij}} = \boldsymbol{x_q} + \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{a=1}^{j} l_{ia}\boldsymbol{q_{ia}}$, the time derivative of $\boldsymbol{x_{ij}}$ is given as follows. Hence, the kinetic energy of the

$$\dot{\boldsymbol{x}}_{\boldsymbol{ij}} = \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \tag{48}$$

Hence, the total kinetic energy of the point masses are derived as follows.

$$\begin{aligned}
\mathcal{T}_p &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{x}}_{\boldsymbol{ij}}\|^2 \\
\mathcal{T}_p &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
\mathcal{T}_p &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
\mathcal{T}_p &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
&\quad + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&\quad + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&\quad + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}
\end{aligned} \tag{49}$$

***Total kinetic energy of first links:***

The position of each first link is

$$\boldsymbol{x_i} = \boldsymbol{x_q} + \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \tag{50}$$

The time derivative is

$$\dot{\boldsymbol{x}}_{\boldsymbol{i}} = \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \tag{51}$$

Hence, the total kinetic energy of the first links' masses are derived as follows.

$$\begin{aligned}
\mathcal{T}_f &= \frac{1}{2}\sum_{i=1}^{m} \|\dot{\boldsymbol{x}}_{\boldsymbol{i}}\|^2 \\
\mathcal{T}_f &= \frac{1}{2}\sum_{i=1}^{m} m_i \left\| \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \right\|^2 \\
\mathcal{T}_f &= \frac{1}{2}\sum_{i=1}^{m} m_i \|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i \|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i \|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
&+ \sum_{i=1}^{m} m_i (\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})) + \sum_{i=1}^{m} m_i \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&+ \sum_{i=1}^{m} m_i \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
\mathcal{T}_f &= \frac{1}{2}\sum_{i=1}^{m} m_i \|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i \|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i \|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} m_i \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})
\end{aligned} \tag{52}$$

Using Eqs. (44, 47, 49, & 52), the total kinetic energy of the system is derived as follows.

$$
\begin{aligned}
\mathcal{T} &= \mathcal{T}_q + \mathcal{T}_r + \mathcal{T}_p + \mathcal{T}_f \\
\mathcal{T} &= \frac{1}{2}m_q\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J\Omega}) + \frac{1}{2}m_r\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}m_r\|\dot{\boldsymbol{R}}\boldsymbol{\rho}\|^2 + \frac{1}{6}m_r l_r^2\|\dot{\boldsymbol{R}}\boldsymbol{e_2}\|^2 + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\dot{\boldsymbol{R}}\boldsymbol{\rho} \\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \frac{1}{2}\sum_{i=1}^{m} m_i\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} = {} & \frac{1}{2}m_q\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T(\boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}m_r\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}m_r\|\dot{\boldsymbol{R}}\boldsymbol{\rho}\|^2 + \frac{1}{6}m_r l_r^2\|\dot{\boldsymbol{R}}\boldsymbol{e_2}\|^2 + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\dot{\boldsymbol{R}}\boldsymbol{\rho} \\
& + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\left\|\sum_{a=1}^{j}l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
& + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\sum_{a=1}^{j}l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\sum_{a=1}^{j}l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\sum_{a=1}^{j}l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
& + \frac{1}{2}\sum_{i=1}^{m}m_i\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}m_i\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}m_i\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
& + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}m_i \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} = &\frac{1}{2}\left(m_q + m_r + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m} m_i\right)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{J}\boldsymbol{\Omega} + \frac{1}{2}m_r\|\dot{\boldsymbol{R}}\boldsymbol{\rho}\|^2 + \frac{1}{6}m_r l_r^2\|\dot{\boldsymbol{R}}\boldsymbol{e_2}\|^2 + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\dot{\boldsymbol{R}}\boldsymbol{\rho} \\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \frac{1}{2}\sum_{i=1}^{m} m_i\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2}\left(m_q + m_r + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m} m_i\right)\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{J}\boldsymbol{\Omega} + \frac{1}{2}m_r\|\dot{\boldsymbol{R}}\boldsymbol{\rho}\|^2 + \frac{1}{6}m_r l_r^2\|\dot{\boldsymbol{R}}\boldsymbol{e_2}\|^2 + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\dot{\boldsymbol{R}}\boldsymbol{\rho} \\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} \\
&+ \frac{1}{2}\sum_{i=1}^{m} m_i\|\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\|^2 + \frac{1}{2}\sum_{i=1}^{m} m_i\|\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 \\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\dot{\boldsymbol{R}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

Consider the following transformations. These transformation are valid in general and easily be proven by considering $m, n = 2$.

***Transformation 1:***

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\left\|\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}}\right\|^2 = \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n} M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}}, \quad \text{where, } M_{ijk} = \left\{\sum_{a=\max\{j,k\}}^{n} m_{ia}\right\} l_{ij}l_{ik} \tag{53}$$

***Transformation 2:***

$$\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} = \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} = \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}, \quad M_{0ij} = \sum_{a=j}^{n} m_{ia}l_{ij} \tag{54}$$

$$\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\sum_{a=1}^{j} l_{ia}\dot{\boldsymbol{q}}_{\boldsymbol{ia}} = \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} = \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}, \quad M_{0ij} = \sum_{a=j}^{n} m_{ia}l_{ij} \tag{55}$$

***Transformation 3:***

$$M_{00} = \left( m_q + m_r + \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m} m_i \right) \tag{56}$$

***Transformation 4:***

$$\sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \sum_{a=1}^{j} l_{ia} \dot{\boldsymbol{q}}_{\boldsymbol{ia}} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{a=j}^{n} m_{ia} \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} \tag{57}$$

$$\begin{aligned}
\mathcal{T} = {} & \frac{1}{2} M_{00} \|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2} \boldsymbol{\Omega}^T \boldsymbol{J} \boldsymbol{\Omega} + \frac{1}{2} m_r \|\dot{\boldsymbol{R}} \boldsymbol{\rho}\|^2 + \frac{1}{6} m_r l_r^2 \|\dot{\boldsymbol{R}} \boldsymbol{e_2}\|^2 + m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \dot{\boldsymbol{R}} \boldsymbol{\rho} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 \\
& + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \|\boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2} \sum_{i=1}^{m} \sum_{j,k=1}^{n} M_{ijk} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} \cdot \dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \\
& + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} M_{0ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{a=j}^{n} m_{ia} \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} M_{0ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \frac{1}{2} \sum_{i=1}^{m} m_i \|\dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 + \frac{1}{2} \sum_{i=1}^{m} m_i \|\boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i \\
& + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} m_i \dot{\boldsymbol{R}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})
\end{aligned}$$

Using the relation, $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\Omega}}$, the above equation is updated as follows

$$\begin{aligned}
\mathcal{T} = {} & \frac{1}{2} M_{00} \|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2} \boldsymbol{\Omega}^T \boldsymbol{J} \boldsymbol{\Omega} + \frac{1}{2} m_r \|\boldsymbol{R}\widehat{\boldsymbol{\Omega}} \boldsymbol{\rho}\|^2 + \frac{1}{6} m_r l_r^2 \|\boldsymbol{R}\widehat{\boldsymbol{\Omega}} \boldsymbol{e_2}\|^2 + m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}} \boldsymbol{\rho} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 \\
& + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \|\boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \frac{1}{2} \sum_{i=1}^{m} \sum_{j,k=1}^{n} M_{ijk} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} \cdot \dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \\
& + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} M_{0ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{a=j}^{n} m_{ia} \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} \sum_{j=1}^{n} M_{0ij} \dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \frac{1}{2} \sum_{i=1}^{m} m_i \|\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\|^2 + \frac{1}{2} \sum_{i=1}^{m} m_i \|\boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}}\|^2 + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i \\
& + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})
\end{aligned}$$

$$
\begin{aligned}
\mathcal{T} = {} & \frac{1}{2}M_{00}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{J}\boldsymbol{\Omega} + \frac{1}{2}m_r\|\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\Omega}\|^2 + \frac{1}{6}m_r l_r^2\|\boldsymbol{R}\widehat{\boldsymbol{e_2}}\boldsymbol{\Omega}\|^2 + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\|\boldsymbol{R}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])\boldsymbol{\Omega}\|^2 \\
& + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}(\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}})^T\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n}M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij} \\
& + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n}m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \frac{1}{2}\sum_{i=1}^{m}m_i\|\boldsymbol{R}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])\boldsymbol{\Omega}\|^2 + \frac{1}{2}\sum_{i=1}^{m}m_i(\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}})^T\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}m_i \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} = {} & \frac{1}{2}M_{00}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{J}\boldsymbol{\Omega} + \frac{1}{2}m_r(\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\Omega})^T\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\Omega} + \frac{1}{6}m_r l_r^2(\boldsymbol{R}\widehat{\boldsymbol{e_2}}\boldsymbol{\Omega})^T\boldsymbol{R}\widehat{\boldsymbol{e_2}}\boldsymbol{\Omega} + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}(\boldsymbol{R}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])\boldsymbol{\Omega})^T\boldsymbol{R}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])\boldsymbol{\Omega} \\
& + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}(\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}})^T\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n}M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij} \\
& + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n}m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \frac{1}{2}\sum_{i=1}^{m}m_i(\boldsymbol{R}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])\boldsymbol{\Omega})^T\boldsymbol{R}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])\boldsymbol{\Omega} + \frac{1}{2}\sum_{i=1}^{m}m_i(\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}})^T\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}m_i \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2}M_{00}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{J}\boldsymbol{\Omega} + \frac{1}{2}m_r\boldsymbol{\Omega}^T\widehat{\boldsymbol{\rho}}^T\boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\Omega} + \frac{1}{6}m_r l_r^2\boldsymbol{\Omega}^T\widehat{\boldsymbol{e_2}}^T\boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{e_2}}\boldsymbol{\Omega} + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{\Omega}^T([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])^T\boldsymbol{R}^T\boldsymbol{R}([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])\boldsymbol{\Omega}\\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\boldsymbol{R}^T\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n}M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n}m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\\
&+ \frac{1}{2}\sum_{i=1}^{m}m_i\boldsymbol{\Omega}^T([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])^T\boldsymbol{R}^T\boldsymbol{R}([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])\boldsymbol{\Omega} + \frac{1}{2}\sum_{i=1}^{m}m_il^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\boldsymbol{R}^T\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}m_i\\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2}M_{00}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{J}\boldsymbol{\Omega} - \frac{1}{2}m_r\boldsymbol{\Omega}^T\widehat{\boldsymbol{\rho}}^2\boldsymbol{\Omega} - \frac{1}{6}m_r l_r^2\boldsymbol{\Omega}^T\widehat{\boldsymbol{e_2}}^2\boldsymbol{\Omega} + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{\Omega}^T([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])^2\boldsymbol{\Omega}\\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n}M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n}m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\\
&- \frac{1}{2}\sum_{i=1}^{m}m_i\boldsymbol{\Omega}^T([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])^2\boldsymbol{\Omega} + \frac{1}{2}\sum_{i=1}^{m}m_il^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}m_i\\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2}M_{00}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\left(\boldsymbol{J} - \frac{1}{2}m_r\widehat{\boldsymbol{\rho}}^2 - \frac{1}{6}m_r l_r^2\widehat{\boldsymbol{e_2}}^2 - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])^2 - \frac{1}{2}\sum_{i=1}^{m}m_i([\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l\boldsymbol{q_b}}])^2\right)\boldsymbol{\Omega} + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}\\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n}M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n}m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\\
&+ \frac{1}{2}\sum_{i=1}^{m}m_il^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}m_i + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})
\end{aligned}
$$

Consider,

$$\bar{\boldsymbol{J}} = \left(\boldsymbol{J} - \frac{1}{2}m_r\widehat{\boldsymbol{\rho}}^2 - \frac{1}{6}m_r l_r^2\widehat{\boldsymbol{e_2}}^2 - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2 - \frac{1}{2}\sum_{i=1}^{m} m_i([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2\right) \tag{58}$$

$$\begin{aligned}
\dot{\bar{\boldsymbol{J}}} &= \frac{d}{dt}\left(\boldsymbol{J} - \frac{1}{2}m_r\widehat{\boldsymbol{\rho}}^2 - \frac{1}{6}m_r l_r^2\widehat{\boldsymbol{e_2}}^2 - \frac{1}{2}\sum_{i=1}^{m} m_{i1}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2 - \frac{1}{2}\sum_{i=1}^{m} m_i([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2\right)\\
&= -\sum_{i=1}^{m} m_{i1}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}} - \sum_{i=1}^{m} m_i([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\\
&= -\sum_{i=1}^{m}(m_{i1} + m_i)l(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\widehat{\boldsymbol{q_t}}\dot{\alpha}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T} &= \frac{1}{2}M_{00}\|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}\\
&+ \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n} M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\cdot\dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\\
&+ \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\\
&+ \frac{1}{2}\sum_{i=1}^{m} m_i l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})
\end{aligned} \tag{59}$$

The total potential energy ($\mathcal{V}$) of the system is given by the summation of the total potential energy of the quadcopter, the total potential energy of the rigid rod which is net mechanism, the total potential energy of the point masses, and total potential energy of the first links' masses.

$$
\begin{aligned}
\mathcal{V} &= \mathcal{V}_q + \mathcal{V}_r + \mathcal{V}_p + \mathcal{V}_f \\
\mathcal{V} &= m_q g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g(\boldsymbol{x_q} + \boldsymbol{R}\boldsymbol{\rho}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g\left(\boldsymbol{x_q} + \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{a=1}^{j} l_{ia}\boldsymbol{q_{ia}}\right) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m} m_i g\left(\boldsymbol{x_q} + \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) \cdot \boldsymbol{e_3} \\
&= m_q g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \sum_{a=1}^{j} l_{ia}\boldsymbol{q_{ia}} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m} m_i g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} \\
&= \left(m_q + m_r + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m} m_i\right) g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \sum_{a=1}^{j} l_{ia}\boldsymbol{q_{ia}} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} \\
&= M_{00} g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \sum_{a=1}^{j} l_{ia}\boldsymbol{q_{ia}} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} \\
&= M_{00} g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \sum_{a=1}^{j} l_{ia}\boldsymbol{e_3} \cdot \boldsymbol{q_{ia}} + \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} \\
&= M_{00} g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} g l_{ij}\boldsymbol{e_3} \cdot \boldsymbol{q_{ij}} + \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} \\
\mathcal{V} &= M_{00} g \boldsymbol{x_q} \cdot \boldsymbol{e_3} + m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij} g \boldsymbol{e_3} \cdot \boldsymbol{q_{ij}} + \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}, \quad \text{(Refer Eq. (54)} \qquad (60)
\end{aligned}
$$

Using Eqs. (59 & 60), the Lagrangian ($\mathcal{L}$) of the system is written as given below.

$$
\begin{aligned}
\mathcal{L} &= \mathcal{T} - \mathcal{V} \\
\mathcal{L} &= \frac{1}{2} M_{00} \|\dot{\boldsymbol{x}}_{\boldsymbol{q}}\|^2 + \frac{1}{2}\boldsymbol{\Omega}^T \bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} \\
&\quad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l^2 \dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \dot{\boldsymbol{q}}_{\boldsymbol{b}} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n} M_{ijk}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \cdot \dot{\boldsymbol{q}}_{\boldsymbol{ik}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} \\
&\quad + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&\quad + \frac{1}{2}\sum_{i=1}^{m} m_i l^2 \dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i + \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \\
&\quad - M_{00} g \boldsymbol{x_q} \cdot \boldsymbol{e_3} - m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij} g \boldsymbol{e_3} \cdot \boldsymbol{q_{ij}} - \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} \qquad (61)
\end{aligned}
$$

According to Lagrange-d'Alembert principle [2], the infinitesimal variation in the action integral ($\mathcal{I}$) is equal to the negative of the infinitesimal variation in the work done ($\mathcal{W}$) by the system, i.e.,

$$\delta\mathcal{I} = -\delta\mathcal{W} \qquad (62)$$

The variations [2] in the cable attitude is defined as $\delta \boldsymbol{q} = \boldsymbol{\xi} \times \boldsymbol{q}$, where, differential curves $\boldsymbol{\xi}, \boldsymbol{\xi} : [t_o, t_f] \to \mathbb{R}^3$, satisfying $\boldsymbol{\xi(t_0)} = \boldsymbol{\xi(t_f)} = \boldsymbol{0}$. Similarly, the variations in quadcopter's attitude and angular velocities are defined as $\delta \boldsymbol{R} = \boldsymbol{R}\hat{\boldsymbol{\eta}}$ and $\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\eta}} + \hat{\boldsymbol{\Omega}}\boldsymbol{\eta}$, $\boldsymbol{\eta} \in \mathbb{R}^{\boldsymbol{3}}$ respectively. Hence, the infinitesimal workdone by the system is given in Eq. (63).

$$
\begin{aligned}
\delta\mathcal{W} &= \int_{t_0}^{t_f} \Big(\boldsymbol{u}\cdot\delta\boldsymbol{x_q} + \boldsymbol{\tau}\cdot\boldsymbol{\eta}\Big)dt \\
\delta\mathcal{W} &= \int_{t_0}^{t_f} \boldsymbol{u}\cdot\delta\boldsymbol{x_q}dt + \int_{t_0}^{t_f}\boldsymbol{\tau}\cdot\boldsymbol{\eta}dt
\end{aligned} \tag{63}
$$

The infinitesimal variation in the action integral is given as follows,

$$
\begin{aligned}
\delta\mathcal{I} &= \int_{t_0}^{t_f} \delta\mathcal{L}\ dt \\
&= \int_{t_0}^{t_f}\Bigg[\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\cdot\delta\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\boldsymbol{x_q}}\cdot\delta\boldsymbol{x_q}\Big) + \sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}_{\boldsymbol{ij}}}\cdot\delta\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{\partial\mathcal{L}}{\partial\boldsymbol{q_{ij}}}\cdot\delta\boldsymbol{q_{ij}}\Big) + \Big(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\Omega} + \frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}\Big) + \Big(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\cdot\delta\dot{\alpha} + \frac{\partial\mathcal{L}}{\partial\alpha}\cdot\delta\alpha\Big)\Bigg]dt
\end{aligned} \tag{64}
$$

Finally, using Eqs. (62, 63, and 64), the following equations are derived.

$$
\int_{t_0}^{t_f}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\cdot\delta\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \frac{\partial\mathcal{L}}{\partial\boldsymbol{x_q}}\cdot\delta\boldsymbol{x_q} + \boldsymbol{u}\cdot\delta\boldsymbol{x_q}\Big)dt = 0 \tag{65}
$$

$$
\int_{t_0}^{t_f}\sum_{i=1}^{m}\sum_{j=1}^{n}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}_{\boldsymbol{ij}}}\cdot\delta\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{\partial\mathcal{L}}{\partial\boldsymbol{q_{ij}}}\cdot\delta\boldsymbol{q_{ij}}\Big)dt = 0 \tag{66}
$$

$$
\int_{t_0}^{t_f}\Big(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\Omega} + \frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R} + \boldsymbol{\tau}\cdot\boldsymbol{\eta}\Big)dt = 0 \tag{67}
$$

$$
\int_{t_0}^{t_f}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\cdot\delta\dot{\alpha} + \frac{\partial\mathcal{L}}{\partial\alpha}\cdot\delta\alpha\Big)dt = 0 \tag{68}
$$

Solve Eq. (65) using integral by parts and rearrange the terms,

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\cdot\int_{t_0}^{t_f}\delta\dot{\boldsymbol{x}}_{\boldsymbol{q}}dt - \int_{t_0}^{t_f}\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\Big)\cdot\delta\boldsymbol{x_q}dt + \int_{t_0}^{t_f}\Big(\frac{\partial\mathcal{L}}{\partial\boldsymbol{x_q}} + \boldsymbol{u}\Big)\cdot\delta\boldsymbol{x_q}dt &= 0 \\
\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\cdot[\cancelto{0}{\delta\boldsymbol{x_q}(t_f)} - \cancelto{0}{\delta\boldsymbol{x_q}(t_0)}] - \int_{t_0}^{t_f}\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\Big)\cdot\delta\boldsymbol{x_q}dt + \int_{t_0}^{t_f}\Big(\frac{\partial\mathcal{L}}{\partial\boldsymbol{x_q}} + \boldsymbol{u}\Big)\cdot\delta\boldsymbol{x_q}dt &= 0 \\
\int_{t_0}^{t_f}\Big(\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\Big) - \frac{\partial\mathcal{L}}{\partial\boldsymbol{x_q}} - \boldsymbol{u}\Big)\cdot\delta\boldsymbol{x_q}dt &= 0 \\
\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}_{\boldsymbol{q}}}\Big) - \frac{\partial\mathcal{L}}{\partial\boldsymbol{x_q}} &= \boldsymbol{u}
\end{aligned} \tag{69}
$$

Using the relation $\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\delta\dot{\boldsymbol{q}} = \widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{\xi}} + \widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{\xi}$ (refer Dev. (2)) and $\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\delta\boldsymbol{q} = \widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{\xi}$ (refer Dev. (3)), Eq. (66) is further resolved as follows. Here, I have not put $i^{th}$ subscript in the equations, but assume that you will define the EOM differently for each link.

$$
\begin{aligned}
\int_{t_0}^{t_f}\left(\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{\xi}}+\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{\xi}+\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{\xi}\right)dt=0\\
\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\int_{t_0}^{t_f}\dot{\boldsymbol{\xi}}dt-\int_{t_0}^{t_f}\frac{d}{dt}\left(\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)\cdot\boldsymbol{\xi}dt+\int_{t_0}^{t_f}\left(\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{\xi}+\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{\xi}\right)dt=0\\
\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot[\cancelto{0}{\boldsymbol{\xi}(t_f)}-\cancelto{0}{\boldsymbol{\xi}(t_0)}]-\int_{t_0}^{t_f}\frac{d}{dt}\left(\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)\cdot\boldsymbol{\xi}dt+\int_{t_0}^{t_f}\left(\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{\xi}+\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{\xi}\right)dt=0\\
-\int_{t_0}^{t_f}\frac{d}{dt}\left(\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)\cdot\boldsymbol{\xi}dt+\int_{t_0}^{t_f}\left(\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{\xi}+\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{\xi}\right)dt=0\\
\frac{d}{dt}\left(\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)-\left(\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}+\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}})\right)=0\\
\cancel{\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}}+\widehat{\boldsymbol{q}}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)-\cancel{\widehat{\dot{\boldsymbol{q}}}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}}-\widehat{\boldsymbol{q}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}})=0\\
\widehat{\boldsymbol{q}_{ij}}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}_{ij}}\right)-\widehat{\boldsymbol{q}_{ij}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}_{ij}}=0
\end{aligned}
\tag{70}
$$

Using relation $\delta\boldsymbol{\Omega}=\dot{\boldsymbol{\eta}}+\boldsymbol{\Omega}\times\boldsymbol{\eta}$, Eq. (67) is resolved as follows.

$$
\int_{t_0}^{t_f}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}}\cdot\delta\boldsymbol{\Omega}+\frac{\partial\mathcal{L}}{\partial\boldsymbol{R}}\cdot\delta\boldsymbol{R}+\boldsymbol{\tau}\cdot\boldsymbol{\eta}\right)dt=0
\tag{71}
$$

$$
\begin{aligned}
\int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \delta \boldsymbol{\Omega} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt &= 0 \\
\int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot (\dot{\boldsymbol{\eta}} + \boldsymbol{\Omega} \times \boldsymbol{\eta}) + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt &= 0 \\
\int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \dot{\boldsymbol{\eta}} + \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{\eta}) + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt &= 0 \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot [\cancelto{0}{\boldsymbol{\eta}(t_f)} - \cancelto{0}{\boldsymbol{\eta}(t_0)}] - \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) \cdot \boldsymbol{\eta} dt + \int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{\eta}) + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt &= 0 \\
- \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) \cdot \boldsymbol{\eta} dt + \int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{\eta}) + \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt &= 0 \\
- \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) \cdot \boldsymbol{\eta} dt + \int_{t_0}^{t_f} \left( \widehat{\boldsymbol{\Omega}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \boldsymbol{\eta} + \boldsymbol{d_R} \cdot \boldsymbol{\eta} + \boldsymbol{\tau} \cdot \boldsymbol{\eta} \right) dt &= 0 \\
\left( \because \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{\eta}) = \widehat{\boldsymbol{\Omega}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \cdot \boldsymbol{\eta}, \text{ Dev.(4), and consider} \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \delta \boldsymbol{R} = \boldsymbol{d_R} \cdot \boldsymbol{\eta} \right) & \\
- \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) \cdot \boldsymbol{\eta} dt + \int_{t_0}^{t_f} \left( \widehat{\boldsymbol{\Omega}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} + \boldsymbol{d_R} + \boldsymbol{\tau} \right) \cdot \boldsymbol{\eta} dt &= 0 \\
- \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) + \widehat{\boldsymbol{\Omega}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} + \boldsymbol{d_R} + \boldsymbol{\tau} &= 0 \\
- \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) - \widehat{\boldsymbol{\Omega}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} + \boldsymbol{d_R} + \boldsymbol{\tau} &= 0 \\
\frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} \right) + \widehat{\boldsymbol{\Omega}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} - \boldsymbol{d_R} &= \boldsymbol{\tau}
\end{aligned} \tag{72}
$$

Solve Eq. (68) using integral by parts and rearrange the terms,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \cdot \int_{t_0}^{t_f} \delta \dot{\alpha} dt - \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) \cdot \delta \alpha dt + \int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \alpha} \right) \cdot \delta \alpha dt &= 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \cdot [\cancelto{0}{\delta \alpha(t_f)} - \cancelto{0}{\delta \alpha(t_0)}] - \int_{t_0}^{t_f} \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) \cdot \delta \alpha dt + \int_{t_0}^{t_f} \left( \frac{\partial \mathcal{L}}{\partial \alpha} \right) \cdot \delta \alpha dt &= 0 \\
\int_{t_0}^{t_f} \left( \frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \alpha} \right) \cdot \delta \alpha dt &= 0 \\
\frac{d}{dt}\left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \alpha} &= 0
\end{aligned} \tag{73}
$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\dot{\boldsymbol{x}}_{\boldsymbol{q}}$***

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}} = M_{00}\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_r\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i \tag{74}$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{x_q}$***

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} = -M_{00}g\boldsymbol{e_3} \tag{75}$$

***First EOM***

From Eqs. (69, 74, & 75), the first dynamical equation of the system is derived as follows,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{x_q}} = \boldsymbol{u}$$

$$\frac{d}{dt}\left(M_{00}\dot{\boldsymbol{x}}_{\boldsymbol{q}} + m_r\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i\right) + M_{00}g\boldsymbol{e_3} = \boldsymbol{u}$$

$$\frac{d}{dt}\left(M_{00}\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\left(m_r\boldsymbol{\rho} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) + \boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\right) + M_{00}g\boldsymbol{e_3} = \boldsymbol{u}$$

Consider

$$\boldsymbol{A_1} = \left(m_r\boldsymbol{\rho} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) \tag{76}$$

$$c_1 = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\right) \tag{77}$$

$$\frac{d}{dt}\left(M_{00}\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{A_1} + c_1\boldsymbol{R}\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\right) + M_{00}g\boldsymbol{e_3} = \boldsymbol{u} \tag{78}$$

The time derivative of $\boldsymbol{A_1}$ is

$$\begin{aligned}\dot{\boldsymbol{A}}_{\boldsymbol{1}} &= \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\boldsymbol{q_t}\dot{\alpha} + \sum_{i=1}^{m} m_i l\boldsymbol{q_t}\dot{\alpha} \\ &= \boldsymbol{q_t}\dot{\alpha}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\right) \\ \dot{\boldsymbol{A}}_{\boldsymbol{1}} &= c_1\dot{\alpha}\boldsymbol{q_t}\end{aligned} \tag{79}$$

We know $\boldsymbol{q_b} = \boldsymbol{q_b} = \begin{bmatrix} -sin\alpha & 0 & -cos\alpha \end{bmatrix}^T$. The time derivative of $\boldsymbol{q_b}$ is

$$\begin{aligned} \dot{\boldsymbol{q}}_{\boldsymbol{b}} &= \begin{bmatrix} -cos\alpha & 0 & sin\alpha \end{bmatrix}^T \dot{\alpha} \\ \dot{\boldsymbol{q}}_{\boldsymbol{b}} &= \boldsymbol{q_t}\dot{\alpha} \end{aligned} \tag{80}$$

Where,

$$\boldsymbol{q_t} = \begin{bmatrix} -cos\alpha & 0 & sin\alpha \end{bmatrix}^T \tag{81}$$

The derivative of $\boldsymbol{q_b}$ w.r.t. $\alpha$ is

$$\begin{aligned} \frac{\partial}{\partial\alpha}\boldsymbol{q_b} &= \begin{bmatrix} -cos\alpha & 0 & sin\alpha \end{bmatrix}^T \\ &= \boldsymbol{q_t} \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{d}{dt}\boldsymbol{q_t} &= \begin{bmatrix} sin\alpha & 0 & cos\alpha \end{bmatrix}^T \dot{\alpha} \\ \dot{\boldsymbol{q}}_{\boldsymbol{t}} &= -\boldsymbol{q_b}\dot{\alpha} \end{aligned} \tag{83}$$

$$\begin{aligned} \ddot{\boldsymbol{q}}_{\boldsymbol{b}} &= \dot{\boldsymbol{q}}_{\boldsymbol{t}}\dot{\alpha} + \boldsymbol{q_t}\ddot{\alpha} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{b}} &= -\dot{\alpha}^2\boldsymbol{q_b} + \boldsymbol{q_t}\ddot{\alpha} \end{aligned} \tag{84}$$

Hence, the main dynamical equation will be,

$$\begin{aligned} M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \dot{\boldsymbol{R}}\widehat{\boldsymbol{\Omega}}\boldsymbol{A_1} + \boldsymbol{R}\widehat{\dot{\boldsymbol{\Omega}}}\boldsymbol{A_1} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\dot{\boldsymbol{A}}_{\boldsymbol{1}} + c_1\dot{\boldsymbol{R}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} + c_1\boldsymbol{R}\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + M_{00}g\boldsymbol{e_3} &= \boldsymbol{u} \\ M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Omega}}\boldsymbol{A_1} - \boldsymbol{R}\widehat{\boldsymbol{A_1}}\dot{\boldsymbol{\Omega}} + c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} + c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} + c_1\boldsymbol{R}(-\dot{\alpha}^2\boldsymbol{q_b} + \boldsymbol{q_t}\ddot{\alpha}) + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + M_{00}g\boldsymbol{e_3} &= \boldsymbol{u} \\ M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Omega}}\boldsymbol{A_1} - \boldsymbol{R}\widehat{\boldsymbol{A_1}}\dot{\boldsymbol{\Omega}} + c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} + c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1\dot{\alpha}^2\boldsymbol{R}\boldsymbol{q_b} + c_1\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + M_{00}g\boldsymbol{e_3} &= \boldsymbol{u} \end{aligned} \tag{85}$$

$$M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{R}\widehat{\boldsymbol{A_1}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + c_1\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2\boldsymbol{A_1} + 2c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1\dot{\alpha}^2\boldsymbol{R}\boldsymbol{q_b} + M_{00}g\boldsymbol{e_3} = \boldsymbol{u} \tag{86}$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\dot{\boldsymbol{q}}_{\boldsymbol{ij}}$***

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_{ij}} = \frac{\partial}{\partial\dot{q}_{ij}}\Big(\frac{1}{2}\sum_{i=1}^{m}\sum_{j,k=1}^{n} M_{ijk}\dot{q}_{ij}\cdot\dot{q}_{ik}\Big) + \frac{\partial}{\partial\dot{q}_{ij}}\Big(\dot{x}_q\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{q}_{ij}\Big) + \frac{\partial}{\partial\dot{q}_{ij}}\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}R\widehat{\Omega}(\rho+\rho_i+lq_b)\cdot l_{ij}\dot{q}_{ij}\Big) + \frac{\partial}{\partial\dot{q}_{ij}}\Big(Rl\dot{q}_b\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{q}_{ij}\Big)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_{ij}} = \sum_{k=1}^{n} M_{ijk}\dot{q}_{ik} + M_{0ij}\dot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}R\widehat{\Omega}(\rho+\rho_i+lq_b) + M_{0ij}lR\dot{q}_b \quad \text{(This derivation is verified. Don't waste more time!)} \tag{87}$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $q_{ij}$***

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial q_{ij}} &= \frac{\partial}{\partial q_{ij}}\Big(-\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}ge_3\cdot q_{ij}\Big)\\ \frac{\partial\mathcal{L}}{\partial q_{ij}} &= -M_{0ij}ge_3\end{aligned} \tag{88}$$

***Second EOM***

From Eqs. (70, 87, & 88), the EOM for $ij^{th}$ link is derived as follows,

$$\begin{aligned}\widehat{q}_{ij}\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}_{ij}}\Big) - \widehat{q}_{ij}\frac{\partial\mathcal{L}}{\partial q_{ij}} &= 0\\
\widehat{q}_{ij}\frac{d}{dt}\Big(\sum_{k=1}^{n} M_{ijk}\dot{q}_{ik} + M_{0ij}\dot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}R\widehat{\Omega}(\rho+\rho_i+lq_b) + M_{0ij}lR\dot{q}_b\Big) - \widehat{q}_{ij}(-M_{0ij}ge_3) &= 0\\
\widehat{q}_{ij}\frac{d}{dt}\Big(\sum_{k=1}^{n} M_{ijk}\dot{q}_{ik} + M_{0ij}\dot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}R\widehat{\Omega}(\rho+\rho_i+lq_b) + M_{0ij}lR\dot{q}_b\Big) + \widehat{q}_{ij}M_{0ij}ge_3 &= 0\\
\widehat{q}_{ij}\Big(\sum_{k=1}^{n} M_{ijk}\ddot{q}_{ik} + M_{0ij}\ddot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}\dot{R}\widehat{\Omega}(\rho+\rho_i+lq_b) - \sum_{a=j}^{n} m_{ia}l_{ij}R(\widehat{\rho+\rho_i+lq_b})\dot{\Omega} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}R\widehat{\Omega}q_t + M_{0ij}l\dot{R}\dot{q}_b + M_{0ij}lR(-\dot{\alpha}^2q_b + q_t\ddot{\alpha})\Big) + \widehat{q}_{ij}M_{0ij}ge_3 &= 0\\
\widehat{q}_{ij}\Big(\sum_{k=1}^{n} M_{ijk}\ddot{q}_{ik} + M_{0ij}\ddot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}R\widehat{\Omega}\widehat{\Omega}(\rho+\rho_i+lq_b) - \sum_{a=j}^{n} m_{ia}l_{ij}R(\widehat{\rho+\rho_i+lq_b})\dot{\Omega} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}R\widehat{\Omega}q_t + M_{0ij}lR\widehat{\Omega}\dot{q}_b - M_{0ij}l\dot{\alpha}^2Rq_b + M_{0ij}lRq_t\ddot{\alpha}\Big) + \widehat{q}_{ij}M_{0ij}ge_3 &= 0\\
\sum_{k=1}^{n} M_{ijk}\widehat{q}_{ij}\ddot{q}_{ik} + M_{0ij}\widehat{q}_{ij}\ddot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{q}_{ij}R\widehat{\Omega}^2(\rho+\rho_i+lq_b) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{q}_{ij}R(\widehat{\rho+\rho_i+lq_b})\dot{\Omega} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{q}_{ij}R\widehat{\Omega}q_t + M_{0ij}l\widehat{q}_{ij}R\widehat{\Omega}\dot{q}_b - M_{0ij}l\dot{\alpha}^2\widehat{q}_{ij}Rq_b + M_{0ij}l\widehat{q}_{ij}Rq_t\ddot{\alpha} + \widehat{q}_{ij}M_{0ij}ge_3 &= 0\end{aligned} \tag{89}$$

Now, we will derive the explicit expressions for $\ddot{q}_{ij}$ quantity. From Eq. (89),

$$\begin{aligned}M_{ijj}\widehat{q}_{ij}\ddot{q}_{ij} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{q}_{ij}\ddot{q}_{ik} + M_{0ij}\widehat{q}_{ij}\ddot{x}_q + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{q}_{ij}R\widehat{\Omega}^2(\rho+\rho_i+lq_b) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{q}_{ij}R(\widehat{\rho+\rho_i+lq_b})\dot{\Omega} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{q}_{ij}R\widehat{\Omega}q_t\\
+M_{0ij}l\widehat{q}_{ij}R\widehat{\Omega}\dot{q}_b - M_{0ij}l\dot{\alpha}^2\widehat{q}_{ij}Rq_b + M_{0ij}l\widehat{q}_{ij}Rq_t\ddot{\alpha} + \widehat{q}_{ij}M_{0ij}ge_3 = 0\end{aligned}$$

Multiple above equation by $\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}$

$$M_{ijj}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b})\dot{\boldsymbol{\Omega}} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}$$
$$+M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0$$

Using the relation $\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} = -\boldsymbol{q}\|\dot{\boldsymbol{q}}\|^2 - \ddot{\boldsymbol{q}}$ (refer Derivation 161 for proof),

$$M_{ijj}(-\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 - \ddot{\boldsymbol{q}}_{\boldsymbol{ij}}) + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b})\dot{\boldsymbol{\Omega}} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}$$
$$+M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0$$

$$-M_{ijj}\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 - M_{ijj}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b})\dot{\boldsymbol{\Omega}} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}$$
$$+M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\dot{\boldsymbol{q}}_{\boldsymbol{b}} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0$$

$$-M_{ijj}\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 - M_{ijj}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b})\dot{\boldsymbol{\Omega}} + \sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}$$
$$+M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0$$

$$-M_{ijj}\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 - M_{ijj}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b})\dot{\boldsymbol{\Omega}}$$
$$+2M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0$$

$$M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b})\dot{\boldsymbol{\Omega}} - M_{ijj}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} \widehat{M_{ijk}\boldsymbol{q}_{\boldsymbol{ij}}^2}\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} - M_{ijj}\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})$$
$$+2M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{R}$***

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \partial \boldsymbol{R} = \frac{\partial}{\partial \boldsymbol{R}} \Bigg( & m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) - m_r g\boldsymbol{R}\boldsymbol{\rho}\cdot\boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3} \\
& - \sum_{i=1}^{m} m_i g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3}\Bigg)\cdot\boldsymbol{R}\widehat{\eta} \\
= \frac{\partial}{\partial \boldsymbol{R}} \Bigg( & m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i + l\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \sum_{i=1}^{m} m_i\boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) - m_r g\boldsymbol{R}\boldsymbol{\rho}\cdot\boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3} \\
& - \sum_{i=1}^{m} m_i g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3}\Bigg)\cdot\boldsymbol{R}\widehat{\eta} \\
= \frac{\partial}{\partial \boldsymbol{R}} \Bigg( & m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i + l\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) - m_r g\boldsymbol{R}\boldsymbol{\rho}\cdot\boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3} \\
& - \sum_{i=1}^{m} m_i g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3}\Bigg)\cdot\boldsymbol{R}\widehat{\eta} \\
= \frac{\partial}{\partial \boldsymbol{R}} \Bigg( & m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i - m_r g\boldsymbol{R}\boldsymbol{\rho}\cdot\boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3} \\
& - \sum_{i=1}^{m} m_i g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3}\Bigg)\cdot\boldsymbol{R}\widehat{\eta} + \cancel{\frac{\partial}{\partial \boldsymbol{R}}\Bigg( l\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + l\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\Bigg)\cdot\boldsymbol{R}\widehat{\eta}}
\end{aligned}
$$

To solve this derivative, just simply replace $\boldsymbol{R}$ with $\boldsymbol{R}\widehat{\boldsymbol{\eta}}$ as done in [3]).

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} \cdot \partial \boldsymbol{R} = \frac{\partial}{\partial \boldsymbol{R}}\Big( m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}$$

$$+ \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i - m_r g \boldsymbol{R}\boldsymbol{\rho} \cdot \boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}$$

$$- \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}\Big) \cdot \boldsymbol{R}\widehat{\eta}$$

$$= m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\eta}}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\eta}}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\eta}} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\eta}}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}$$

$$+ \boldsymbol{R}\widehat{\boldsymbol{\eta}} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\eta}}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\eta}} l \dot{\boldsymbol{q}}_{\boldsymbol{b}} \sum_{i=1}^{m} m_i - m_r g \boldsymbol{R}\widehat{\boldsymbol{\eta}}\boldsymbol{\rho} \cdot \boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}\widehat{\boldsymbol{\eta}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}$$

$$- \sum_{i=1}^{m} m_i g \boldsymbol{R}\widehat{\boldsymbol{\eta}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}$$

$$= -m_r \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}\boldsymbol{\eta} - \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{\eta} - l\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{\eta} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{\eta} \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}$$

$$- \boldsymbol{R} l \widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{\eta} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{\eta} - l\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{\eta} \sum_{i=1}^{m} m_i + m_r g \boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\eta} \cdot \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}[\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{\eta} \cdot \boldsymbol{e_3}$$

$$+ \sum_{i=1}^{m} m_i g \boldsymbol{R}[\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{\eta} \cdot \boldsymbol{e_3}$$

$$= -m_r \boldsymbol{\eta}^T \widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}^T \boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{\eta}^T[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]^T \boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} - l\boldsymbol{\eta}^T \widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}^T \boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij}\boldsymbol{\eta}^T[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]^T \boldsymbol{R}^T \dot{\boldsymbol{q}}_{\boldsymbol{ij}}$$

$$- l\boldsymbol{\eta}^T \widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}^T \boldsymbol{R}^T \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^{m} m_i \boldsymbol{\eta}^T[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]^T \boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} - l\boldsymbol{\eta}^T \widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}^T \boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} \sum_{i=1}^{m} m_i + m_r g \boldsymbol{\eta}^T \widehat{\boldsymbol{\rho}}^T \boldsymbol{R}^T \boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{\eta}^T[\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]^T \boldsymbol{R}^T \boldsymbol{e_3}$$

$$+ \sum_{i=1}^{m} m_i g \boldsymbol{\eta}^T[\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]^T \boldsymbol{R}^T \boldsymbol{e_3}$$

$$= \boldsymbol{\eta}^T\Big( m_r \widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}} \boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T \dot{\boldsymbol{q}}_{\boldsymbol{ij}}$$

$$+ l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T \dot{\boldsymbol{x}}_{\boldsymbol{q}} \sum_{i=1}^{m} m_i - m_r g \widehat{\boldsymbol{\rho}}\boldsymbol{R}^T \boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g[\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T \boldsymbol{e_3}$$

$$- \sum_{i=1}^{m} m_i g[\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T \boldsymbol{e_3}\Big)$$

Hence, the expression of $\boldsymbol{d_R}$ is

$$\begin{aligned}
\boldsymbol{d_R} = {}& m_r\widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}[\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}}+\boldsymbol{lq_b})]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}[\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}}+\boldsymbol{lq_b})]\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
& + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i[\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}}+\boldsymbol{lq_b})]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m} m_i - m_r g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g[(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}}+\boldsymbol{lq_b})]\boldsymbol{R}^T\boldsymbol{e_3} \\
& - \sum_{i=1}^{m} m_i g[(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}}+\boldsymbol{lq_b})]\boldsymbol{R}^T\boldsymbol{e_3}
\end{aligned} \tag{90}$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{\Omega}$***

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\boldsymbol{\Omega}} = {}& \frac{\partial}{\partial\boldsymbol{\Omega}}\Bigg(\frac{1}{2}\boldsymbol{\Omega}^T\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\Bigg) \\
= {}& \frac{\partial}{\partial\boldsymbol{\Omega}}\Bigg(\frac{1}{2}\boldsymbol{\Omega}^T\bar{\boldsymbol{J}}\boldsymbol{\Omega} - m_r\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\rho}}\boldsymbol{\Omega} - \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{\Omega} - \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{\Omega} \\
& - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{\Omega}\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{\Omega} - \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{\Omega}\Bigg) \\
= {}& \frac{\partial}{\partial\boldsymbol{\Omega}}\Bigg(\frac{1}{2}\boldsymbol{\Omega}^T\bar{\boldsymbol{J}}\boldsymbol{\Omega} - m_r\boldsymbol{\Omega}^T\widehat{\boldsymbol{\rho}}^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{\Omega}^T(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{\Omega}^T(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
& - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}\boldsymbol{\Omega}^T(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^{m} m_i\boldsymbol{\Omega}^T(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m} m_i\boldsymbol{\Omega}^T(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Bigg) \\
= {}& \bar{\boldsymbol{J}}\boldsymbol{\Omega} - m_r\widehat{\boldsymbol{\rho}}^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
& - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})^T l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
= {}& \bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
& + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\widehat{\boldsymbol{\rho_i}+l}\boldsymbol{q_b})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}
\end{aligned} \tag{91}$$

***Third EOM***
From Eqs. (72, 90, 91) the third dynamical equation of the system is derived as follows,

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}}\right) + \widehat{\boldsymbol{\Omega}}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}} - d_{\boldsymbol{R}} = \boldsymbol{\tau} \\
&\frac{d}{dt}\Bigg(\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Bigg) \\
&+ \widehat{\boldsymbol{\Omega}}\Bigg(\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Bigg) \\
&-\Bigg(m_r\widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&\quad + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m} m_i - m_r g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})]\boldsymbol{R}^T\boldsymbol{e_3} \\
&-\sum_{i=1}^{m} m_i g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})]\boldsymbol{R}^T\boldsymbol{e_3}\Bigg) = \boldsymbol{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}\Bigg(\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Bigg) \\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\qquad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Bigg) \\
&- m_r\widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&\quad - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^{m} m_i[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \widehat{l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\boldsymbol{R}^T}\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m} m_i + m_rg\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\boldsymbol{e_3} \\
&+ \sum_{i=1}^{m} m_ig[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + m_r\widehat{\boldsymbol{\rho}}\dot{\boldsymbol{R}}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\dot{\boldsymbol{R}}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\dot{\boldsymbol{R}}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&+ \sum_{i=1}^{m} m_i(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\dot{\boldsymbol{R}}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{\boldsymbol{q_t}}l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Big) \\
&- m_r\widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&\quad - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^{m} m_i[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \widehat{l\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m} m_i + m_r g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g[(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})]\boldsymbol{R}^T\boldsymbol{e_3} \\
&+ \sum_{i=1}^{m} m_i g[(\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}})]\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + m_r\widehat{\boldsymbol{\rho}}(-\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T)\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})(-\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T)\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\\
&\qquad + \sum_{i=1}^m\sum_{j=1}^n\sum_{a=j}^n m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^m\sum_{j=1}^n\sum_{a=j}^n m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^m\sum_{j=1}^n\sum_{a=j}^n m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})(-\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T)\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\\
&+ \sum_{i=1}^m m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m m_il\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})(-\widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T)\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^m m_il\dot{\alpha}\widehat{\boldsymbol{q_t}}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + m_r\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\\
&\qquad + \sum_{i=1}^m\sum_{j=1}^n\sum_{a=j}^n m_{ia}l_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^m m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^m m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\Big)\\
&- m_r\widehat{\widehat{\boldsymbol{\Omega}}\boldsymbol{\rho}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{i=1}^m\sum_{j=1}^n m_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^m\sum_{j=1}^n m_{ij} - \sum_{i=1}^m\sum_{j=1}^n\sum_{a=j}^n m_{ia}l_{ij}[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\\
&\quad - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\sum_{i=1}^m\sum_{j=1}^n M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^m m_i[\widehat{\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} - \widehat{l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\boldsymbol{R}^T}\dot{\boldsymbol{x}}_q\sum_{i=1}^m m_i + m_rg\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T\boldsymbol{e_3} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\boldsymbol{e_3}\\
&+ \sum_{i=1}^m m_ig[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i}} + l\boldsymbol{q_b})]\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\bar{J}\dot{\Omega} + \dot{\bar{J}}\Omega + m_r\widehat{\rho}R^T\ddot{x}_q - \cancelto{0}{m_r\widehat{\rho}\widehat{\Omega}R^T\dot{x}_q} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\rho + \widehat{\rho_i + lq_b})R^T\ddot{x}_q + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\dot{\alpha}\widehat{q_t}R^T\dot{x}_q - \cancelto{1}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\rho + \widehat{\rho_i + lq_b})\widehat{\Omega}R^T\dot{x}_q} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\rho + \widehat{\rho_i + lq_b})l\ddot{q}_b + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\dot{\alpha}\widehat{q_t}l\dot{q}_b \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\rho + \widehat{\rho_i + lq_b})R^T\ddot{q}_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{q_t}R^T\dot{q}_{ij} - \cancelto{2}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\rho + \widehat{\rho_i + lq_b})\widehat{\Omega}R^T\dot{q}_{ij}} \\
&+ \sum_{i=1}^{m} m_i(\rho + \widehat{\rho_i + lq_b})R^T\ddot{x}_q + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{q_t}R^T\dot{x}_q - \cancelto{3}{\sum_{i=1}^{m} m_i(\rho + \widehat{\rho_i + lq_b})\widehat{\Omega}R^T\dot{x}_q} + \sum_{i=1}^{m} m_i(\rho + \widehat{\rho_i + lq_b})l\ddot{q}_b + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{q_t}l\dot{q}_b \\
&+ \widehat{\Omega}\bar{J}\Omega + \cancelto{0}{m_r\widehat{\Omega}\widehat{\rho}R^T\dot{x}_q} + \cancelto{1}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})R^T\dot{x}_q} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})l\dot{q}_b \\
&\quad + \cancelto{2}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})R^T\dot{q}_{ij}} + \cancelto{3}{\sum_{i=1}^{m} m_i\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})R^T\dot{x}_q} + \sum_{i=1}^{m} m_i\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})l\dot{q}_b \\
&- \cancelto{0}{m_r\widehat{\widehat{\Omega}\rho}R^T\dot{x}_q} - \cancelto{1}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}[\widehat{\Omega}(\rho + \widehat{\rho_i} + lq_b)]R^T\dot{x}_q} - l\widehat{\dot{q}_b}R^T\dot{x}_q\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - \cancelto{2}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}[\widehat{\Omega}(\rho + \widehat{\rho_i} + lq_b)]R^T\dot{q}_{ij}} \\
&\quad - l\widehat{\dot{q}_b}R^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{q}_{ij} - \cancelto{3}{\sum_{i=1}^{m} m_i[\widehat{\Omega}(\rho + \widehat{\rho_i} + lq_b)]R^T\dot{x}_q} - l\widehat{\dot{q}_b}R^T\dot{x}_q\sum_{i=1}^{m} m_i + m_r g\widehat{\rho}R^T e_3 + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g[(\rho + \widehat{\rho_i} + lq_b)]R^T e_3 \\
&+ \sum_{i=1}^{m} m_i g[(\rho + \widehat{\rho_i} + lq_b)]R^T e_3 = \tau
\end{aligned}
$$

$$
\begin{aligned}
&\bar{J}\dot{\Omega} + \dot{\bar{J}}\Omega + m_r\widehat{\rho}R^T\ddot{x}_q + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\rho + \widehat{\rho_i + lq_b})R^T\ddot{x}_q + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\dot{\alpha}\widehat{q_t}R^T\dot{x}_q + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\rho + \widehat{\rho_i + lq_b})l\ddot{q}_b + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\dot{\alpha}\widehat{q_t}l\dot{q}_b \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\rho + \widehat{\rho_i + lq_b})R^T\ddot{q}_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{q_t}R^T\dot{q}_{ij} + \sum_{i=1}^{m} m_i(\rho + \widehat{\rho_i + lq_b})R^T\ddot{x}_q + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{q_t}R^T\dot{x}_q + \sum_{i=1}^{m} m_i(\rho + \widehat{\rho_i + lq_b})l\ddot{q}_b + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{q_t}l\dot{q}_b \\
&\quad + \widehat{\Omega}\bar{J}\Omega + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})l\dot{q}_b + \sum_{i=1}^{m} m_i\widehat{\Omega}(\rho + \widehat{\rho_i + lq_b})l\dot{q}_b - l\widehat{\dot{q}_b}R^T\dot{x}_q\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - l\widehat{\dot{q}_b}R^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{q}_{ij} - l\widehat{\dot{q}_b}R^T\dot{x}_q\sum_{i=1}^{m} m_i \\
&\quad + m_r g\widehat{\rho}R^T e_3 + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g[(\rho + \widehat{\rho_i} + lq_b)]R^T e_3 + \sum_{i=1}^{m} m_i g[(\rho + \widehat{\rho_i} + lq_b)]R^T e_3 = \tau
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} + \Big(m_r\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})\boldsymbol{R}^T + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})\boldsymbol{R}^T\Big)\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l\dot{\alpha}\widehat{\boldsymbol{q_t}} l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij} l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\ddot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{\boldsymbol{q_t}} l\dot{\boldsymbol{q}}_{\boldsymbol{b}} \\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\boldsymbol{q}}_{\boldsymbol{b}} - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - l\widehat{\dot{\boldsymbol{q}}_{\boldsymbol{b}}}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m} m_i \\
&+ \Big(m_r g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})]\boldsymbol{R}^T + \sum_{i=1}^{m} m_i g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})]\boldsymbol{R}^T\Big)\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

Consider

$$
\boldsymbol{A_2} = \Big(m_r g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})]\boldsymbol{R}^T + \sum_{i=1}^{m} m_i g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})]\boldsymbol{R}^T\Big) \tag{92}
$$

And also put, $\dot{\boldsymbol{q}}_{\boldsymbol{b}} = \boldsymbol{q_t}\dot{\alpha}$, $\ddot{\boldsymbol{q}}_{\boldsymbol{b}} = -\dot{\alpha}^2\boldsymbol{q_b} + \boldsymbol{q_t}\ddot{\alpha}$.

$$
\begin{aligned}
&\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l(-\dot{\alpha}^2\boldsymbol{q_b} + \boldsymbol{q_t}\ddot{\alpha}) + \cancel{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l\dot{\alpha}^2\widehat{\boldsymbol{q_t}} l\boldsymbol{q_t}} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij} l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \Big(\cancelto{1}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T} + \cancelto{0}{\sum_{i=1}^{m} m_i l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T} \cancelto{0}{- l\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{R}^T\sum_{i=1}^{m} m_i}\Big)\dot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l(-\dot{\alpha}^2\boldsymbol{q_b} + \boldsymbol{q_t}\ddot{\alpha}) + \cancel{\sum_{i=1}^{m} m_i l\dot{\alpha}^2\widehat{\boldsymbol{q_t}} l\boldsymbol{q_t}} \\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\alpha}\boldsymbol{q_t} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\alpha}\boldsymbol{q_t} \cancelto{1}{- l\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{R}^T\dot{\boldsymbol{x}}_{\boldsymbol{q}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}} - l\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&+ \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\alpha}^2\boldsymbol{q_b} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\boldsymbol{q_t}\ddot{\alpha} \\
&\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia} l_{ij} l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\alpha}^2\boldsymbol{q_b} + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\boldsymbol{q_t}\ddot{\alpha} \\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\alpha}\boldsymbol{q_t} + \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} +} l\boldsymbol{q_b})l\dot{\alpha}\boldsymbol{q_t} - l\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{R}^T\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&+ \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} + \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\Big)\boldsymbol{q_t}\ddot{\alpha} \\
&+ \Big(\cancelto{0}{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}l\dot{\alpha}\widehat{\boldsymbol{q_t}}\boldsymbol{R}^T - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}l\widehat{\boldsymbol{q_t}}\dot{\alpha}\boldsymbol{R}^T}\Big)\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\Big)\dot{\alpha}^2\boldsymbol{q_b} \\
&+ \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \widehat{\boldsymbol{\Omega}}\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\Big)\dot{\alpha}\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
$$

Consider

$$
\boldsymbol{A_3} = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\Big) \tag{93}
$$

$$
\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{A_3}\boldsymbol{q_t}\ddot{\alpha} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \boldsymbol{A_3}\dot{\alpha}^2\boldsymbol{q_b} + \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \widehat{\boldsymbol{\Omega}}\boldsymbol{A_3}\dot{\alpha}\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau}
$$

***Fourth EOM***

***The partial derivatives of $\mathcal{L}$ w.r.t. $\dot{\alpha}$***

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\alpha}} =& \frac{\partial}{\partial\dot{\alpha}}\Big(\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\sum_{i=1}^{m} m_il^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i \\
&+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\Big)
\end{aligned} \tag{94}
$$

put, $\dot{\boldsymbol{q}}_{\boldsymbol{b}} = \boldsymbol{q_t}\dot{\alpha}$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{\alpha}} =& \frac{\partial}{\partial\dot{\alpha}}\Big(\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2(\boldsymbol{q_t}\dot{\alpha})^T(\boldsymbol{q_t}\dot{\alpha}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\sum_{i=1}^{m} m_il^2(\boldsymbol{q_t}\dot{\alpha})^T(\boldsymbol{q_t}\dot{\alpha}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\sum_{i=1}^{m} m_i \\
&+ \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\Big)
\end{aligned} \tag{95}
$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} =\frac{\partial}{\partial \dot{\alpha}}\Bigg(\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2\dot{\alpha}^2 + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l\boldsymbol{q_t}\dot{\alpha}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\alpha}\boldsymbol{q_t}^T\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + l\boldsymbol{R}\boldsymbol{q_t}\dot{\alpha}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\sum_{i=1}^{m} m_i l^2\dot{\alpha}^2 + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l\boldsymbol{q_t}\dot{\alpha}\sum_{i=1}^{m} m_i$$
$$+ l\dot{\alpha}\boldsymbol{q_t}^T\sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\Bigg) \tag{96}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2\dot{\alpha} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l\boldsymbol{q_t}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\boldsymbol{q_t}^T\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + l\boldsymbol{R}\boldsymbol{q_t}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{i=1}^{m} m_i l^2\dot{\alpha} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l\boldsymbol{q_t}\sum_{i=1}^{m} m_i + l\boldsymbol{q_t}^T\sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) \tag{97}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} =\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\Bigg)l\dot{\alpha} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}\boldsymbol{q_t}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\Bigg) + l\boldsymbol{q_t}\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\Bigg) + l\boldsymbol{R}\boldsymbol{q_t}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \tag{98}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} =c_1 l\dot{\alpha} + c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}\boldsymbol{q_t} + l\boldsymbol{q_t}\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\Bigg) + l\boldsymbol{R}\boldsymbol{q_t}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \tag{99}$$

***The partial derivatives of $\mathcal{L}$ w.r.t. $\alpha$***

$$\frac{\partial \mathcal{L}}{\partial \alpha} =\frac{\partial}{\partial \alpha}\Bigg(\frac{1}{2}\boldsymbol{\Omega}^T\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\sum_{i=1}^{m} m_i l^2\dot{\boldsymbol{q}}_{\boldsymbol{b}}^T\dot{\boldsymbol{q}}_{\boldsymbol{b}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\sum_{i=1}^{m} m_i$$
$$+ \boldsymbol{R}l\dot{\boldsymbol{q}}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3} - \sum_{i=1}^{m} m_i g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3}\Bigg) \tag{100}$$

put, $\dot{\boldsymbol{q}}_{\boldsymbol{b}} = \boldsymbol{q_t}\dot{\alpha}$

$$\frac{\partial \mathcal{L}}{\partial \alpha} =\frac{\partial}{\partial \alpha}\Bigg(\frac{1}{2}\boldsymbol{\Omega}^T\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l^2(\boldsymbol{q_t}\dot{\alpha})^T(\boldsymbol{q_t}\dot{\alpha}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\sum_{i=1}^{m} m_i l^2(\boldsymbol{q_t}\dot{\alpha})^T(\boldsymbol{q_t}\dot{\alpha}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\sum_{i=1}^{m} m_i$$
$$+ \boldsymbol{R}l(\boldsymbol{q_t}\dot{\alpha})\cdot\sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3} - \sum_{i=1}^{m} m_i g\boldsymbol{R}(\boldsymbol{\rho}+\boldsymbol{\rho_i}+l\boldsymbol{q_b})\cdot\boldsymbol{e_3}\Bigg) \tag{101}$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& \frac{\partial}{\partial \alpha}\bigg(\frac{1}{2}\boldsymbol{\Omega}^T \bar{\boldsymbol{J}}\boldsymbol{\Omega} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l^2 \boldsymbol{q_t}^T \boldsymbol{q_t}\dot{\alpha}^2 + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\alpha}\boldsymbol{q_t}^T \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_t} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\sum_{i=1}^{m} m_i l^2 \boldsymbol{q_t}^T\boldsymbol{q_t}\dot{\alpha}^2 + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \boldsymbol{q_t}\dot{\alpha}\sum_{i=1}^{m} m_i \\
&+ l\dot{\alpha}\boldsymbol{q_t}^T \sum_{i=1}^{m} m_i \boldsymbol{R}^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} - \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}\bigg)
\end{aligned} \tag{102}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& \frac{\partial}{\partial \alpha}\bigg(\frac{1}{2}\boldsymbol{\Omega}^T \bar{\boldsymbol{J}}\boldsymbol{\Omega} + \cancel{\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} l^2 \dot{\alpha}^2} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\alpha}\boldsymbol{q_t}^T \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_t} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \cancel{\frac{1}{2}\sum_{i=1}^{m} m_i l^2 \dot{\alpha}^2} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \boldsymbol{q_t}\dot{\alpha}\sum_{i=1}^{m} m_i \\
&+ l\dot{\alpha}\boldsymbol{q_t}^T \sum_{i=1}^{m} m_i \widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} - \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}\bigg)
\end{aligned} \tag{103}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& \frac{\partial}{\partial \alpha}\bigg(\frac{1}{2}\boldsymbol{\Omega}^T \bar{\boldsymbol{J}}\boldsymbol{\Omega} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\alpha}\boldsymbol{q_t} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&+ l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_t} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i \boldsymbol{R}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R} l \boldsymbol{q_t}\dot{\alpha}\sum_{i=1}^{m} m_i + l\dot{\alpha}\boldsymbol{q_t} \cdot \sum_{i=1}^{m} m_i \widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3} - \sum_{i=1}^{m} m_i g \boldsymbol{R}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \cdot \boldsymbol{e_3}\bigg)
\end{aligned} \tag{104}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \alpha}\bar{\boldsymbol{J}} &= \frac{\partial}{\partial \alpha}\bigg(\boldsymbol{J} - \frac{1}{2}m_r\widehat{\boldsymbol{\rho}}^2 - \frac{1}{6}m_r l_r^2\widehat{\boldsymbol{e_2}}^2 - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b}])^2 - \frac{1}{2}\sum_{i=1}^{m} m_i([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b}])^2\bigg) \\
&= -\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b}) l\widehat{\boldsymbol{q_t}} - \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b}) l\widehat{\boldsymbol{q_t}} \\
&= -\bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b}) l + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l}\boldsymbol{q_b}) l\bigg)\widehat{\boldsymbol{q_t}} \\
\frac{\partial}{\partial \alpha}\bar{\boldsymbol{J}} &= -\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& \frac{1}{2}\boldsymbol{\Omega}^T(-\boldsymbol{A_3}\widehat{\boldsymbol{q_t}})\boldsymbol{\Omega} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} + l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}(-\boldsymbol{q_b})\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} + l\dot{\alpha}(-\boldsymbol{q_b}) \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \cancel{l\dot{\alpha}\boldsymbol{q_t} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t}} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&+ l\dot{\alpha}\boldsymbol{R}(-\boldsymbol{q_b}) \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l(-\boldsymbol{q_b})\dot{\alpha}\sum_{i=1}^{m} m_i + l\dot{\alpha}(-\boldsymbol{q_b}) \cdot \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \cancel{l\dot{\alpha}\boldsymbol{q_t} \cdot \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t}} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}l\boldsymbol{q_t} \cdot \boldsymbol{e_3} - \sum_{i=1}^{m} m_i g\boldsymbol{R}l\boldsymbol{q_t} \cdot \boldsymbol{e_3}
\end{aligned} \tag{106}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& -\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} - l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - l\dot{\alpha}\boldsymbol{q_b} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&- l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_b} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} - \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\boldsymbol{q_b}\dot{\alpha}\sum_{i=1}^{m} m_i - l\dot{\alpha}\boldsymbol{q_b} \cdot \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}l\boldsymbol{q_t} \cdot \boldsymbol{e_3} - \sum_{i=1}^{m} m_i g\boldsymbol{R}l\boldsymbol{q_t} \cdot \boldsymbol{e_3}
\end{aligned} \tag{107}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& -\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} - l\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij} - l\dot{\alpha}\boldsymbol{q_b} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&- l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_b} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \sum_{i=1}^{m} m_i\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} - \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}l\boldsymbol{q_b}\dot{\alpha}\sum_{i=1}^{m} m_i - l\dot{\alpha}\boldsymbol{q_b} \cdot \sum_{i=1}^{m} m_i\widehat{\boldsymbol{\Omega}}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}g\boldsymbol{R}l\boldsymbol{q_t} \cdot \boldsymbol{e_3} - \sum_{i=1}^{m} m_i g\boldsymbol{R}l\boldsymbol{q_t} \cdot \boldsymbol{e_3}
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& -\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + \dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\right) - \dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\right) + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&- l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_b} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - l\dot{\alpha}\boldsymbol{q_b} \cdot \widehat{\boldsymbol{\Omega}}\left(\sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) - \left(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\right)g\boldsymbol{R}\boldsymbol{q_t} \cdot \boldsymbol{e_3}
\end{aligned} \tag{109}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \alpha} =& -\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - \dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_b}c_1 + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q_t} \cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - l\dot{\alpha}\boldsymbol{R}\boldsymbol{q_b} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&- l\dot{\alpha}\boldsymbol{q_b} \cdot \widehat{\boldsymbol{\Omega}}\left(\sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) - c_1 g\boldsymbol{R}\boldsymbol{q_t} \cdot \boldsymbol{e_3}
\end{aligned} \tag{110}
$$

The dynamics along $\alpha$ is derived as follows.

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0 \\
&\frac{d}{dt}\Bigg(c_1 l\dot{\alpha} + c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} + l\boldsymbol{q}_{\boldsymbol{t}}\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg) + l\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\Bigg) \\
&-\Bigg(-\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A}_{\boldsymbol{3}}\widehat{\boldsymbol{q}_{\boldsymbol{t}}}\boldsymbol{\Omega} + c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}} - \dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}c_1 + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q}_{\boldsymbol{t}}\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} - l\dot{\alpha}\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&- l\dot{\alpha}\boldsymbol{q}_{\boldsymbol{b}}\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg) - c_1 g\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\cdot\boldsymbol{e}_{\boldsymbol{3}}\Bigg) = 0
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
&c_1 l\ddot{\alpha} + c_1\ddot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} + \cancelto{1}{c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}} + \cancelto{2}{c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}(-\boldsymbol{q}_{\boldsymbol{b}}\dot{\alpha})} + \cancelto{0}{l(-\boldsymbol{q}_{\boldsymbol{b}}\dot{\alpha})\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg)} \\
&+ l\boldsymbol{q}_{\boldsymbol{t}}\cdot\dot{\widehat{\boldsymbol{\Omega}}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg) + l\boldsymbol{q}_{\boldsymbol{t}}\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} + \sum_{i=1}^{m} m_i l\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha}\Bigg) + \cancelto{3}{l\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}} \\
&\cancelto{4}{- l\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}\dot{\alpha}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}} + l\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} \\
&+ \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A}_{\boldsymbol{3}}\widehat{\boldsymbol{q}_{\boldsymbol{t}}}\boldsymbol{\Omega} \cancelto{1}{- c_1\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}} + \cancelto{2}{\dot{\alpha}\dot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}c_1} \cancelto{3}{- \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}l\boldsymbol{q}_{\boldsymbol{t}}\cdot l_{ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}} + \cancelto{4}{l\dot{\alpha}\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\dot{\boldsymbol{q}}_{\boldsymbol{ij}}} \\
&+ \cancelto{0}{l\dot{\alpha}\boldsymbol{q}_{\boldsymbol{b}}\cdot\widehat{\boldsymbol{\Omega}}\Bigg(\sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg)} + c_1 g\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\cdot\boldsymbol{e}_{\boldsymbol{3}} = 0
\end{aligned}
$$

$$
c_1 l\ddot{\alpha} + c_1\ddot{\boldsymbol{x}}_{\boldsymbol{q}}\cdot\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} + l\boldsymbol{q}_{\boldsymbol{t}}\cdot\dot{\widehat{\boldsymbol{\Omega}}}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg) + l\boldsymbol{q}_{\boldsymbol{t}}\cdot\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}l + \sum_{i=1}^{m} m_i l\Bigg) + l\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\cdot\sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A}_{\boldsymbol{3}}\widehat{\boldsymbol{q}_{\boldsymbol{t}}}\boldsymbol{\Omega} + c_1 g\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\cdot\boldsymbol{e}_{\boldsymbol{3}} = 0
$$

Consider,

$$
\boldsymbol{A}_{\boldsymbol{4}} = \Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho}+\boldsymbol{\rho}_{\boldsymbol{i}}+l\boldsymbol{q}_{\boldsymbol{b}})\Bigg) \tag{112}
$$

$$c_1 l\ddot{\alpha} + c_1\ddot{\boldsymbol{x}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\boldsymbol{q_t} + l\boldsymbol{q_t} \cdot \widehat{\dot{\boldsymbol{\Omega}}}\boldsymbol{A_4} + \cancel{l\boldsymbol{q_t} \cdot \widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha}c_1} + l\boldsymbol{R}\boldsymbol{q_t} \cdot \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + c_1 g\boldsymbol{R}\boldsymbol{q_t} \cdot \boldsymbol{e_3} = 0$$

$$c_1 l\ddot{\alpha} + c_1\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - l\boldsymbol{q_t}^T\widehat{\boldsymbol{A_4}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + c_1 g\boldsymbol{q_t}^T\boldsymbol{R}^T\boldsymbol{e_3} = 0 \tag{113}$$

$$c_1\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - l\boldsymbol{q_t}^T\widehat{\boldsymbol{A_4}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + c_1 l\ddot{\alpha} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + c_1 g\boldsymbol{q_t}^T\boldsymbol{R}^T\boldsymbol{e_3} = 0 \tag{114}$$

In brief, the EOM of the system are as follows with full expressions.

***Quadcopter's Translational Dynamics:***

$$M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{R}\widehat{\boldsymbol{A_1}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + c_1\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2\boldsymbol{A_1} + 2c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1\dot{\alpha}^2\boldsymbol{R}\boldsymbol{q_b} + M_{00}g\boldsymbol{e_3} = \boldsymbol{u}$$

$$M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{R}\widehat{\boldsymbol{A_1}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}(M_{0i1}\ddot{\boldsymbol{q}}_{\boldsymbol{i1}} + M_{0i2}\ddot{\boldsymbol{q}}_{\boldsymbol{i2}} + \cdots + M_{0in}\ddot{\boldsymbol{q}}_{\boldsymbol{in}}) + c_1\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2\boldsymbol{A_1} + 2c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1\dot{\alpha}^2\boldsymbol{R}\boldsymbol{q_b} + M_{00}g\boldsymbol{e_3} = \boldsymbol{u} \tag{115}$$

$$\begin{aligned} M_{00}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{R}\widehat{\boldsymbol{A_1}}\dot{\boldsymbol{\Omega}} + (M_{011}\ddot{\boldsymbol{q}}_{\boldsymbol{11}} + M_{012}\ddot{\boldsymbol{q}}_{\boldsymbol{12}} + \cdots + M_{01n}\ddot{\boldsymbol{q}}_{\boldsymbol{1n}}) + (M_{021}\ddot{\boldsymbol{q}}_{\boldsymbol{21}} + M_{022}\ddot{\boldsymbol{q}}_{\boldsymbol{22}} + \cdots + M_{02n}\ddot{\boldsymbol{q}}_{\boldsymbol{2n}}) + \cdots \\ +(M_{0m1}\ddot{\boldsymbol{q}}_{\boldsymbol{m1}} + M_{0m2}\ddot{\boldsymbol{q}}_{\boldsymbol{m2}} + \cdots + M_{0mn}\ddot{\boldsymbol{q}}_{\boldsymbol{mn}}) + c_1\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} + \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2\boldsymbol{A_1} + 2c_1\dot{\alpha}\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1\dot{\alpha}^2\boldsymbol{R}\boldsymbol{q_b} + M_{00}g\boldsymbol{e_3} = \boldsymbol{u} \end{aligned} \tag{116}$$

***Links' Attitude Dynamics:***

$$\begin{aligned} M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})\dot{\boldsymbol{\Omega}} - M_{ijj}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_t}\ddot{\alpha} - M_{ijj}\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) \\ +2M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 M_{0ij}g\boldsymbol{e_3} = 0 \end{aligned}$$

Consider

$$\boldsymbol{K_{ij}} = \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \tag{117}$$

Hence, the above expression updated to

$$M_{0ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{ij}}\dot{\boldsymbol{\Omega}} - M_{ijj}\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \sum_{k=1,k\neq j}^{n} M_{ijk}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{ik}} + M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{ijj}\boldsymbol{q}_{\boldsymbol{ij}}\|\dot{\boldsymbol{q}}_{\boldsymbol{ij}}\|^2 + \sum_{a=j}^{n} m_{ia}l_{ij}\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{i}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{0ij}l\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{0ij}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2M_{0ij}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

***For link 11 dynamics***

$$M_{011}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{11}}\dot{\boldsymbol{\Omega}} - M_{111}\ddot{\boldsymbol{q}}_{\boldsymbol{11}} + \sum_{k=2}^{n} M_{11k}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{1k}} + M_{011}l\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{111}\boldsymbol{q}_{\boldsymbol{11}}\|\dot{\boldsymbol{q}}_{\boldsymbol{11}}\|^2 + \sum_{a=1}^{n} m_{1a}l_{11}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{1}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{011}l\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{011}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2M_{011}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

***For link 12 dynamics***

$$M_{012}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{12}}\dot{\boldsymbol{\Omega}} - M_{122}\ddot{\boldsymbol{q}}_{\boldsymbol{12}} + \sum_{k=1,k\neq 2}^{n} M_{12k}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{1k}} + M_{012}l\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{122}\boldsymbol{q}_{\boldsymbol{12}}\|\dot{\boldsymbol{q}}_{\boldsymbol{12}}\|^2 + \sum_{a=2}^{n} m_{1a}l_{12}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{1}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{012}l\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{012}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2M_{012}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

***For link 1n dynamics***

$$M_{01n}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{1n}}\dot{\boldsymbol{\Omega}} - M_{1nn}\ddot{\boldsymbol{q}}_{\boldsymbol{1n}} + \sum_{k=1,k\neq n}^{n} M_{1nk}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{1k}} + M_{01n}l\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{1nn}\boldsymbol{q}_{\boldsymbol{1n}}\|\dot{\boldsymbol{q}}_{\boldsymbol{1n}}\|^2 + \sum_{a=n}^{n} m_{1a}l_{1n}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{1}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{01n}l\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{01n}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2M_{01n}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

***For link 21 dynamics***

$$M_{021}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{21}}\dot{\boldsymbol{\Omega}} - M_{211}\ddot{\boldsymbol{q}}_{\boldsymbol{21}} + \sum_{k=1,k\neq 1}^{n} M_{21k}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{2k}} + M_{021}l\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{211}\boldsymbol{q}_{\boldsymbol{21}}\|\dot{\boldsymbol{q}}_{\boldsymbol{21}}\|^2 + \sum_{a=1}^{n} m_{2a}l_{21}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{2}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{021}l\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{021}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2M_{021}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

***For link 22 dynamics***

$$M_{022}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{22}}\dot{\boldsymbol{\Omega}} - M_{222}\ddot{\boldsymbol{q}}_{\boldsymbol{22}} + \sum_{k=1,k\neq 2}^{n} M_{22k}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{2k}} + M_{022}l\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{222}\boldsymbol{q}_{\boldsymbol{22}}\|\dot{\boldsymbol{q}}_{\boldsymbol{22}}\|^2 + \sum_{a=2}^{n} m_{2a}l_{22}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{2}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{022}l\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{022}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2M_{022}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

***For link 2n dynamics***

$$M_{02n}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - \boldsymbol{K}_{\boldsymbol{2n}}\dot{\boldsymbol{\Omega}} - M_{2nn}\ddot{\boldsymbol{q}}_{\boldsymbol{2n}} + \sum_{k=1,k\neq n}^{n} M_{2nk}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\ddot{\boldsymbol{q}}_{\boldsymbol{2k}} + M_{02n}l\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}}\ddot{\alpha} - M_{2nn}\boldsymbol{q}_{\boldsymbol{2n}}\|\dot{\boldsymbol{q}}_{\boldsymbol{2n}}\|^2 + \sum_{a=n}^{n} m_{2a}l_{2n}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho} + \boldsymbol{\rho}_{\boldsymbol{2}} + l\boldsymbol{q}_{\boldsymbol{b}}) + 2M_{02n}l\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q}_{\boldsymbol{t}}\dot{\alpha} - M_{02n}l\dot{\alpha}^2\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\boldsymbol{q}_{\boldsymbol{b}} + \widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2M_{02n}g\boldsymbol{e}_{\boldsymbol{3}} = 0$$

**For link m1 dynamics**

$$M_{0m1}\widehat{q}_{m1}^2\ddot{x}_q - K_{m1}\dot{\Omega} - M_{m11}\ddot{q}_{m1} + \sum_{k=1,k\neq 1}^{n} M_{m1k}\widehat{q}_{m1}^2\ddot{q}_{mk} + M_{0m1}l\widehat{q}_{m1}^2 Rq_t\ddot{\alpha} - M_{m11}q_{m1}\|\dot{q}_{m1}\|^2 + \sum_{a=1}^{n} m_{ma}l_{m1}\widehat{q}_{m1}^2 R\widehat{\Omega}^2(\rho + \rho_m + lq_b) + 2M_{0m1}l\widehat{q}_{m1}^2 R\widehat{\Omega}q_t\dot{\alpha} - M_{0m1}l\dot{\alpha}^2\widehat{q}_{m1}^2 Rq_b + \widehat{q}_{m1}^2 M_{0m1}g$$

**For link m2 dynamics**

$$M_{0m2}\widehat{q}_{m2}^2\ddot{x}_q - K_{m2}\dot{\Omega} - M_{m22}\ddot{q}_{m2} + \sum_{k=1,k\neq 2}^{n} M_{m2k}\widehat{q}_{m2}^2\ddot{q}_{mk} + M_{0m2}l\widehat{q}_{m2}^2 Rq_t\ddot{\alpha} - M_{m22}q_{m2}\|\dot{q}_{m2}\|^2 + \sum_{a=2}^{n} m_{ma}l_{m2}\widehat{q}_{m2}^2 R\widehat{\Omega}^2(\rho + \rho_m + lq_b) + 2M_{0m2}l\widehat{q}_{m2}^2 R\widehat{\Omega}q_t\dot{\alpha} - M_{0m2}l\dot{\alpha}^2\widehat{q}_{m2}^2 Rq_b + \widehat{q}_{m2}^2 M_{0m2}g$$

**For link mn dynamics**

$$M_{0mn}\widehat{q}_{mn}^2\ddot{x}_q - K_{mn}\dot{\Omega} - M_{mnn}\ddot{q}_{mn} + \sum_{k=1,k\neq n}^{n} M_{mnk}\widehat{q}_{mn}^2\ddot{q}_{mk} + M_{0mn}l\widehat{q}_{mn}^2 Rq_t\ddot{\alpha} - M_{mnn}q_{mn}\|\dot{q}_{mn}\|^2 + \sum_{a=n}^{n} m_{ma}l_{mn}\widehat{q}_{mn}^2 R\widehat{\Omega}^2(\rho + \rho_m + lq_b) + 2M_{0mn}l\widehat{q}_{mn}^2 R\widehat{\Omega}q_t\dot{\alpha} - M_{0mn}l\dot{\alpha}^2\widehat{q}_{mn}^2 Rq_b + \widehat{q}_{mn}^2 M_{0n}$$

***Quadcopter's Rotational Dynamics:***

$$A_2\ddot{x}_q + \bar{J}\dot{\Omega} + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{a=j}^{n} m_{ia}l_{ij}(\widehat{\rho + \rho_i + lq_b})R^T\ddot{q}_{ij} + A_3q_t\ddot{\alpha} + \dot{\bar{J}}\Omega - A_3\dot{\alpha}^2 q_b + \widehat{\Omega}\bar{J}\Omega + \widehat{\Omega}A_3\dot{\alpha}q_t + A_2e_3 = \tau$$

Consider

$$K_{R_{ij}} = \sum_{a=j}^{n} m_{ia}l_{ij}(\widehat{\rho + \rho_i + lq_b})$$

Let verify that is it possible to consider $K_{R_{ij}}$ operator or not.
Consider m=3, n=3
We need to verify that are following two expressions are the same or not..

$$E_1 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{a=j}^{3} m_{ia}l_{ij}(\widehat{\rho + \rho_i + lq_b})R^T\ddot{q}_{ij}$$
$$E_{ij} = K_{R_{ij}}R^T\ddot{q}_{ij}$$

Let's expand, $E_1$.

$$E_1 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{a=j}^{3} m_{ia} l_{ij} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{ij}} \tag{118}$$

$$E_1 = \sum_{i=1}^{3}\left(\sum_{a=1}^{3} m_{ia} l_{i1} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{i1}} + \sum_{a=2}^{3} m_{ia} l_{i2} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{i2}} + \sum_{a=3}^{3} m_{ia} l_{i3} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{i3}}\right) \tag{119}$$

$$E_1 = \sum_{a=1}^{3} m_{1a} l_{11} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_1}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{11}} + \sum_{a=2}^{3} m_{1a} l_{12} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_1}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{12}} + \sum_{a=3}^{3} m_{1a} l_{13} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_1}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{13}} \tag{120}$$

$$+ \sum_{a=1}^{3} m_{2a} l_{21} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_2}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{21}} + \sum_{a=2}^{3} m_{2a} l_{22} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_2}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{22}} + \sum_{a=3}^{3} m_{2a} l_{23} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_2}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{23}} \tag{121}$$

$$+ \sum_{a=1}^{3} m_{3a} l_{31} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_3}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{31}} + \sum_{a=2}^{3} m_{3a} l_{32} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_3}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{32}} + \sum_{a=3}^{3} m_{3a} l_{33} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_3}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{33}} \tag{122}$$

$$\tag{123}$$

Now, test $E_{ij}$,

$$E_{11} = \boldsymbol{K}_{\boldsymbol{R}_{11}} \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{11}} = \sum_{a=1}^{3} m_{1a} l_{11} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_1}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{11}} \tag{124}$$

$$E_{23} = \boldsymbol{K}_{\boldsymbol{R}_{23}} \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{23}} = \sum_{a=3}^{3} m_{2a} l_{23} (\widehat{\boldsymbol{\rho} + \boldsymbol{\rho_2}}) \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{23}} \tag{125}$$

$$\tag{126}$$

Hence, the quadcopter's rotational dynamics is updated as follows.

$$\begin{gathered}
\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n} \boldsymbol{K}_{\boldsymbol{R}_{ij}} \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \boldsymbol{A_3}\boldsymbol{q_t}\ddot{\alpha} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \boldsymbol{A_3}\dot{\alpha}^2\boldsymbol{q_b} + \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \widehat{\boldsymbol{\Omega}}\boldsymbol{A_3}\dot{\alpha}\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau} \\
\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} + \sum_{i=1}^{m}\sum_{j=1}^{n} \boldsymbol{K}_{\boldsymbol{R}_{ij}} \boldsymbol{R}^T \ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \boldsymbol{A_3}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \boldsymbol{A_3}\dot{\alpha}^2\boldsymbol{q_b} + \widehat{\boldsymbol{\Omega}}\boldsymbol{A_3}\dot{\alpha}\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau}
\end{gathered} \tag{127}$$

***Quadcopter's Attitude Dynamics:***

$$
\begin{aligned}
\boldsymbol{A_2}\ddot{\boldsymbol{x}}_{\boldsymbol{q}} &+ \boldsymbol{K}_{\boldsymbol{R}_{11}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{11}} + \boldsymbol{K}_{\boldsymbol{R}_{12}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{12}} + \cdots + \boldsymbol{K}_{\boldsymbol{R}_{1n}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{1n}} \\
&+ \boldsymbol{K}_{\boldsymbol{R}_{21}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{21}} + \boldsymbol{K}_{\boldsymbol{R}_{22}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{22}} + \cdots + \boldsymbol{K}_{\boldsymbol{R}_{2n}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{2n}} \\
&+ \boldsymbol{K}_{\boldsymbol{R}_{m1}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{m1}} + \boldsymbol{K}_{\boldsymbol{R}_{m2}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{m2}} + \cdots + \boldsymbol{K}_{\boldsymbol{R}_{mn}}\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{mn}} \\
&\qquad + \bar{\boldsymbol{J}}\dot{\boldsymbol{\Omega}} + \boldsymbol{A_3}\boldsymbol{q_t}\ddot{\alpha} + \widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \boldsymbol{A_3}\dot{\alpha}^2\boldsymbol{q_b} + \widehat{\boldsymbol{\Omega}}\boldsymbol{A_3}\dot{\alpha}\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} = \boldsymbol{\tau}
\end{aligned}
\tag{128}
$$

***First Link's orientation dynamics:***

$$
c_1\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} - l\boldsymbol{q_t}^T\widehat{\boldsymbol{A_4}}\dot{\boldsymbol{\Omega}} + \sum_{i=1}^{m}\sum_{j=1}^{n} M_{0ij}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{ij}} + c_1l\ddot{\alpha} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + c_1g\boldsymbol{q_t}^T\boldsymbol{R}^T\boldsymbol{e_3} = 0
$$

$$
\begin{aligned}
c_1\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{x}}_{\boldsymbol{q}} &+ M_{011}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{11}} + M_{012}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{12}} + \cdots + M_{01n}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{1n}} \\
&+ M_{021}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{21}} + M_{022}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{22}} + \cdots + M_{02n}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{2n}} \\
&+ M_{0m1}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{m2}} + M_{0m2}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{m2}} + \cdots + M_{0mn}l\boldsymbol{q_t}^T\boldsymbol{R}^T\ddot{\boldsymbol{q}}_{\boldsymbol{mn}} \\
&\qquad - l\boldsymbol{q_t}^T\widehat{\boldsymbol{A_4}}\dot{\boldsymbol{\Omega}} + c_1l\ddot{\alpha} + \frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q_t}}\boldsymbol{\Omega} + c_1g\boldsymbol{q_t}^T\boldsymbol{R}^T\boldsymbol{e_3} = \boldsymbol{0}
\end{aligned}
\tag{129}
$$

Finally, the dynamics of the system is written in terms of matrix form are follows.

$$
\left[\begin{array}{ccccccc}
M_{00} & \begin{bmatrix} M_{011} & M_{012} & \cdots & M_{01n} \end{bmatrix} & \begin{bmatrix} M_{021} & M_{022} & \cdots & M_{02n} \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} M_{0m1} & M_{0m2} & \cdots & M_{0mn} \end{bmatrix} & -\boldsymbol{R}\widehat{\boldsymbol{A}}_{\boldsymbol{1}} & c_1 \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\hline
M_{011}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 & \begin{bmatrix} -M_{111} & M_{112}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 & \cdots & M_{11n}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{11}} & M_{011} l \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
M_{012}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2 & \begin{bmatrix} M_{121}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2 & -M_{122} & \cdots & M_{12n}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2 \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{12}} & M_{012} l \widehat{\boldsymbol{q}}_{\boldsymbol{12}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
M_{01n}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2 & \begin{bmatrix} M_{1n1}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2 & M_{1n2}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2 & \cdots & -M_{1nn} \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{1n}} & M_{01n} l \widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\hline
M_{021}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} -M_{211} & M_{212}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 & \cdots & M_{21n}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{21}} & M_{021} l \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
M_{022}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2 & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} M_{221}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2 & -M_{222} & \cdots & M_{22n}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{22}} & M_{022} l \widehat{\boldsymbol{q}}_{\boldsymbol{22}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
M_{02n}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2 & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} M_{2n1}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2 & M_{2n2}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2 & \cdots & -M_{2nn} \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{2n}} & M_{02n} l \widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\hline
M_{0m1}\widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2 & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} -M_{m11} & M_{m12}\widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2 & \cdots & M_{m1n}\widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{m1}} & M_{0m1} l \widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
M_{0m2}\widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2 & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} M_{m21}\widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2 & -M_{m22} & \cdots & M_{m2n}\widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2 \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{m2}} & M_{0m2} l \widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
M_{0mn}\widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2 & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} 0_3 & 0_3 & \cdots & 0_3 \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} M_{mn1}\widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2 & M_{mn2}\widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2 & \cdots & -M_{mnn} \end{bmatrix} & -\boldsymbol{K}_{\boldsymbol{mn}} & M_{0mn} l \widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q}_{\boldsymbol{t}} \\
\hline
\boldsymbol{A}_{\boldsymbol{2}} & \begin{bmatrix} \boldsymbol{K}_{\boldsymbol{R}_{11}}\boldsymbol{R}^T & \boldsymbol{K}_{\boldsymbol{R}_{12}}\boldsymbol{R}^T & \cdots & \boldsymbol{K}_{\boldsymbol{R}_{1n}}\boldsymbol{R}^T \end{bmatrix} & \begin{bmatrix} \boldsymbol{K}_{\boldsymbol{R}_{21}}\boldsymbol{R}^T & \boldsymbol{K}_{\boldsymbol{R}_{22}}\boldsymbol{R}^T & \cdots & \boldsymbol{K}_{\boldsymbol{R}_{2n}}\boldsymbol{R}^T \end{bmatrix} & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} \boldsymbol{K}_{\boldsymbol{R}_{m1}}\boldsymbol{R}^T & \boldsymbol{K}_{\boldsymbol{R}_{m2}}\boldsymbol{R}^T & \cdots & \boldsymbol{K}_{\boldsymbol{R}_{mn}}\boldsymbol{R}^T \end{bmatrix} & \bar{\boldsymbol{J}} & \boldsymbol{A}_{\boldsymbol{3}}\boldsymbol{q}_{\boldsymbol{t}} \\
\hline
c_1 \boldsymbol{q}_{\boldsymbol{t}}^T \boldsymbol{R}^T & \begin{bmatrix} M_{011}l & M_{012}l & \cdots & M_{01n}l \end{bmatrix} \boldsymbol{q}_{\boldsymbol{t}}^T \boldsymbol{R}^T & \begin{bmatrix} M_{021}l & M_{022}l & \cdots & M_{02n}l \end{bmatrix} \boldsymbol{q}_{\boldsymbol{t}}^T \boldsymbol{R}^T & \begin{bmatrix} \cdot & \cdot & \cdots & \cdot \end{bmatrix} & \begin{bmatrix} M_{0m1}l & M_{0m2}l & \cdots & M_{0mn}l \end{bmatrix} \boldsymbol{q}_{\boldsymbol{t}}^T \boldsymbol{R}^T & -l\boldsymbol{q}_{\boldsymbol{t}}^T \widehat{\boldsymbol{A}}_{\boldsymbol{4}} & c_1 l
\end{array}\right]
\begin{bmatrix}
\ddot{\boldsymbol{x}}_{\boldsymbol{q}} \\
\begin{bmatrix} \ddot{\boldsymbol{q}}_{\boldsymbol{11}} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{12}} \\ \vdots \\ \ddot{\boldsymbol{q}}_{\boldsymbol{1n}} \end{bmatrix} \\
\begin{bmatrix} \ddot{\boldsymbol{q}}_{\boldsymbol{21}} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{22}} \\ \vdots \\ \ddot{\boldsymbol{q}}_{\boldsymbol{2n}} \end{bmatrix} \\
\begin{bmatrix} \cdot \\ \cdot \\ \vdots \\ \cdot \end{bmatrix} \\
\begin{bmatrix} \ddot{\boldsymbol{q}}_{\boldsymbol{m1}} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{m2}} \\ \vdots \\ \ddot{\boldsymbol{q}}_{\boldsymbol{mn}} \end{bmatrix} \\
\dot{\boldsymbol{\Omega}} \\
\ddot{\alpha}
\end{bmatrix}
\tag{130}
$$

$$
+\left[\begin{array}{c}
+\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2\boldsymbol{A_1} + 2c_1\dot\alpha\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1\dot\alpha^2\boldsymbol{R}\boldsymbol{q_b} + M_{00}g\boldsymbol{e_3} \\
\hline
-M_{111}\boldsymbol{q_{11}}\|\dot{\boldsymbol{q}}_{\boldsymbol{11}}\|^2 + \sum_{a=1}^n m_{1a}l_{11}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_1}+l\boldsymbol{q_b}) + 2M_{011}l\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{011}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2M_{011}g\boldsymbol{e_3} \\
-M_{122}\boldsymbol{q_{12}}\|\dot{\boldsymbol{q}}_{\boldsymbol{12}}\|^2 + \sum_{a=2}^n m_{1a}l_{12}\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_1}+l\boldsymbol{q_b}) + 2M_{012}l\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{012}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{12}}^2M_{012}g\boldsymbol{e_3} \\
\vdots \\
-M_{1nn}\boldsymbol{q_{1n}}\|\dot{\boldsymbol{q}}_{\boldsymbol{1n}}\|^2 + \sum_{a=n}^n m_{1a}l_{1n}\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_1}+l\boldsymbol{q_b}) + 2M_{01n}l\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{01n}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{1n}}^2M_{01n}g\boldsymbol{e_3} \\
\hline
-M_{211}\boldsymbol{q_{21}}\|\dot{\boldsymbol{q}}_{\boldsymbol{21}}\|^2 + \sum_{a=1}^n m_{2a}l_{21}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_2}+l\boldsymbol{q_b}) + 2M_{021}l\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{021}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2M_{021}g\boldsymbol{e_3} \\
-M_{222}\boldsymbol{q_{22}}\|\dot{\boldsymbol{q}}_{\boldsymbol{22}}\|^2 + \sum_{a=2}^n m_{2a}l_{22}\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_2}+l\boldsymbol{q_b}) + 2M_{022}l\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{022}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{22}}^2M_{022}g\boldsymbol{e_3} \\
\vdots \\
-M_{2nn}\boldsymbol{q_{2n}}\|\dot{\boldsymbol{q}}_{\boldsymbol{2n}}\|^2 + \sum_{a=n}^n m_{2a}l_{2n}\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_2}+l\boldsymbol{q_b}) + 2M_{02n}l\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{02n}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{2n}}^2M_{02n}g\boldsymbol{e_3} \\
\hline
-M_{m11}\boldsymbol{q_{m1}}\|\dot{\boldsymbol{q}}_{\boldsymbol{m1}}\|^2 + \sum_{a=1}^n m_{ma}l_{m1}\widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_m}+l\boldsymbol{q_b}) + 2M_{0m1}l\widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{0m1}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{m1}}^2M_{0m1}g\boldsymbol{e_3} \\
-M_{m22}\boldsymbol{q_{m2}}\|\dot{\boldsymbol{q}}_{\boldsymbol{m2}}\|^2 + \sum_{a=2}^n m_{ma}l_{m2}\widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_m}+l\boldsymbol{q_b}) + 2M_{0m2}l\widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{0m2}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{m2}}^2M_{0m2}g\boldsymbol{e_3} \\
\vdots \\
-M_{mnn}\boldsymbol{q_{mn}}\|\dot{\boldsymbol{q}}_{\boldsymbol{mn}}\|^2 + \sum_{a=n}^n m_{ma}l_{mn}\widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2(\boldsymbol{\rho}+\boldsymbol{\rho_m}+l\boldsymbol{q_b}) + 2M_{0mn}l\widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2\boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot\alpha - M_{0mn}l\dot\alpha^2\widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2\boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{mn}}^2M_{0mn}g\boldsymbol{e_3} \\
\hline
+\widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \boldsymbol{A_3}\dot\alpha^2\boldsymbol{q_b} + \widehat{\boldsymbol{\Omega}}\boldsymbol{A_3}\dot\alpha\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} \\
\hline
+\frac{1}{2}\boldsymbol{\Omega}^T\boldsymbol{A_3}\widehat{\boldsymbol{q}}_{\boldsymbol{t}}\boldsymbol{\Omega} + c_1 g\boldsymbol{q_t}^T\boldsymbol{R}^T\boldsymbol{e_3}
\end{array}\right]
=
\left[\begin{array}{c}
\boldsymbol{u} \\ \hline \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \hline \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \hline \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \hline \boldsymbol{\tau} \\ \hline 0
\end{array}\right]
\tag{131}
$$

where,

$$
M_{00} = \left(m_q + m_r + \sum_{i=1}^m\sum_{j=1}^n m_{ij} + \sum_{i=1}^m m_i\right), \qquad M_{ijk} = \left\{\sum_{a=\max\{j,k\}}^n m_{ia}\right\} l_{ij}l_{ik}, \qquad M_{0ij} = \sum_{a=j}^n m_{ia}l_{ij} \tag{132}
$$

$$
\bar{\boldsymbol{J}} = \left(\boldsymbol{J} - \frac{1}{2}m_r\widehat{\boldsymbol{\rho}}^2 - \frac{1}{6}m_r l_r^2\widehat{\boldsymbol{e_2}}^2 - \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^n m_{ij}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2 - \frac{1}{2}\sum_{i=1}^m m_i([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2\right) \tag{133}
$$

$$
\boldsymbol{A_1} = \left(m_r\boldsymbol{\rho} + \sum_{i=1}^m\sum_{j=1}^n m_{ij}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^m m_i(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) \tag{134}
$$

$$
c_1 = \left(\sum_{i=1}^m\sum_{j=1}^n m_{ij}l + \sum_{i=1}^m m_i l\right) \tag{135}
$$

$$\boldsymbol{A_2} = \left( m_r g \widehat{\boldsymbol{\rho}} \boldsymbol{R}^T + \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} g [\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}] \boldsymbol{R}^T + \sum_{i=1}^{m} m_i g [\widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})}] \boldsymbol{R}^T \right) \tag{136}$$

$$\boldsymbol{A_3} = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})} l + \sum_{i=1}^{m} m_i \widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})} l \right) \tag{137}$$

$$\boldsymbol{K_{ij}} = \sum_{a=j}^{n} m_{ia} l_{ij} \widehat{\boldsymbol{q}}_{\boldsymbol{ij}}^2 \boldsymbol{R} \widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})} \tag{138}$$

$$\boldsymbol{K_{R_{ij}}} = \sum_{a=j}^{n} m_{ia} l_{ij} \widehat{(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})} \tag{139}$$

### 1.3.1 Special Case 1:

Consider, $m = 3$, and $n = 1$.

$$\begin{bmatrix} M_{00} & M_{011} & M_{021} & M_{031} & -\boldsymbol{R}\widehat{\boldsymbol{A}}_{\boldsymbol{1}} & c_1 \boldsymbol{R}\boldsymbol{q_t} \\ M_{011}\widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 & -M_{111} & 0_3 & 0_3 & -\boldsymbol{K_{11}} & M_{011} l \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q_t} \\ M_{021}\widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 & 0_3 & -M_{211} & 0_3 & -\boldsymbol{K_{21}} & M_{021} l \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q_t} \\ M_{031}\widehat{\boldsymbol{q}}_{\boldsymbol{31}}^2 & 0_3 & 0_3 & -M_{311} & -\boldsymbol{K_{m1}} & M_{031} l \widehat{\boldsymbol{q}}_{\boldsymbol{31}}^{\boldsymbol{2}} \boldsymbol{R}\boldsymbol{q_t} \\ \boldsymbol{A_2} & \boldsymbol{K_{R_{11}}}\boldsymbol{R}^T & \boldsymbol{K_{R_{21}}}\boldsymbol{R}^T & \boldsymbol{K_{R_{31}}}\boldsymbol{R}^T & \bar{\boldsymbol{J}} & \boldsymbol{A_3}\boldsymbol{q_t} \\ c_1 \boldsymbol{q_t}^T \boldsymbol{R}^T & M_{011} l \boldsymbol{q_t}^T \boldsymbol{R}^T & M_{021} l \boldsymbol{q_t}^T \boldsymbol{R}^T & M_{031} l \boldsymbol{q_t}^T \boldsymbol{R}^T & -l\boldsymbol{q_t}^T \widehat{\boldsymbol{A_4}} & c_1 l \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{x}}_q \\ \ddot{\boldsymbol{q}}_{\boldsymbol{11}} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{21}} \\ \ddot{\boldsymbol{q}}_{\boldsymbol{31}} \\ \dot{\boldsymbol{\Omega}} \\ \ddot{\alpha} \end{bmatrix} \tag{140}$$

$$+ \begin{bmatrix} +\boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2 \boldsymbol{A_1} + 2c_1 \dot{\alpha} \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t} - c_1 \dot{\alpha}^2 \boldsymbol{R}\boldsymbol{q_b} + M_{00} g \boldsymbol{e_3} \\ -M_{111}\boldsymbol{q_{11}} \|\dot{\boldsymbol{q}}_{\boldsymbol{11}}\|^2 + m_{11} l_{11} \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2 (\boldsymbol{\rho} + \boldsymbol{\rho_1} + l\boldsymbol{q_b}) + 2M_{011} l \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{011} l \dot{\alpha}^2 \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 \boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{11}}^2 M_{011} g \boldsymbol{e_3} \\ -M_{211}\boldsymbol{q_{21}} \|\dot{\boldsymbol{q}}_{\boldsymbol{21}}\|^2 + m_{21} l_{21} \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2 (\boldsymbol{\rho} + \boldsymbol{\rho_2} + l\boldsymbol{q_b}) + 2M_{021} l \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{021} l \dot{\alpha}^2 \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 \boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{21}}^2 M_{021} g \boldsymbol{e_3} \\ -M_{311}\boldsymbol{q_{31}} \|\dot{\boldsymbol{q}}_{\boldsymbol{31}}\|^2 + m_{31} l_{31} \widehat{\boldsymbol{q}}_{\boldsymbol{31}}^2 \boldsymbol{R}\widehat{\boldsymbol{\Omega}}^2 (\boldsymbol{\rho} + \boldsymbol{\rho_3} + l\boldsymbol{q_b}) + 2M_{031} l \widehat{\boldsymbol{q}}_{\boldsymbol{31}}^2 \boldsymbol{R}\widehat{\boldsymbol{\Omega}}\boldsymbol{q_t}\dot{\alpha} - M_{031} l \dot{\alpha}^2 \widehat{\boldsymbol{q}}_{\boldsymbol{31}}^2 \boldsymbol{R}\boldsymbol{q_b} + \widehat{\boldsymbol{q}}_{\boldsymbol{31}}^2 M_{031} g \boldsymbol{e_3} \\ +\widehat{\boldsymbol{\Omega}}\bar{\boldsymbol{J}}\boldsymbol{\Omega} + \dot{\bar{\boldsymbol{J}}}\boldsymbol{\Omega} - \boldsymbol{A_3}\dot{\alpha}^2 \boldsymbol{q_b} + \widehat{\boldsymbol{\Omega}}\boldsymbol{A_3}\dot{\alpha}\boldsymbol{q_t} + \boldsymbol{A_2}\boldsymbol{e_3} \\ +\frac{1}{2}\boldsymbol{\Omega}^T \boldsymbol{A_3} \widehat{\boldsymbol{q_t}} \boldsymbol{\Omega} + c_1 g \boldsymbol{q_t}^T \boldsymbol{R}^T \boldsymbol{e_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{\tau} \\ 0 \end{bmatrix} \tag{141}$$

where,

$$M_{00} = \left( m_q + m_r + \sum_{i=1}^{m} m_{i1} + \sum_{i=1}^{m} m_i \right), \qquad M_{ijk} = m_{i1} l_{ij} l_{ik}, \qquad M_{0ij} = m_{i1} l_{ij} \tag{142}$$

$$\bar{\boldsymbol{J}} = \left(\boldsymbol{J} - \frac{1}{2}m_r\widehat{\boldsymbol{\rho}}^2 - \frac{1}{6}m_r l_r^2\widehat{\boldsymbol{e_2}}^2 - \frac{1}{2}\sum_{i=1}^{m} m_{i1}([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2 - \frac{1}{2}\sum_{i=1}^{m} m_i([\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}])^2\right) \tag{143}$$

$$\boldsymbol{A_1} = \left(m_r\boldsymbol{\rho} + \sum_{i=1}^{m} m_{i1}(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b}) + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \boldsymbol{\rho_i} + l\boldsymbol{q_b})\right) \tag{144}$$

$$c_1 = \left(\sum_{i=1}^{m} m_{i1}l + \sum_{i=1}^{m} m_i l\right) \tag{145}$$

$$\boldsymbol{A_2} = \left(m_r g\widehat{\boldsymbol{\rho}}\boldsymbol{R}^T + \sum_{i=1}^{m} m_{i1}g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})]\boldsymbol{R}^T + \sum_{i=1}^{m} m_i g[(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})]\boldsymbol{R}^T\right) \tag{146}$$

$$\boldsymbol{A_3} = \left(\sum_{i=1}^{m} m_{i1}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l + \sum_{i=1}^{m} m_i(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}})l\right) \tag{147}$$

$$\boldsymbol{K_{i1}} = m_{i1}l_{i1}\widehat{\boldsymbol{q}}_{\boldsymbol{i1}}^2\boldsymbol{R}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \tag{148}$$

$$\boldsymbol{K_{R_{i1}}} = m_{i1}l_{i1}(\boldsymbol{\rho} + \widehat{\boldsymbol{\rho_i} + l\boldsymbol{q_b}}) \tag{149}$$

# A Useful Derivations

## A.1 Useful Properties

1. Dot cross product: $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$
2. Triple cross product: $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$
3. Trace of matrix:
   (a) $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the trace $Tr(\boldsymbol{A}) = a_{11} + a_{22} + a_{33}$.
   (b) $Tr(\boldsymbol{I_n}) = n\ tr[A] = \sum_{i=1}^{n} [A]_{ii}\ tr[AB] = tr[BA] = tr[B^T A^T] = tr[A^T B^T] \sum_{i=1}^{n} [A]_{ii}$
   (c) $Tr(\boldsymbol{A} + \boldsymbol{B}) = Tr(\boldsymbol{A}) + Tr(\boldsymbol{B})$
   (d) $Tr(\boldsymbol{AB}) = Tr(\boldsymbol{BA})$
   (e) $Tr(c\boldsymbol{A}) = c\ Tr(\boldsymbol{A})$, where $c$ is a scalar
   (f) $Tr(\boldsymbol{R}) = 2cos\theta + 1$, where, $\boldsymbol{R}$ is a rotation matrix
   (g) $|\theta| = arccos\left(\frac{Tr(\boldsymbol{R})-1}{2}\right)$
4. Derivative of Rotation matrix: $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\Omega}$
5. Hat map: $\widehat{\cdot} : \mathbb{R}^3 \to SO(3)$ is defined by the condition $\widehat{\boldsymbol{V}}\boldsymbol{Y} = \boldsymbol{V} \times \boldsymbol{Y}$. $\widehat{V} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$
6. Vee map: $(\cdot)^\vee : SO(3) \to \mathbb{R}^3$ is defined by the condition $\boldsymbol{M}^\vee = \boldsymbol{V} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $M = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$
7. Derivatives of rotation matrix: $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\Omega}$
8. Proof: $tr(\boldsymbol{A}\widehat{\boldsymbol{x}}) = -(\boldsymbol{A} - \boldsymbol{A^T})^\vee \cdot \boldsymbol{x}$

$$
\begin{aligned}
tr(\boldsymbol{A}\widehat{\boldsymbol{x}}) &= \frac{1}{2} tr[\widehat{\boldsymbol{x}}(\boldsymbol{A} - \boldsymbol{A^T})] && (150) \\
&= -\boldsymbol{x^T}(\boldsymbol{A} - \boldsymbol{A^T})^\vee && (151) \\
&= -(\boldsymbol{A} - \boldsymbol{A^T})^\vee \cdot \boldsymbol{x} && (152)
\end{aligned}
$$

9. $(\boldsymbol{A}\widehat{\boldsymbol{x}}\boldsymbol{A}^T = \widehat{\boldsymbol{Ax}})$
10. $\widehat{\boldsymbol{a}}\widehat{\boldsymbol{b}} - \widehat{\boldsymbol{b}}\widehat{\boldsymbol{a}} = \widehat{\widehat{\boldsymbol{a}}\boldsymbol{b}}$

## A.2 Useful Derivations on $\mathbb{S}^2$

Some useful formulas for system evolving on two sphere $\mathbb{S}^2$. Consider the attitude of the object evolving on two spheres as $\boldsymbol{q} \in \mathbb{S}^2$, and its angular velocity as $\boldsymbol{\omega} \in \mathbb{R}^3$.

1. $\boldsymbol{q} \cdot \boldsymbol{\omega} = 0$ (Orthogonality relationship between attitude and angular velocity)
2. $\dot{\boldsymbol{q}} = \boldsymbol{\omega} \times \boldsymbol{q}$ (Kinematic relationship)
3. Derivation of $\|\dot{\boldsymbol{q}}\|^2 = \|\boldsymbol{\omega}\|^2$

$$\begin{aligned}
\|\dot{\boldsymbol{q}}\|^2 &= \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}} \\
&= (\boldsymbol{\omega} \times \boldsymbol{q}) \cdot (\boldsymbol{\omega} \times \boldsymbol{q}) \\
&= \boldsymbol{\omega} \cdot (\boldsymbol{q} \times (\boldsymbol{\omega} \times \boldsymbol{q})) \\
&= \boldsymbol{\omega} \cdot (\boldsymbol{\omega}(\boldsymbol{q} \cdot \boldsymbol{q}) - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{\omega})) \\
&= \boldsymbol{\omega} \cdot (\boldsymbol{\omega} - 0)\|\boldsymbol{q}\|^2 \\
&= \|\boldsymbol{\omega}\|^2
\end{aligned} \tag{153}$$

4. Derivation of $\ddot{\boldsymbol{q}} = \dot{\boldsymbol{\omega}} \times \boldsymbol{q} - \boldsymbol{q}\|\boldsymbol{\omega}\|^2$

$$\begin{aligned}
\dot{\boldsymbol{q}} = \boldsymbol{\omega} \times \boldsymbol{q} \rightarrow \ddot{\boldsymbol{q}} &= \dot{\boldsymbol{\omega}} \times \boldsymbol{q} + \boldsymbol{\omega} \times \dot{\boldsymbol{q}} \\
&= \dot{\boldsymbol{\omega}} \times \boldsymbol{q} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{q}) \\
&= \dot{\boldsymbol{\omega}} \times \boldsymbol{q} + \boldsymbol{\omega}\cancelto{\mathbf{0}}{(\boldsymbol{\omega} \cdot \boldsymbol{q})} - \boldsymbol{q}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \\
\ddot{\boldsymbol{q}} &= \dot{\boldsymbol{\omega}} \times \boldsymbol{q} - \boldsymbol{q}\|\boldsymbol{\omega}\|^2
\end{aligned} \tag{154}$$

5. Derivation for $\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} = \mathbf{0}$

$$\begin{aligned}
\boldsymbol{\omega} \cdot \boldsymbol{q} &= 0 \\
\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} + \boldsymbol{\omega} \cdot \dot{\boldsymbol{q}} &= 0 \\
\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} &= -\boldsymbol{\omega} \cdot \dot{\boldsymbol{q}} \\
\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} &= -\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \boldsymbol{q}) \\
\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} &= \cancelto{0}{-\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \boldsymbol{q})} \\
\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} &= 0
\end{aligned}$$

6. Derivation of $\boldsymbol{q} \times (\boldsymbol{q} \times \dot{\boldsymbol{\omega}}) = \hat{\boldsymbol{q}}^{\mathbf{2}}\dot{\boldsymbol{\omega}} = -\dot{\boldsymbol{\omega}}$

$$\begin{aligned}
\boldsymbol{q} \times (\boldsymbol{q} \times \dot{\boldsymbol{\omega}}) &= \boldsymbol{q}(\dot{\boldsymbol{\omega}} \cdot \boldsymbol{q}) - \dot{\boldsymbol{\omega}}(\boldsymbol{q} \cdot \boldsymbol{q}) \\
\text{Using the property, } \dot{\boldsymbol{\omega}} \cdot \boldsymbol{q} = 0, &\text{(see for the derivation 5)} \\
\boldsymbol{q} \times (\boldsymbol{q} \times \dot{\boldsymbol{\omega}}) &= -\dot{\boldsymbol{\omega}}
\end{aligned} \tag{155}$$

7. Derivation of $\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} = \dot{\boldsymbol{\omega}}$

$$
\begin{aligned}
\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} &= \widehat{\boldsymbol{q}}(\dot{\boldsymbol{\omega}} \times \boldsymbol{q} - \boldsymbol{q}\|\boldsymbol{\omega}\|^2) \\
\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} &= -\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\dot{\boldsymbol{\omega}} - \cancel{\widehat{\boldsymbol{q}}\boldsymbol{q}\|\boldsymbol{\omega}\|^2} \\
\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} &= -\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\dot{\boldsymbol{\omega}} \\
\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} &= \dot{\boldsymbol{\omega}}
\end{aligned} \tag{156}
$$

8. Derivation of $\boldsymbol{e}_{\dot{\boldsymbol{q}}} \cdot \boldsymbol{q} = 0$

$$
\begin{aligned}
\boldsymbol{e}_{\dot{\boldsymbol{q}}} \cdot \boldsymbol{q} &= (\dot{\boldsymbol{q}} - (\boldsymbol{q}_d \times \dot{\boldsymbol{q}}_d) \times \boldsymbol{q}) \cdot \boldsymbol{q} \\
&= \cancelto{0}{\dot{\boldsymbol{q}} \cdot \boldsymbol{q}} - \cancelto{0}{((\boldsymbol{q}_d \times \dot{\boldsymbol{q}}_d) \times \boldsymbol{q}) \cdot \boldsymbol{q})} \\
\boldsymbol{e}_{\dot{\boldsymbol{q}}} \cdot \boldsymbol{q} &= 0
\end{aligned} \tag{157}
$$

9. Derivation of $-\hat{\boldsymbol{q}}^2\boldsymbol{e}_{\dot{\boldsymbol{q}}} = \boldsymbol{e}_{\dot{\boldsymbol{q}}}$

$$
\begin{aligned}
-\hat{\boldsymbol{q}}^2\boldsymbol{e}_{\dot{\boldsymbol{q}}} &= -(\boldsymbol{q} \times (\boldsymbol{q} \times \boldsymbol{e}_{\dot{\boldsymbol{q}}})) \\
&= -(\boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{e}_{\dot{\boldsymbol{q}}}) - \boldsymbol{e}_{\dot{\boldsymbol{q}}}) \\
&= \boldsymbol{e}_{\dot{\boldsymbol{q}}} - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{e}_{\dot{\boldsymbol{q}}}) \\
-\hat{\boldsymbol{q}}^2\boldsymbol{e}_{\dot{\boldsymbol{q}}} &= \boldsymbol{e}_{\dot{\boldsymbol{q}}} \qquad \text{(We know } \boldsymbol{e}_{\dot{\boldsymbol{q}}} \cdot \boldsymbol{q} = 0 \text{ from Eq. (157))}
\end{aligned} \tag{158}
$$

10. If $\boldsymbol{x} \cdot \boldsymbol{q} = 0$ for any $\boldsymbol{x} \in \mathbb{R}^3$ then for any $\boldsymbol{y} \in \mathbb{R}^3$, the following relation hold true [4].

$$
\boldsymbol{x} \cdot \boldsymbol{y} = -\hat{\boldsymbol{q}}^2\boldsymbol{x} \cdot \boldsymbol{y} = -\hat{\boldsymbol{q}}^2\boldsymbol{y} \cdot \boldsymbol{x} \tag{159}
$$

11. The value of $\boldsymbol{e}_{\boldsymbol{q}}$ is

$$
\begin{aligned}
\boldsymbol{e}_{\boldsymbol{q}} &= \boldsymbol{e}_{\boldsymbol{q}}(\boldsymbol{q} \cdot \boldsymbol{q}_d) - \boldsymbol{q}_d \\
\boldsymbol{q}_d \cdot \boldsymbol{e}_{\boldsymbol{q}} &= \boldsymbol{q}_d \cdot (\boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{q}_d) - \boldsymbol{q}_d) \\
&= (\boldsymbol{q}_d \cdot \boldsymbol{q})(\boldsymbol{q} \cdot \boldsymbol{q}_d) - (\boldsymbol{q}_d \cdot \boldsymbol{q}_d) \\
\boldsymbol{q}_d \cdot \boldsymbol{e}_{\boldsymbol{q}} &= (\boldsymbol{q} \cdot \boldsymbol{q}_d)(\boldsymbol{q} \cdot \boldsymbol{q}_d) - 1
\end{aligned} \tag{160}
$$

12. Derivation of $\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} = -\boldsymbol{q}\|\dot{\boldsymbol{q}}\|^2 - \ddot{\boldsymbol{q}}$ and $\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} = -\boldsymbol{q}\|\dot{\boldsymbol{q}}\|^2 - \ddot{\boldsymbol{q}}$.
We know $\boldsymbol{q} \cdot \dot{\boldsymbol{q}} = 0 \rightarrow \frac{d}{dt} \rightarrow \dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} + \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}} = 0 \rightarrow \dot{\boldsymbol{q}} \cdot \ddot{\boldsymbol{q}} = -(\dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{q}})$

$$
\begin{aligned}
\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} &= \boldsymbol{q} \times (\boldsymbol{q} \times \ddot{\boldsymbol{q}}) \\
&= \boldsymbol{q}(\boldsymbol{q} \cdot \ddot{\boldsymbol{q}}) - \ddot{\boldsymbol{q}}(\boldsymbol{q} \cdot \boldsymbol{q}) \\
\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} &= -\boldsymbol{q}\|\dot{\boldsymbol{q}}\|^2 - \ddot{\boldsymbol{q}}
\end{aligned} \tag{161}
$$

Similarly, $\widehat{\boldsymbol{q}}\widehat{\boldsymbol{q}}\ddot{\boldsymbol{q}} = -\boldsymbol{q}\|\dot{\boldsymbol{q}}\|^2 - \ddot{\boldsymbol{q}}$

## A.3 Useful Derivations on $\mathbb{SO}(3)$

1. Derivation of $\dot{\boldsymbol{e}}_{\boldsymbol{R}} = \boldsymbol{C}(\boldsymbol{R}_d^T\boldsymbol{R})\boldsymbol{e}_{\boldsymbol{\Omega}}$.

$$
\begin{aligned}
\dot{\boldsymbol{e}}_{\boldsymbol{R}} &= \frac{d}{dt}\frac{1}{2}(\boldsymbol{R}_d^T\boldsymbol{R} - \boldsymbol{R}^T\boldsymbol{R}_d)^\vee \\
&= \frac{1}{2}(\dot{\boldsymbol{R}}_d^T\boldsymbol{R} + \boldsymbol{R}_d^T\dot{\boldsymbol{R}} - \dot{\boldsymbol{R}}^T\boldsymbol{R}_d - \boldsymbol{R}^T\dot{\boldsymbol{R}}_d)^\vee \\
&= \frac{1}{2}((\boldsymbol{R}_d\widehat{\boldsymbol{\Omega}}_d)^T\boldsymbol{R} + \boldsymbol{R}_d^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}} - (\boldsymbol{R}\widehat{\boldsymbol{\Omega}})^T\boldsymbol{R}_d - \boldsymbol{R}^T\boldsymbol{R}_d\widehat{\boldsymbol{\Omega}}_d)^\vee \\
&= \frac{1}{2}(\widehat{\boldsymbol{\Omega}}_d^T\boldsymbol{R}_d^T\boldsymbol{R} + \boldsymbol{R}_d^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}} - \widehat{\boldsymbol{\Omega}}^T\boldsymbol{R}^T\boldsymbol{R}_d - \boldsymbol{R}^T\boldsymbol{R}_d\widehat{\boldsymbol{\Omega}}_d)^\vee \\
&= \frac{1}{2}(-\widehat{\boldsymbol{\Omega}}_d\boldsymbol{R}_d^T\boldsymbol{R} + \boldsymbol{R}_d^T\boldsymbol{R}\widehat{\boldsymbol{\Omega}} + \widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\boldsymbol{R}_d - \boldsymbol{R}^T\boldsymbol{R}_d\widehat{\boldsymbol{\Omega}}_d)^\vee \\
&= \frac{1}{2}(\boldsymbol{R}_d^T\boldsymbol{R}(\widehat{\boldsymbol{\Omega}} - \boldsymbol{R}^T\boldsymbol{R}_d\widehat{\boldsymbol{\Omega}}_d\boldsymbol{R}_d^T\boldsymbol{R}) + (\widehat{\boldsymbol{\Omega}} - \boldsymbol{R}^T\boldsymbol{R}_d\widehat{\boldsymbol{\Omega}}_d\boldsymbol{R}_d^T\boldsymbol{R})\boldsymbol{R}^T\boldsymbol{R}_d)^\vee \\
&\text{Using the properties, } (\boldsymbol{A}\widehat{\boldsymbol{x}}\boldsymbol{A}^T = \widehat{\boldsymbol{A}\boldsymbol{x}}) \\
&= \frac{1}{2}(\boldsymbol{R}_d^T\boldsymbol{R}\widehat{\boldsymbol{e}}_{\boldsymbol{\Omega}} + \widehat{\boldsymbol{e}}_{\boldsymbol{\Omega}}\boldsymbol{R}^T\boldsymbol{R}_d)^\vee \\
&\text{Using the properties, } \widehat{\boldsymbol{x}}\boldsymbol{A} + \boldsymbol{A}^T\widehat{\boldsymbol{x}} = (\{tr(\boldsymbol{A})\boldsymbol{I} - \boldsymbol{A}\}\boldsymbol{x})^\wedge \\
&= \frac{1}{2}((\{tr(\boldsymbol{R}^T\boldsymbol{R}_d)\boldsymbol{I} - \boldsymbol{R}^T\boldsymbol{R}_d\}\boldsymbol{e}_{\boldsymbol{\Omega}})^\wedge)^\vee \\
&= \frac{1}{2}(tr[\boldsymbol{R}^T\boldsymbol{R}_d]I - \boldsymbol{R}^T\boldsymbol{R}_d)\boldsymbol{e}_{\boldsymbol{\Omega}} \\
\dot{\boldsymbol{e}}_{\boldsymbol{R}} &= \boldsymbol{C}(\boldsymbol{R}^T\boldsymbol{R}_d)\boldsymbol{e}_{\boldsymbol{\Omega}}
\end{aligned}
\tag{162}
$$

where it is shown in [5] that the function $\boldsymbol{C}(\boldsymbol{R}^T\boldsymbol{R}_d)$ satisfies the property $\|\boldsymbol{C}\ (\boldsymbol{R}^T\ \boldsymbol{R}_d)\| \leq 1$ for any rotation matrix in $\mathbb{SO}(3)$.

2. Derivation of $\dot{\boldsymbol{e}}_{\boldsymbol{\Omega}} = \dot{\boldsymbol{\Omega}} + \widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\boldsymbol{R}_d\boldsymbol{\Omega}_d - \boldsymbol{R}^T\boldsymbol{R}_d\dot{\boldsymbol{\Omega}}_d$.

$$
\begin{aligned}
\dot{\boldsymbol{e}}_{\boldsymbol{\Omega}} &= \frac{d}{dt}(\boldsymbol{\Omega} - \boldsymbol{R}^T\boldsymbol{R}_d\boldsymbol{\Omega}_d) \\
&= \dot{\boldsymbol{\Omega}} - \dot{\boldsymbol{R}}^T\boldsymbol{R}_d\boldsymbol{\Omega}_d - \boldsymbol{R}^T\dot{\boldsymbol{R}}_d\boldsymbol{\Omega}_d - \boldsymbol{R}^T\boldsymbol{R}_d\dot{\boldsymbol{\Omega}}_d \\
&= \dot{\boldsymbol{\Omega}} - (\boldsymbol{R}\widehat{\boldsymbol{\Omega}})^T\boldsymbol{R}_d\boldsymbol{\Omega}_d - \boldsymbol{R}^T\boldsymbol{R}_d\cancelto{\mathbf{0}}{\widehat{\boldsymbol{\Omega}}_d\boldsymbol{\Omega}_d} - \boldsymbol{R}^T\boldsymbol{R}_d\dot{\boldsymbol{\Omega}}_d \\
&= \dot{\boldsymbol{\Omega}} - \widehat{\boldsymbol{\Omega}}^T\boldsymbol{R}^T\boldsymbol{R}_d\boldsymbol{\Omega}_d - \boldsymbol{R}^T\boldsymbol{R}_d\dot{\boldsymbol{\Omega}}_d \\
\dot{\boldsymbol{e}}_{\boldsymbol{\Omega}} &= \dot{\boldsymbol{\Omega}} + \widehat{\boldsymbol{\Omega}}\boldsymbol{R}^T\boldsymbol{R}_d\boldsymbol{\Omega}_d - \boldsymbol{R}^T\boldsymbol{R}_d\dot{\boldsymbol{\Omega}}_d
\end{aligned}
\tag{163}
$$

3. Derivation of configuration error function, $\dot{\boldsymbol{\Psi}}_{\boldsymbol{R}}(\boldsymbol{R}, \boldsymbol{R}_d) = \boldsymbol{e}_{\boldsymbol{R}} \cdot \boldsymbol{e}_{\boldsymbol{\Omega}}$.

$$
\begin{aligned}
\dot{\Psi}_{R}(R, R_d) &= \frac{d}{dt}\frac{1}{2} tr[I - R_d^T R] \\
&= \frac{1}{2} tr\Big[\widehat{\Omega}_d R_d^T R - R_d^T R\widehat{\Omega}\Big] \\
&= \frac{1}{2} tr\Big[R_d^T R(R^T R_d \widehat{\Omega}_d R_d^T R) - R_d^T R\widehat{\Omega}\Big] \\
&\text{using relation, } A\widehat{x}A^T = \widehat{Ax} \\
&= \frac{1}{2} tr\Big[R_d^T R(R^T R_d \Omega_d)^\wedge - R_d^T R\widehat{\Omega}\Big] \\
&= -\frac{1}{2} tr\Big[R_d^T R(\widehat{\Omega} - (R^T R_d \Omega_d)^\wedge)\Big] \\
&= -\frac{1}{2} tr\Big[R_d^T R(\Omega - (R^T R_d \Omega_d))^\wedge\Big] \\
&= -\frac{1}{2} tr\Big[R_d^T R\widehat{e_\Omega}\Big] \\
&(\text{ using the property } tr(A\widehat{x}) = -(A - A^T)^\vee \cdot x) \\
&= -\frac{1}{2}(-(R_d^T R - (R_d^T R)^T))^\wedge \cdot e_\Omega \\
&= \frac{1}{2}(R_d^T R - (R_d^T R)^T)^\wedge \cdot e_\Omega \\
\dot{\Psi}_{R}(R, R_d) &= e_R \cdot e_\Omega
\end{aligned} \tag{164}
$$

## A.4 Useful Derivations used for Lagrangian

1. Derivation of $u \cdot (\xi \times q) = (\widehat{q}u) \cdot \xi$.

$$
\begin{aligned}
u \cdot (\xi \times q) &= -u \cdot (q \times \xi) \\
&= -u \cdot (\widehat{q}\xi) \\
&= -(q\xi)^T u \\
&= -(\xi^T \widehat{q}^T) u \\
&= (\xi^T \widehat{q}) u \\
&= \xi \cdot (\widehat{q}u) \\
u \cdot (\xi \times q) &= (\widehat{q}u) \cdot \xi
\end{aligned}
$$

Similarly, $u \cdot (\xi \times q) = (\widehat{q}u) \cdot \xi$

2. Derivation of $\frac{\partial \mathcal{L}}{\partial \dot{q}_p} \cdot \delta\dot{q}_p = \widehat{q}_p \frac{\partial \mathcal{L}}{\partial \dot{q}_p} \cdot \dot{\xi}_p + \widehat{\dot{q}}_p \frac{\partial \mathcal{L}}{\partial \dot{q}_p} \cdot \xi_p$ and $\frac{\partial \mathcal{L}}{\partial \dot{q}_c} \cdot \delta\dot{q}_c = \widehat{q}_c \frac{\partial \mathcal{L}}{\partial \dot{q}_c} \cdot \dot{\xi}_c + \widehat{\dot{q}}_c \frac{\partial \mathcal{L}}{\partial \dot{q}_c} \cdot \xi_c$.

   We know, $\delta q_p = \xi_p \times q_p \rightarrow \delta\dot{q}_p = \dot{\xi}_p \times q_p + \xi_p \times \dot{q}_p$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \cdot \delta \dot{\boldsymbol{q}}_{\boldsymbol{p}} &= \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \cdot (\dot{\boldsymbol{\xi}}_{\boldsymbol{p}} \times \boldsymbol{q}_{\boldsymbol{p}}) + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \cdot (\boldsymbol{\xi}_{\boldsymbol{p}} \times \dot{\boldsymbol{q}}_{\boldsymbol{p}}) \\
&= (\dot{\boldsymbol{\xi}}_{\boldsymbol{p}} \times \boldsymbol{q}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} + (\boldsymbol{\xi}_{\boldsymbol{p}} \times \dot{\boldsymbol{q}}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \\
&= (\widehat{\dot{\boldsymbol{\xi}}}_{\boldsymbol{p}} \boldsymbol{q}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} + (\widehat{\boldsymbol{\xi}}_{\boldsymbol{p}} \dot{\boldsymbol{q}}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \\
&= -(\widehat{\boldsymbol{q}}_{\boldsymbol{p}} \dot{\boldsymbol{\xi}}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} - (\dot{\boldsymbol{q}}_p \widehat{\boldsymbol{\xi}}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \\
&= -\dot{\boldsymbol{\xi}}_p^T \widehat{\boldsymbol{q}}_{\boldsymbol{p}}^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} - \boldsymbol{\xi}_{\boldsymbol{p}}^T \dot{\boldsymbol{q}}_p^T \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \\
\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \cdot \delta \dot{\boldsymbol{q}}_{\boldsymbol{p}} &= \widehat{\boldsymbol{q}}_{\boldsymbol{p}} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \cdot \dot{\boldsymbol{\xi}}_{\boldsymbol{p}} + \widehat{\dot{\boldsymbol{q}}}_{\boldsymbol{p}} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{p}}} \cdot \boldsymbol{\xi}_{\boldsymbol{p}} \qquad (165)
\end{aligned}$$

Similarly, $\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{c}}} \cdot \delta \dot{\boldsymbol{q}}_{\boldsymbol{c}} = \widehat{\boldsymbol{q}}_{\boldsymbol{c}} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{c}}} \cdot \dot{\boldsymbol{\xi}}_{\boldsymbol{c}} + \widehat{\dot{\boldsymbol{q}}}_{\boldsymbol{c}} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{c}}} \cdot \boldsymbol{\xi}_{\boldsymbol{c}}$

3. Derivation of $\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \cdot \boldsymbol{\delta q}_{\boldsymbol{p}} = \widehat{\boldsymbol{q}}_{\boldsymbol{p}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \cdot \boldsymbol{\xi}_{\boldsymbol{p}}$ and $\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{c}}} \cdot \boldsymbol{\delta q}_{\boldsymbol{c}} = \widehat{\boldsymbol{q}}_{\boldsymbol{c}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{c}}} \cdot \boldsymbol{\xi}_{\boldsymbol{c}}$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \cdot \boldsymbol{\delta q}_{\boldsymbol{p}} &= \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \cdot (\boldsymbol{\xi}_{\boldsymbol{p}} \times \boldsymbol{q}_{\boldsymbol{p}}) \\
&= (\boldsymbol{\xi}_{\boldsymbol{p}} \times \boldsymbol{q}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \\
&= -(\widehat{\boldsymbol{q}}_{\boldsymbol{p}} \boldsymbol{\xi}_{\boldsymbol{p}})^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \\
&= -\boldsymbol{\xi}_{\boldsymbol{p}}^T \widehat{\boldsymbol{q}}_{\boldsymbol{p}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \cdot \boldsymbol{\delta q}_{\boldsymbol{p}} &= \widehat{\boldsymbol{q}}_{\boldsymbol{p}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{p}}} \cdot \boldsymbol{\xi}_{\boldsymbol{p}} \qquad (166)
\end{aligned}$$

Similarly, $\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{c}}} \cdot \boldsymbol{\delta q}_{\boldsymbol{c}} = \widehat{\boldsymbol{q}}_{\boldsymbol{c}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}_{\boldsymbol{c}}} \cdot \boldsymbol{\xi}_{\boldsymbol{c}}$

4. Derivation of $\frac{\partial \mathcal{L}}{\partial \Omega_q} \cdot (\boldsymbol{\Omega_q} \times \boldsymbol{\eta_q}) = \widehat{\boldsymbol{\Omega}}_{\boldsymbol{q}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}_q} \cdot \boldsymbol{\eta_q}$.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \Omega_q} \cdot (\boldsymbol{\Omega_q} \times \boldsymbol{\eta_q}) &= (\widehat{\boldsymbol{\Omega}}_{\boldsymbol{q}} \boldsymbol{\eta_q})^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}_q} \\
&= \boldsymbol{\eta}_{\boldsymbol{q}}^T \widehat{\boldsymbol{\Omega}}_{\boldsymbol{q}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}_q} \\
&= \boldsymbol{\eta_q} \cdot \widehat{\boldsymbol{\Omega}}_{\boldsymbol{q}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}_q} \\
\frac{\partial \mathcal{L}}{\partial \Omega_q} \cdot (\boldsymbol{\Omega_q} \times \boldsymbol{\eta_q}) &= \widehat{\boldsymbol{\Omega}}_{\boldsymbol{q}}^T \frac{\partial \mathcal{L}}{\partial \boldsymbol{\Omega}_q} \cdot \boldsymbol{\eta_q} \qquad (167)
\end{aligned}
$$

# References


[1] Francesco Bullo and Andrew D. Lewis. *Geometric Control of Mechanical Systems*, volume 49 of *Texts in Applied Mathematics*. Springer Verlag, New York-Heidelberg-Berlin, 2004.

[2] Taeyoung Lee, Melvin Leok, and N Harris McClamroch. *Global Formulations of Lagrangian and Hamiltonian Dynamics on Manifolds*. Springer, 2017.

[3] Farhad A Goodarzi and Taeyoung Lee. Stabilization of a rigid body payload with multiple cooperative quadrotors. *Journal of Dynamic Systems, Measurement, and Control*, 138(12):121001, 2016.

[4] Tse-Huai Wu, Brien Flewelling, Fred Leve, and Taeyoung Lee. Spacecraft relative attitude formation tracking on so (3) based on line-of-sight measurements. In *2013 American Control Conference*, pages 4820–4825. IEEE, 2013.

[5] Taeyoung Lee, Melvin Leok, and N Harris McClamroch. Geometric tracking control of a quadrotor uav on se (3). In *49th IEEE conference on decision and control (CDC)*, pages 5420–5425. IEEE, 2010.