Matrices can be thought of as a collection of similar dimension vectors either stacked horizontally or vertically. They can also be defined as a rectangular grid of numbers, symbols, and expressions arranged in rows and columns. The dimensions of a matrix are represented as *R* X *C*, where *R* is the number of rows and *C* is the number of columns. This *R* X *C* notation is also called the **order** of the matrix. Let us see an example of a matrix.

4 Columns

$$\text{2 Rows} \quad \begin{bmatrix} 2 & 5 & 1 & 4 \\ 6 & 3 & -2 & 0 \end{bmatrix}$$

Dimensions : (2 x 4)

This is a ***2 by 4*** matrix. This can be thought of as a collection of 4, 2-D vectors stacked vertically one after another or a collection of 2 , 4-D vectors kept one over another.

Traditionally, a matrix in the abstract is named ***A***. Matrix entries (also called elements or components) are denoted by a lower-case ***a***, and a particular entry is referenced by its row index (labeled i) and its column index (labeled j).

For example, -2 is the entry in row 2 and column 3 in the matrix above, so another way of saying that would be a_{23} = -2. More generally, the element in the ***ith*** row and ***jth*** column is labeled a_{ij}, and called the ***ij-entry*** or ***ij-component***.

A little more formally than before, we can denote a 3x3 matrix like this:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

In the above matrix ***A*** , a_{ij} is an entry in the ith row and jth column.The sequence of numbers A_i = (a_{i1}, ..., a_{im}) is the ***ith*** row of ***A***, and the sequence of numbers A^j = (a_{1j}, ..., a_{nj}) is the ***jth*** column of ***A***.

Null Matrix :

A matrix that has all elements 0 is called a **null matrix**. It can be of any order. For example, we could have a null matrix of the order 2 X 3. It's also a **singular matrix**, since it does not have an inverse and its determinant is 0.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Matrix

A **row matrix** is a matrix with only one row. Its order would be 1 X *C*, where *C* is the number of columns. For example, here's a row matrix of the order 1 X 5:

$$\begin{bmatrix} 3 & 5 & 7 & 9 & 11 \end{bmatrix}$$

Column Matrix

A **column matrix** is a matrix with only one column. It is represented by an order of *R* X 1, where R is the number of rows. Here's a column matrix of the order 3 X 1:

$$\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

Square Matrix :

A matrix is said to be square if it has the same number of rows as columns. To designate the size of a square matrix with n rows and n columns, it is called n-square. For example, the matrix below is 3-square.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Diagonal Matrix

A **diagonal matrix** is a ***square*** matrix where all the elements are 0 except for those in the diagonal from the top left corner to the bottom right corner. Let's take a look at a diagonal matrix of order 4 X 4:

$$\begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Transpose of a Matrix

The transpose of a matrix is created by converting its rows into columns; that is, row 1 becomes column 1, row 2 becomes column 2, etc. The transpose of a matrix is indicated with a superscripted T , e.g. the transpose of matrix A is A^T. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Then its transpose is given by

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Multiplication of Matrices :

We can only multiply matrices if the number of columns in the first matrix is the same as the number of rows in the second matrix. Let us see a few examples:

a) Multiplying a 2 × 3 matrix by a 3 × 4 matrix is possible and it gives a 2 × 4 matrix.

b) Multiplying a 7 × 1 matrix by a 1 × 2 matrix is okay; it gives a 7 × 2 matrix

c) A 4 × 3 matrix times a 2 × 3 matrix is NOT possible.

If ***A*** is a ***m × n*** matrix and ***B*** is a ***n × p*** matrix, then the product ***AB*** is an ***m × p*** matrix.The coordinates of ***AB*** are determined by taking the **inner product (or dot product)** of each ***row of A*** and each ***column in B***. That is, if A_1,A_2 ...,A_m are the row vectors of matrix A, and B^1, B^2,...,B^p are the column vectors of B, then ab_{ik} of ***AB*** equals $A_i \cdot B^k$. The example below illustrates.

$$\begin{array}{c} \vec{a_1}\rightarrow \\ \vec{a_2}\rightarrow \end{array} \underset{A}{\begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix}} \cdot \begin{array}{c} \begin{array}{cc} \vec{b_1} & \vec{b_2} \\ \downarrow & \downarrow \end{array} \\ \underset{B}{\begin{bmatrix} 3 & 3 \\ 5 & 2 \end{bmatrix}} \end{array} = \underset{C}{\begin{bmatrix} \vec{a_1}\cdot\vec{b_1} & \vec{a_1}\cdot\vec{b_2} \\ \vec{a_2}\cdot\vec{b_1} & \vec{a_2}\cdot\vec{b_2} \end{bmatrix}}$$

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 5 & 2 \end{bmatrix} B = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} AB = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 0 & 26 \end{bmatrix}$$

$$ab_{11} = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 2(3) + 1(-1) + 4(1) = 9$$

$$ab_{12} = \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 2(4) + 1(4) + 4(2) = 16$$

$$ab_{21} = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 1(3) + 5(-1) + 2(1) = 0$$

$$ab_{22} = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 1(2) + 5(4) + 2(2) = 26$$

Identity Matrix :

The identity matrix is a square matrix that has 1’s along the main diagonal and 0’s for all other entries. This matrix is often written simply as ***I***, and is special in that it acts like 1 in matrix multiplication.It is always a square matrix.For example, consider the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

This is a 2×4 matrix since there are 2 rows and 4 columns. This can be verified that I^2A=A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

And AI^4 =A

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Symmetric Matrix

A matrix whose transpose is the same as the original matrix is called a **symmetric matrix**. Only a square matrix can be a symmetric matrix. In the following example the given matrix *A* is a 3 X 3 symmetric matrix, since it's the same as its transpose A^T.

$$A = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 8 & 6 \\ 9 & 6 & 5 \end{bmatrix} ; A^T = \begin{bmatrix} 5 & 1 & 9 \\ 1 & 8 & 6 \\ 9 & 6 & 5 \end{bmatrix}$$

Orthogonal Matrix

A matrix ***A*** is orthogonal if AA^T = A^TA = I. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{bmatrix}$$

is orthogonal because

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix :

The determinant can be viewed as a function whose ***input*** is a ***square matrix*** and whose ***output*** is a ***number***. If ***n*** is the number of rows and columns in the matrix , we can call our matrix an ***n×n*** matrix. The simplest square matrix is a ***1×1*** matrix, which isn't very interesting since it contains just a single number. The determinant of a ***1×1*** matrix is that number itself. Moving up in complexity, the next square matrix is a 2×2 matrix, which we can write as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We calculate the determinant of this matrix as follows. We proceed along the first row, starting with the upper left component ***a***. We multiply the component ***a*** by the determinant of the "***submatrix***" formed by ***ignoring a's row and column***. In this case, this submatrix is the 1×1 matrix consisting of d, and its determinant is just d. So the first term of the determinant is ***ad***.

Next, we proceed to the second component of the first row, which is the upper right component ***b***. We multiply b by the determinant of the ***submatrix*** formed by ***ignoring b's row and column***, which is ***c***. So, the next term of the determinant is ***bc***. The total determinant is simply the first term ***ad*** **minus** the second term ***bc***. We denote this as

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

Okay, that was a lot of work for a simple fact.But the reason for going through this process was to make calculating a 3×3 (and larger) determinant easy. We calculate the determinant of a 3×3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

in the exact same way. We proceed along the first row and multiply each component by the determinant of the submatrix formed by ignoring that component's row and column. Through this procedure we calculate three terms, one for ***a***, one for ***b***, and one for ***c***. Each of these terms is added together, only with alternating signs (i.e., the first term minus the second term plus the third term). We can now write down the determinant of a 3×3 matrix.

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= aei + bfg + cdh - afh - bdi - ceg$$

The determinant of a matrix ***A*** is also written as **|*A*|**.

Inverse of a Matrix :

For a square matrix A, the inverse matrix A^{-1} is a matrix that when multiplied by A yields the Identity matrix of the vector space. AA^{-1}=I $A^{-1}A$=I.

A^{-1} can be multiplied to the left or right of A, and still yield I. Non-square matrices do not have inverses.That does not mean that all square matrices have inverses. A square matrix which has an inverse is called ***invertible*** or ***nonsingular***, and a square matrix without an inverse is called ***non invertible*** or ***singular***.

OK, how do we calculate the inverse? Well, for a 2x2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{2\,x4 - 1x3} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: A × A^{-1} = **I**

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 2x0.8 - 3x0.2 & -2x0.6 + 3x0.4 \\ 1x0.8 - 4x0.2 & -1x0.6 + 4x0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigenvectors and Eigenvalues of a Matrix :

An eigenvector is a nonzero vector that satisfies the equation

Av = λv

where **A** is ***n x n square*** matrix, **λ** is a ***scalar***, and ***v*** is the nx1 eigenvector. ***λ*** is called an eigenvalue. Eigenvalues and eigenvectors are also known as, respectively, characteristic roots and characteristic vectors, or latent roots and latent vectors.Let us solve the above equation.

Av - λv=0

Av - λIv= 0 where I is an identity matrix of same rank as A. (A-λ**I**)v =0 we need the identity matrix because adding a matrix and a scalar is undefined.

If **v** is non-zero, this equation will only have a solution if

|**A**-λ·**I**|=0

This equation is called the ***characteristic*** equation of **A**, and is an n^{th} order polynomial in λ with n roots. These roots are called the eigenvalues of **A**. Let us explore an example to understand the concept. Consider the following matrix to calculate its eigenvalues and corresponding eigenvectors.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assuming x =[x_1 , x_2] is the eigenvectors of the above matrix and λ is the eigenvalue we can write

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in order to solve for λ, x1 and x2. This statement is equivalent to the system of equations

$2x_1 + x_2 = \lambda x_1$

$x_1 + 2x_2 = \lambda x_2$

which can be rearranged as

$(2 - \lambda)x_1 + x_2 = 0$
$x_1 + (2 - \lambda)x_2 = 0$
A necessary and sufficient condition for this system to have a nonzero vector $[x_1, x_2]$ is that the determinant of the coefficient matrix

$$\begin{bmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{bmatrix}$$

to be zero. Or

$(2 - \lambda)(2 - \lambda) - 1 \cdot 1 = 0$

$\lambda 2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

There are two values of λ that satisfy the last equation; thus there are two eigenvalues of the original matrix A and these are $\lambda_1 = 3$, $\lambda_2 = 1$.

We can find eigenvectors which correspond to these eigenvalues by plugging λ back in to the equations above and solving for x1 and x2. To find an eigenvector corresponding to λ = 3, start with

$(2 - \lambda)x_1 + x_2 = 0$

and substitute to get

$(2 - 3)x_1 + x_2 = 0$

which reduces and rearranges to

$x_1 = x_2$

There are an infinite number of values for x_1 which satisfy this equation; the only restriction is that not all the components in an eigenvector can equal zero. So if $x_1 = 1$, then $x_2 = 1$

and one of the eigenvectors corresponding to λ = 3 is [1, 1].

Finding an eigenvector for λ = 1 works the same way.

$(2 - 1)x_1 + x_2 = 0$

$x_1 = -x_2$

So one of the eigenvectors for λ = 1 is [1,−1].

<u>Diagonalization of a Matrix :</u>

We say that two square matrices A and B are ***similar*** provided there exists an invertible matrix P so that

$A = PBP^{-1}$

If A is similar to B, then A and B have the same eigenvalues.

We also say that a matrix A is diagonalizable if it is similar to a diagonal matrix.

In other words an nxn matrix A is called diagonalizable if $A = PDP^{-1}$ for some diagonal nxn matrix D and for some nxn invertible matrix P

If an n × n matrix A has n linearly independent eigenvectors the A can be reproduced as a diagonal matrix D such that

$A = PDP^{-1}$,

Where the columns of P are n linearly independent eigenvectors of A and, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Steps for diagonalizing an nxn square matrix A

1. Find n eigenvalues (repeated or not) for A and form a diagonal matrix D with eigenvalues on the diagonal;
2. Find n linearly independent eigenvectors corresponding to these eigenvalues, if possible, and form an invertible P ∈ R^{nxn} ;
3. $A = PDP^{-1}$

Let us see an example to illustrate these steps. Let us diagonalize

$$A = \begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix}$$

Step 1: Find the characteristic polynomial:

$$p(\lambda) = det\begin{pmatrix} 2-\lambda & 5 \\ -1 & -4-\lambda \end{pmatrix} = (2-\lambda)(-4-\lambda) - 5(-1) = \lambda^2 + 2\lambda - 3$$

Step 2.: Find the roots of p(λ).

λ_1=1 λ_2=-3

Step 3.: Find the eigenvectors separately for each eigenvalue.

Eigenvector for λ_1=1:

This eigenvector v_1= (x, y) solves Av_1 = $\lambda_1 v_1$, or (A $-\lambda_1$I)v_1 = 0. Hence

$$\begin{pmatrix} 1 & 5 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

x + 5y = 0

−x − 5y = 0

These two equations are not independent, you can throw away one of them, and give any nontrivial solution to x + 5y = 0. For example x = −5, y = 1 would do, hence

$$\mathbf{v}_1 = \begin{pmatrix} -5 \\ 1 \end{pmatrix}.$$

Eigenvector for λ_2=-3:

Similarly, we need

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

solving (A $-\lambda_2$I)v_2 = 0

(these x, y are not the same ones as above), i.e.

$$\begin{pmatrix} 5 & 5 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

5x + 5y = 0

−x − y = 0 so for example x = 1, y = −1 would do.

Hence

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is one of the solution for λ_2=-3

Step 4.: Finally, we write up the diagonal form $A = PDP^{-1}$.

Clearly

$$P = \begin{bmatrix} -5 & 1 \\ 1 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1/4 & -1/4 \\ -1/4 & -5/4 \end{bmatrix}$$

Hence

$$A=\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -1/4 & -1/4 \\ -1/4 & -5/4 \end{bmatrix}$$

is the diagonal decomposition of A.