Mathematical Models in Ecology & Evolution 2019

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27^{th} July, 2019

Problem 6.8

Problem statement: Given a model of infectious disease spread in a fixed population comprising either susceptible S or infected I individuals, whose rates of change are represented as $dS/dt = -acSI + \sigma I$, and $dI/dt = acSI - \sigma I$, where a is the probability of transmission, c is the per-capita contact rate between infected and susceptible individuals, and σ is the recovery rate, we want to (a) prove that the proportion of infected individuals $P = I/(S+I)$ satisfies the differential equation

$$dP/dt = \alpha P(1-P) - \sigma P \tag{1}$$

when $\alpha = ac(S+I)$, (b) Determine the equilibria for P and their validity, (c) Determine the local stability of the equilibria and its implications, (d) Sketch the shape of the differential equation to determine the global stability of the equilibria, and (e) Determine the the general solution of P.

(a) Proving the differential equation of P

Starting from equation (3), and substituting the value of P:

$$dP/dt = \frac{d\frac{I}{(S+I)}}{dt} \tag{2}$$

Applying the quotient rule:

$$dP/dt = \frac{dI}{dt}(I+S) - I\frac{d(I+S)}{dt} \tag{3}$$

and then expanding $Id(I+S)/dt$:

$$dP/dt = \frac{dI}{dt}(I+S) - I\frac{dI}{dt} - I\frac{dS}{dt} \tag{4}$$

and further cancelling oppositely signed $I(dI/dt)$ leaves us the following equation.

$$dP/dt = S\frac{dI}{dt} - I\frac{dS}{dt} \tag{5}$$

Substituting the values of dI/dt and dS/dt as given:

$$dP/dt = \frac{S(acSI - \sigma I) - I(-acSI + \sigma I)}{(I+S)^2} \tag{6}$$

and expanding:

$$dP/dt = \frac{S^2acI - \sigma SI + acSI^2 - \sigma I^2}{(I+S)^2} \tag{7}$$

leaves us:

$$dP/dt = \frac{ac(S+I)\cdot SI}{(I+S)^2} - \frac{\sigma I(I+S)}{(I+S)^2} \tag{8}$$

which, given $\alpha = ac(S+I)$, and $1-P = S/(I+S)$, and on converting all $I/(S+I)$ to P, satisfies the condition $dP/dt = \alpha P(1-P) - \sigma P$.

(b) Equilibria of P

One obvious equilibrium of P is 0, since the infection does not spontaneously arise. The upper equilibrium of P can be found from the osbervation that at equilibrium the rate of change of infection status is 0, i.e., $dP/dt = 0$. Hence at equilibrium:

$$\alpha P(1-P) = \sigma P \tag{9}$$

and the second equilibrium value of P, $\hat{P} = 1 - (\sigma/\alpha)$. Given that P is a proportion, σ/α cannot be greeater than 1, or the rate of recovery cannot be greater than the infectivity. In a special case of the parameters, any starting P is an equilibrium when infectivity and recovery rates are equal.

(c) Local stability of equilibria

Beginning with (3), the differential equation of P, we differentiate $f(P)$ w.r.t. P so:

$$\frac{df(P)}{dP} = \frac{d}{dP}\alpha P(1-P) - \sigma P \tag{10}$$

In *Mathematica* the command `D[\Alpha P (1 - P) - \Sigma P, P]` yields the derivative $\alpha(1-P) - \alpha P - \sigma$, which is defined near the equilibria as:

$$r \equiv df/dP|_{P=\hat{P}} = \alpha(1-\hat{P}) - \alpha\hat{P} - \sigma \tag{11}$$

At each of the two equilibria, $\hat{P} = 0$ and $\hat{P} = 1 - (\sigma/\alpha)$, the corresponding r is obtained by substituting values of $\hat{P}$: $r_0 = \alpha - \sigma$, and $r_{1-(\sigma/\alpha)} = \sigma - \alpha$.

At $\hat{P} = 0$, r is always positive since $\sigma < \alpha$, making it an unstable equilibrium, while r is always negative at $\hat{P} = 1 - (\sigma/\alpha)$, making it a stable equilibrium. There are no oscillations in this one-variable, continuous-time model where $f(n)$ is not a function of time.

(d) Global stability of equilibria

Assuming values of $\alpha = 0.8$ and $\sigma = 0.1$, the global stability of (3) is shown in Figure 1 derived from *Mathematica* using:

```
f = \[Alpha] P (1 - P) - \[Sigma] P;

Plot[f /. \[Alpha] -> 0.8 /. \[Sigma] -> 0.1, {P, 0, 1},
 PlotTheme -> "Classic",
 PlotStyle -> {Black},
 AxesLabel -> {"P", "dP/dt"},
 PlotLabel -> "dP/dt ~ P, \[Alpha] = 0.8, \[Sigma] = 0.1"]
```

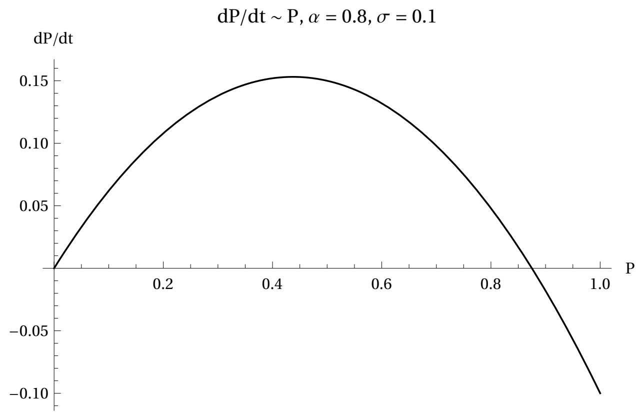

Figure 1: Differential equation dP/dt versus P, for values of $\alpha = 0.8$, $\sigma = 0.1$. The equilibrium at P = 0 is unstable, while the equilibrium at P = 0.875 is stable.

(e) General solution for $dP/dt = \alpha P(1-P) - \sigma P$

The general solution can be found with a separation of variables, where $f(P) = \alpha P(1-P) - \sigma P$, and $g(t) = 1$. The itnegrals for evaluation are

$$\int \frac{1}{f(P)} dP = \int \frac{1}{\alpha P(1-P) - \sigma} dP \tag{12}$$

and $\int g(t)dt = \int 1dt = t + c$. The first integral (14) is solved by passing it to *Mathematica* using

```
iP = Integrate[1/(\[Alpha] P (1 - P) - \[Sigma] P), P];

FullSimplify[iP]
```

which yields

$$\frac{ln(P) - ln[(P-1)(\alpha+\sigma)]}{\alpha - \sigma} \tag{13}$$

This is equated to $t + c$, and the whole equation solved for $P(t)$ using *Mathematica*:

```
eq2 = iP == t + c;

FullSimplify[Solve[eq2, P]]
```

which gives the general solution

$$P(t) = \frac{e^{(c+t)\alpha}(\alpha - \sigma)}{\alpha e^{(c+t)\alpha} - e^{(c+t)\sigma}} \tag{14}$$