

Mathematical Models in Ecology & Evolution 2019

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Problem 3.16

Problem statement: Defining the fitnesses of three diploid phenotypes as follows, $W_{AA} = 1 + s$, $W_{Aa} = 1 + hs$, and $W_{aa} = 1$, we want to show that the recursion equation for the frequency p of allele A at time $t + 1$:

$$p(t + 1) = p(t)^2 \frac{W_{AA}}{\bar{W}} + \frac{1}{2}(2p(t)q(t)) \frac{W_{Aa}}{\bar{W}} \quad (1)$$

can be used to obtain the following continuous-time differential equation:

$$\frac{dp}{dt} = sp(1 - p)(p + h(1 - 2p)) \quad (2)$$

and that this is equivalent to:

$$\frac{dp}{dt} = p(1 - p)(p(W_{AA} - W_{Aa}) + (1 - p)(W_{Aa} - W_{aa})) \quad (3)$$

Part 1:

1. We begin by writing the difference equation form of (1):

$$\Delta p = p(t + 1) - p(t) \quad (4)$$

which works out to:

$$\Delta p = p(t)^2 \frac{W_{AA}}{\bar{W}} + \frac{1}{2}(2p(t)q(t)) \frac{W_{Aa}}{\bar{W}} - p(t) \quad (5)$$

2. Replacing $p(t)$ and $q(t)$ with p and q , and W_{AA} , W_{Aa} , and W_{aa} by the values above, and by fraction subtraction:

$$\Delta p = \frac{p^2(1 + s) + pq(1 + hs) - p(\bar{W})}{\bar{W}} \quad (6)$$

3. Replacing q with $1 - p$, and expanding \bar{W} in the numerator:

$$\Delta p = \frac{p^2(1 + s) + pq(1 + hs) - p(p^2(1 + s) + (1 - p)^2 + 2p(1 - p)(1 + hs))}{\bar{W}} \quad (7)$$

4. Factorising by $p(1 - p)$:

$$\Delta p = \frac{p(1-p)[p(1+s) + ((1+hs)(1-2p)) - (1-p)]}{\overline{W}} \quad (8)$$

which reduces the numerator on expansion to:

$$\Delta p = \frac{p(1-p)[s(p+h-2ph)]}{\overline{W}} \quad (9)$$

and finally upon expanding \overline{W} in the denominator:

$$\Delta p = \frac{sp(1-p)[p+h(1-2p)]}{1+s(p^2+2ph-2p^2h)} \quad (10)$$

5. Over small timesteps Δt , the rate of change of p is:

$$\frac{\Delta p}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{s\Delta t p(1-p)[p+h(1-2p)]}{1+s\Delta t(p^2+2ph-2p^2h)} \cdot \frac{1}{\Delta t} \quad (11)$$

and at the limit of $\Delta t = 0$, we obtain equation (2):

$$\frac{dp}{dt} = sp(1-p)[p+h(1-2p)] \quad (12)$$

Part 2:

6. Working from equation (3), and substituting the values of the phenotype fitnesses:

$$\frac{dp}{dt} = p(1-p)(p(1+s-1-hs) + (1-p)(1+hs-1)) \quad (13)$$

$$p(1-p)(ps-2phs+hs) \quad (14)$$

yields the continuous-time equation (12):

$$sp(1-p)(p+h(1-2p)) \quad (15)$$
