Mathematical Models in Ecology & Evolution 2019

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16th June, 2019

Problem 3.16

Problem statement: Defining the fitnesses of three diploid phenotypes as follows, $W_{AA} = 1 + s$, $W_{Aa} = 1 + hs$, and $W_{aa} = 1$, we want to show that the recursion equation for the frequency p of allele A at time $t + 1$:

$$p(t+1) = p(t)^2 \frac{W_{AA}}{\overline{W}} + \frac{1}{2}(2p(t)q(t)\frac{W_{Aa}}{\overline{W}}) \tag{1}$$

can be used to obtain the following continuous-time differential equation:

$$\frac{dp}{dt} = sp(1-p)(p + h(1-2p)) \tag{2}$$

and that this is equivalent to:

$$\frac{dp}{dt} = p(1-p)(p(W_{AA} - W_{Aa}) + (1-p)(W_{Aa} - W_{aa})) \tag{3}$$

Part 1:

1. We begin by writing the difference equation form of (1):

$$\Delta p = p(t+1) - p(t) \tag{4}$$

which works out to:

$$\Delta p = p(t)^2 \frac{W_{AA}}{\overline{W}} + \frac{1}{2}(2p(t)q(t)\frac{W_{Aa}}{\overline{W}}) - p(t) \tag{5}$$

2. Replacing $p(t)$ and $q(t)$ with p and q, and W_{AA}, W_{Aa}, and W_{aa} by the values above, and by fraction subtraction:

$$\Delta p = \frac{p^2(1+s) + pq(1+hs) - p(\overline{W})}{\overline{W}} \tag{6}$$

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3. Replacing q with $1-p$, and expanding $\overline{W}$ in the numerator:

$$\Delta p = \frac{p^2(1+s) + p(1-p)(1+hs) - p(p^2(1+s) + (1-p)^2 + 2p(1-p)(1+hs))}{\overline{W}} \tag{7}$$

4. Factorising by $p(1-p)$ as follows, first by p, and then by $1-p$:

$$\Delta p = \frac{p[p(1+s)(1-p) + (1-p)(1+hs-1+p-2p(1+hs)]}{\overline{W}} \tag{8}$$

$$\Delta p = \frac{p(1-p)[p(1+s) + ((1+hs)(1-2p)) - (1-p)]}{\overline{W}} \tag{9}$$

which reduces the numerator on expansion to:

$$\Delta p = \frac{p(1-p)[s(p+h-2ph)]}{\overline{W}} \tag{10}$$

and finally upon exapanding $\overline{W}$ in the denominator:

$$\Delta p = \frac{sp(1-p)[p+h(1-2p)]}{1+s(p^2+2ph-2p^2h)} \tag{11}$$

5. Over small timesteps Δt, the rate of change of p is:

$$\frac{\Delta p}{\Delta t} = \lim_{\Delta t \to 0} \frac{s\Delta t p(1-p)[p+h(1-2p)]}{1+s\Delta t(p^2+2ph-2p^2h)} \cdot \frac{1}{\Delta t} \tag{12}$$

and at the limit of $\Delta t = 0$, we obtain equation (2):

$$\frac{dp}{dt} = sp(1-p)[p+h(1-2p)] \tag{13}$$

Part 2:

6. Working from equation (3), and substituting the values of the phenotype fitnesses:

$$\frac{dp}{dt} = p(1-p)(p(1+s-1-hs) + (1-p)(1+hs-1)) \tag{14}$$

$$p(1-p)(ps-2phs+hs) \tag{15}$$

yields the continuous-time equation (12):

$$sp(1-p)(p+h(1-2p)) \tag{16}$$