Some title

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(Q17, MyLab HW 2): Note that the solution here corresponds to no two people having the same birthday. The question instead is asking for no two people having the birth month. Adjust the solution accordingly.

Let A be the event that two people (in a population) share a birthday. This implies A^c is the the event that NO two people share a birthday. Taking their probabilities, we have $P(A) = 1–P(A^c)$.

Lets consider two people in a population. The first person has 365 out of 365 days options to pick their birthday. If we suppose the second person does not share a birthday, this means they can only pick 364 out of the 365 days. We can multiply these together to get the probability they do not have the same birthday is $\frac{364}{365}$, i.e. ,

$$P(A^c) = \frac{365}{365} \times \frac{364}{365}$$

So the chance they do have the same birthday is

$$P(A) = 1 - \frac{364}{365} \approx 0.28$$

Next, lets consider three people. The first person has 365 days to pick their birthday, the second will have 364, and the third will have 363 out of a total of 365 days. The chance that no one shares a birthday with each other is

$$P(A^c) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365}$$

So the chance they do have the same birthday is

$$P(A) = 1 - \left(\frac{364}{365} \times \frac{363}{365}\right)$$

There is a pattern forming. For any n people, the chance that **no one** shares a birthday is

$$P(A^c) = \frac{365}{365} \times \frac{365-1}{365} \times \frac{365-2}{365} \dots \frac{365-n+1}{365}$$

and so the probability that atleast 2 people share a birthday is

$$P(A) = 1 - \frac{365}{365} \times \frac{365-1}{365} \times \frac{365-2}{365} \ldots \frac{365-n+1}{365}$$

Simplifying this equation, we have

$$P(A) = 1 - \frac{365!}{365^n (365-n)!}$$

To solve the problem, set the probability to 0.5 and solve for n. e.e.,

$$0.5 = 1 - \frac{365!}{365^n (365-n)!}$$

This is not easy. There is an approximate solution which is

$$n \approx= \sqrt{2\ln(2)}\sqrt{365}$$

(Q5, Theoretical, Chapter 1) Determine the number of vectors $(x_1, x_2, \dots, x_n)$ such that each x_i is either 0 or 1 and $\sum x_i \geq k$

To understand this question, let's set a value for k. Suppose $k = 1$. Then the question is asking how many vectors exist such that if you add up their elements, i.e.,

$$\sum x_i \geq 1$$

Lets break the inequality down to equalities. Lets ask how many vectors there are such that

$$\sum x_i = 1$$

Lets list a few: $(1, 0, 0, \dots, 0)$ adds to one. Another vector $(0, 1, 0, 0, \dots, 0)$ also adds to one. A third vector is $(0, 0, 1, 0, \dots, 0)$ also adds to one. In general, you have n elements and you want to **choose** just one element $x_i = 1$ (otherwise the sum will not be one). So how many vectors? Precisely

$$\binom{n}{1}$$

Next, lets see how many vectors such that $\sum x_i = 2$. A candidate for this is $(1, 1, 0, 0, \dots, 0)$. Another vector is $(1, 0, 1, 0, \dots, 0)$. In this case, we have n total elements and we want to pick two of them to be equal to one. This is precisely

$$\binom{n}{2}$$

Continuing this, we see that there would be $\binom{n}{3}$ number of vectors such that $\sum x_i = 3$.

And so, going back to the inequality (with assuming $k = 1$) we have

$$\sum x_i \geq 1 => \underbrace{\binom{n}{1}}_{\sum x_i = 1} + \underbrace{\binom{n}{2}}_{\sum x_i = 2} + \underbrace{\binom{n}{3}}_{\sum x_i = 3} \dots = \sum_{i=1}^{n} \binom{n}{i}$$

To complete the question though, note that k is arbitrary. It does not have to start at 1. Therefore, the final answer is

$$\sum_{i=k}^{n} \binom{n}{i}$$

Notice carefully where the i index starts.

(Q14, Self Test, Chapter 1) Determine the number of vectors $(x_1, x_2, ..., x_n)$ such that each x_i is positive and $\sum x_i \leq k$

This is a similar problem but not quite exact. First, the elements of the vector x_i can be any positive value. The second differnce is in the inequality (asking for less than instead of greater than).

Let's pick a value for both n and k to simplify things. Say $n = 3$ and $k = 7$. Let's also look at the equality rather than the inequality, and so we are asked to find the number of vectors such that

$$x_1 + x_2 + x_3 = 7$$

One possible solution is $(2, 1, 4)$. Thinking *way outside the box* this can be visualized as "inserting + symbols in a row of 7 dots". Did that make sense? Let me write it down

$$\bullet\bullet + \bullet + \bullet\bullet\bullet\bullet = 7$$

Infact, the number of vectors such that their elements add up to 7 is precisely insert 2 addition signs in a row of 7 dots. Try it, here is a row of dots. Insert two addition signs whereever you want (and then count the dots).

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

In other words, you have a total of "9" objects (the dots + addition signs), of which "2" are addition signs. The number of ways you can arrange this is precisely $\binom{9}{2}$. In **general**, a solution to the equation

$$x_1 + x_2 + ... + x_n = k$$

corresponds to placing $n-1$ addition signs in a row of k dots. The number of ways of placing $n-1$ addition signs in a row of k bullets is

$$\binom{k+n-1}{n-1}$$

where $k+n-1$ corresponds to the "total" number of objects (i.e., bullets + addition signs). Another way to think about this is that if you have $k+n-1$ "total" objects, then instead of picking $n-1$ addition signs, you can equivalently pick k bullets instead to group, and so we have

$$\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$$

Lets finish the solution. The question is not asking how many vectors such that $\sum x_i = k$, but instead the inequality $\sum x_i \leq k$. So we just have to sum them all up.

The total number of vectors such as $\sum x_i \leq k$ is equal to

$$\sum_{i=1}^{k} \binom{k+n-1}{i}$$

(Q14 MyLab HW 3 (or textbook question 3.77)) A **and** B **alternate rolling a pair of dice, stopping when** A **rolls the sum 10 or when** B **rolls the sum 7. Assuming** A **rolls first, find the probability the final roll is made by** A**.**

For each A and B, there are 36 outcomes (the result of two dice). Out of these, there are 3 outcomes in which the sum is 10 i.e. $\{(4,6),(6,4),(5,5)\}$. Similarly for B, there are six outcomes which results in a 7. So we have

$$P(\text{rolling a ten}) = \frac{3}{36} = \frac{1}{12} \quad \text{and} \quad P(\text{rolling a six}) = \frac{6}{36} = \frac{1}{6}$$

Let A_n denote the event that the nth roll for A wins the game. That is, the nth roll results in a ten. So A_1 refers to rolling a ten on the first, A_2 refers to rolling a ten on A's second roll, and so on. It should be obvious then that $P(A_1) = P(\text{roll } 10) = \frac{1}{12}$.

What about A_2? This event can only happen if the first roll did not equal ten **and** also that B failed to roll 7 **and** rolling a ten on A's second roll. Therefore,

$$P(A_2) = \underbrace{(1 - P(\text{roll } 10)) \cdot (1 - P(\text{roll } 10))}_{\text{first failed roll by A and B}} \cdot P(\text{rolling a ten})$$

Here, we made use of the formula that $P(E \text{ and } F) = P(E)P(F)$ for independent events. Certainly, A and B are indepedent events.

For A_3, we repeat the same logic

$$P(A_3) = \underbrace{(1 - P(\text{roll } 10)) \cdot (1 - P(\text{roll } 6))}_{\text{first failed roll by A and B}} \cdot \underbrace{(1 - P(\text{roll } 10)) \cdot (1 - P(\text{roll } 6))}_{\text{2nd failed roll by A and B}} \cdot P(\text{roll ten})$$

And so we have a general formula

$$\begin{aligned} P(A_n) &= ((1 - P(\text{roll } 10)) \cdot (1 - P(\text{roll } 6)))^{n-1} \cdot P(\text{roll ten}) \\ &= \left(\left(1 - \frac{1}{12}\right)\left(1 - \frac{1}{6}\right)\right)^{n-1} \cdot \frac{1}{12} \\ &= \left(\frac{55}{72}\right)^{n-1} \cdot \frac{1}{12} \end{aligned}$$

To find out the total probability of A_1 OR A_2 OR A_3, we add them all:

$$\sum_{n=1}^{\infty} \left(\frac{55}{72}\right)^{n-1} \cdot \frac{1}{12} = \frac{6}{17}$$

The infinite sum was calculated using the geometric series formula with the first term being $\frac{1}{12}$ and the common ratio being $\frac{55}{72}$.

A closet contains 11 pairs of shoes. If 7 shoes are randomly selected, what is the probability that there will be (a) no complete pair? (b) exactly 1 complete pair?

The closet contains a total of 22 shoes and we can safely assume that each shoe is equally likely to be picked. Therefore, in **total** there are $\binom{22}{7}$ ways to pick 7 shoes out of the 22 (ofcourse, here we could potential pick pairs). In order to **avoid** a pair, lets count the number of ways there are to do that.

First, let's count the number of ways you can pick 7 **pairs** from 11 total. This is equal to $\binom{11}{7}$. For **each** pair, we want to avoid selecting both shoes. This means we are either going to select the left shoe or the right shoe. In other words, there are 2^7 ways to do that (i.e., for each pair, there are exactly two choices - left and right). To get the probability, we can use the counting definition of probability.

$$P(\text{no complete pair}) = \frac{\binom{11}{7} \cdot 2^7}{\binom{22}{7}} = 0.2476$$

For part (b), it's a similar process. We know there are exactly $\binom{22}{7}$ ways to pick 7 shoes (perhaps all of them are pairs, perhaps none of them are pairs), but that's the total number. Now how many number of ways are there to pick **one pair** from 11 pairs? **Well this number is exactly 11**. After selecting a pair, we have 5 shoes to select and we must ensure none of these 5 form a pair. So similar to (a), there are $\binom{10}{5}$ ways to pick pairs (notice now we have 10 pairs left to choose from), and for **each pair** there are 2^5 ways to ensure we either select the left shoe or the right shoe. And so all together we have

$$\frac{11 \cdot \binom{10}{5} \cdot 2^5}{\binom{22}{7}}$$

Caution

NOTE: THE MYLAB SOLUTION WILL BE OFF BY A FACTOR OF TWO (SO DIVIDE MY ANSWER BY 2). THEY DO SOME PERMUTATION STUFF THAT I DO NOT AGREE WITH. IN THE EXAM, EITHER SOLUTION WILL BE FINE

Let Q_n denote the probability that no run of 3 consecutive heads appears in n tosses of a fair coin. Show that the following equations are true. Find Q_8.

The formula given is

$$\begin{aligned} Q_n &= \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3} \\ Q_0 &= Q_1 = Q_2 = 1 \end{aligned}$$

First of all, lets understand the question. The variable Q_n denotes the probability that you **do not** get 3 consecutive heads in a row in n coin tosses. So obviously $Q_0 = Q_1 = Q_2 = 1$ since it's impossible to get three heads in a row if you don't even toss the coin three times.

Lets focus on Q_4 for now. The event Q_3 can be visually described as $Q_4 = \bullet \mid \bullet \mid \bullet \mid \bullet$ where each $\bullet$ is a slot that can be H or T.

The key to this question is understanding recurrence relations (from calculus). The event Q_4 can happen in multiple ways (**depending on how the *string* ends**).

$$\begin{aligned} Q_4 &= \bullet \mid \bullet \mid \bullet \mid T \quad \leftarrow \text{(ends with T)} \\ &= \underbrace{\bullet \mid \bullet \mid \bullet}_{\text{this has to be } Q_3 \text{ (i.e., no 3 heads in a row)}} \mid T \\ P(Q_4) &= Q_3 \cdot P(T) = \frac{1}{2}Q_3 \end{aligned}$$

However, there are OTHER ways that Q_4 can happen

$$\begin{aligned} Q_4 &= \bullet \mid \bullet \mid H \mid T \quad \leftarrow \text{(ends with H T)} \\ &= \underbrace{\bullet \mid \bullet}_{\text{this now has to be } Q_2} \mid H \mid T \\ P(Q_4) &= Q_2 \cdot P(H) \cdot P(T) = \frac{1}{2} \cdot \frac{1}{2}Q_2 = \frac{1}{4}Q_2 \end{aligned}$$

and lastly

$$\begin{aligned} Q_4 &= \bullet \mid H \mid H \mid T \quad \leftarrow \text{(ends with H H T)} \\ &= \underbrace{\bullet}_{\text{this now has to be } Q_1} \mid H \mid H \mid T \\ P(Q_4) &= Q_1 \cdot P(H) \cdot P(H) \cdot P(T) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}Q_1 = \frac{1}{8}Q_1 \end{aligned}$$

And so putting it all together we have

$$Q_4 = (\bullet \mid \bullet \mid \bullet \mid T) + (\bullet \mid \bullet \mid H \mid T) + (\bullet \mid H \mid H \mid T)$$

and taking the probability we have

$$P(Q_4) = \frac{1}{2}Q_3 + \frac{1}{4}Q_2 + \frac{1}{8}Q_1$$

I encourage you to think about how Q_5 looks. Draw out the 5 "slots" (i.e. a string of 5) and look at the possible endings of the string. Either it will end with T, HT, HHT. Generalizing this, we have

$$Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}$$