SICP - Exercise 1.13

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We need to prove by induction that

$$\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

where:

$$\phi = \frac{1+\sqrt{5}}{2}$$
$$\psi = \frac{1-\sqrt{5}}{2}$$

We observe that ϕ and ψ are the roots of the equation:

$$x^2 - x - 1 = 0$$

A key simplification arising out of this equation by re-arranging the terms is $x = 1 + \frac{1}{x}$. Since ϕ and ψ are roots, it follows that:

$$\phi = 1 + \frac{1}{\phi}$$
$$\psi = 1 + \frac{1}{\psi}$$

Step 1 of mathematical induction is validating the base cases. We will do so for Fib(0), Fib(1) and Fib(2).

LHS:

$$\mathrm{Fib}(0) = 0$$
$$\mathrm{Fib}(1) = 1$$
$$\mathrm{Fib}(2) = 1$$

RHS:

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = 0$$
$$\frac{\phi^1 - \psi^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$
$$\frac{\phi^2 - \psi^2}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1$$

The base cases verify correctly. Now, we move on to the inductive step. We assume it is true for $n = k$ and $n = k - 1$ i.e. the following are true:

$$\begin{aligned} \text{Fib}(k) &= \frac{\phi^k - \psi^k}{\sqrt{5}} \\ \text{Fib}(k-1) &= \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} \end{aligned}$$

Now, we need to prove that the statement for $n = k + 1$ i.e. the following is true:

$$\text{Fib}(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$$

$$\begin{aligned} \text{Fib}(k+1) &= \text{Fib}(k) + \text{Fib}(k-1) \\ &= \frac{\phi^k - \psi^k}{\sqrt{5}} + \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} \\ &= \frac{\phi^k + \phi^{k-1} - (\psi^k + \psi^{k-1})}{\sqrt{5}} \\ &= \frac{\phi^k\left(1 + \frac{1}{\phi}\right) - \left(\psi^k\left(1 + \frac{1}{\psi}\right)\right)}{\sqrt{5}} \\ &= \frac{\phi^k \phi - \psi^k \psi}{\sqrt{5}} \\ \text{Fib}(k+1) &= \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} \end{aligned}$$

This has been proved. We also need to show the other part which says that $\text{Fib}(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$. This is essentially equivalent to showing that the difference between the two is less than 0.5. From our previous proof, we can rearrange to obtain the following:

$$\frac{\phi^n}{\sqrt{5}} - \text{Fib}(n) = \frac{\psi^n}{\sqrt{5}} \implies \frac{\psi^n}{\sqrt{5}} < \frac{1}{2} \implies \psi^n < \frac{\sqrt{5}}{2}$$

Now, $\psi = \frac{1-\sqrt{5}}{2} = -0.618$. This gives us $|\psi| < 1 \implies |\psi|^n < 1$. The RHS of our inequality is greater than 1 and this completes the proof.