IPM forumalations for thesis

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1 Original Problem

1.i Primal

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \leq \boldsymbol{u}, \\ & \mathbf{x} \in \mathbb{R}^m_{\geq 0} \end{aligned} \tag{1}$$

1.ii Primal Standard Form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} + \mathbf{x}_u = \boldsymbol{u}, \\ & \mathbf{x}, \mathbf{x}_u \in \mathbb{R}^m_{\geq 0}, \mathbb{R}^m_{\geq 0} \end{aligned} \tag{2}$$

1.iii Dual

$$\begin{aligned} \max_{\mathbf{y}, \mathbf{y}_u} \quad & \mathbf{b}^T\mathbf{y} - \mathbf{u}^T\mathbf{y}_u \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} - \mathbf{y}_u \leq \mathbf{c}, \\ & \mathbf{y}, \mathbf{y}_u \in \mathbb{R}^n, \mathbb{R}^m_{\geq 0} \end{aligned} \tag{3}$$

1.iv Dual Standard form

$$\begin{aligned} \max_{\mathbf{y}, \mathbf{y}_u} \quad & \mathbf{b}^T\mathbf{y} - \mathbf{u}^T\mathbf{y}_u \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} + \mathbf{y}_u + \mathbf{z} = \mathbf{c}, \\ & \mathbf{y}, \mathbf{y}_u, \mathbf{z} \in \mathbb{R}^n, \mathbb{R}^m_{\geq 0}, \mathbb{R}^m_{\geq 0} \end{aligned} \tag{4}$$

But if we want $\mathbf{y}$ variables to be free, then dualize the standard from, then standardize again:

$$\begin{aligned} \max_{\mathbf{y}, \mathbf{y}_u} \quad & \mathbf{b}^T\mathbf{y} + \mathbf{u}^T\mathbf{y}_u \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} + \mathbf{y}_u + \mathbf{z}_1 = \mathbf{c}, \\ & \mathbf{y}_u + \mathbf{z}_2 = \mathbf{0}, \\ & \mathbf{y}, \mathbf{y}_u, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^m_{\geq 0}, \mathbb{R}^m_{\geq 0} \end{aligned} \tag{5}$$

2 Solver Forms

2.i Long Step Path Following Method

Ref: Numerical Optimization (Alg. 14.2)

Due to the formulation, we have to work with the standard form.

2.i.1 Primal

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax = b, \\
& x \in \mathbb{R}^n_{\geq 0}
\end{aligned}
\tag{6}
$$

$$
A = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \quad | \quad x = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_u \end{bmatrix} \quad | \quad b = \begin{bmatrix} \mathbf{b} \\ \mathbf{u} \end{bmatrix} \quad | \quad c = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}
$$

2.i.2 Dual

$$
\begin{aligned}
\max \quad & b^T y \\
\text{s.t.} \quad & A^T \lambda + s = c, \\
& \lambda, s \in \mathbb{R}^m, \mathbb{R}^n_{\geq 0}
\end{aligned}
\tag{7}
$$

$$
A = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \quad | \quad \lambda = \begin{bmatrix} \mathbf{y} \\ \mathbf{y}_u \end{bmatrix} \quad | \quad b = \begin{bmatrix} \mathbf{b} \\ \mathbf{u} \end{bmatrix} \quad | \quad c = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix} \quad | \quad s = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}
$$

2.ii Big KKT

$$
\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ -r_c \end{bmatrix}
$$

2.iii Small KKT

$$
\begin{bmatrix} -X^{-1}S & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} -r_d + X^{-1} r_c \\ -r_p \end{bmatrix}
$$

$$
d_s = -X^{-1}(r_c + S d_x)
$$

2.iv No KKT

Let $L := AS^{-1}XA^T$. Then,

$$
\begin{aligned}
d_y &= L^+(-r_p - AS^{-1}X r_d + AS^{-1} r_c) \\
d_x &= -S^{-1}(r_c - X(r_d + A^T d - y)) \\
d_s &= -X^{-1}(r_c + S d_x)
\end{aligned}
$$