4) a) $y^T P y = y^T (A^T A) y = (y^T A^T) A y$

$= (Ay)^T A y$

Ay is a vector ∴ $(Ay)^T Ay$ is the norm of Ay which is ≥ 0

∴ $y^T P y \geq 0 \quad \forall y$

$z^t Q z = z^t (A A^T) z = (A^T z)^T (A^T z) \geq 0$

If $Pv = \lambda v \qquad v^T P v = \lambda v^T v \geq 0$

∴ $v^T v \geq 0 \Rightarrow \lambda \geq 0$

∴ Eigenvalues of P & Q are non-negative

b) $Pu = \lambda u \qquad QAu = (AA^T) A u$

$= A(A^T A) u = A P u = \lambda (Au)$

∴ Au is an eigenvector of Q with eigenvalue λ.

If $Qv = \mu v \qquad P A^T v = (A^T A) A^T v = A^T (A A^T) v$

$= A^T Q v = \mu A^T v$

∴ $A^T v$ is an eigenvector of Q with eigenvalue μ.

u is of size $n \times 1$; v is of size $m \times 1$

c) $u_i = \frac{A^T v_i}{\|A^T v_i\|} \qquad Q v_i = \mu_i v_i$

$A u_i = \frac{(A A^T) v_i}{\|A^T v_i\|} = \frac{Q v_i}{\|A^T v_i\|} = \frac{\mu_i}{\|A^T v_i\|} v_i$

$= \sigma_i v_i \qquad \sigma_i = \frac{\mu_i}{\|A^T v_i\|} \qquad \|A^T v_i\| \geq 0$ & $\mu_i \geq 0$

∴ Q is +ve semi-definite.

∴ $\sigma_i \geq 0$

Q4] d) If $m \le n$

T is of size $m \times n$ $\quad T = \begin{bmatrix} \gamma_1 & 0 & \cdots & & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2 & & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & & \gamma_m & 0 & \cdots & 0 \end{bmatrix}$ $\quad n-m$

$UT = [\gamma_1 v_1 \; \cdots \; \gamma_m v_m \; 0 \cdots 0]$

Let $v_1, \ldots, v_k$ be a complete set of linearly independent unit norm eigenvectors of P with non-0 eigenvalues.

$\frac{A\cancel{v_1}}{\|Av_1\|}, \ldots, \frac{Av_k}{\|Av_k\|}$ is a set of linearly independent

unit norm eigen-vectors of Q with non-0 eigenvalues.

Proof: We know from 4(b) that they are eigenvectors

& from 4(c) that $\frac{Av_i}{\|Av_i\|}$ $\cancel{= \gamma_i v}$

If $\frac{Av_1}{\|Av_1\|}, \ldots, \frac{Av_k}{\|Av_k\|}$ are not linearly independent then so aren't $\frac{A^TAv_1}{\|Av_1\|}, \ldots, \frac{A^TAv_k}{\|Av_k\|} = \frac{T_1 v_1}{\|Av_1\|}, \ldots, \frac{T_k v_k}{\|Av_k\|}$

which is a contradiction.

Similarly if $v_1, \ldots, v_t$ are non-0 eigenval eigenvectors of Q which are linearly independent then $\frac{A^T v_i}{\|A^T v_i\|}$ are the same for

P. $\therefore t = k$. From (4c) $Av_i = \gamma_i v_i$

$t \le m$; There will be $n-t$ eigenvectors of P which are independent with 0 eigenvalue. They are $v_{t+1}, \ldots, v_n$

$AV = [Av_1, \ldots, Av_t, Av_{t+1}, \ldots, Av_n]$

$= [\gamma_1 v_1, \ldots, \gamma_t v_t, 0, \ldots, 0] = UT$

$A(VV^T) = UTV^T$ $\qquad VV^T = I$ (as $v_i^T v_j = 0$ if $i \ne j$, 1 otherwise)

$A = UTV^T$

For $m > n$; run the same argument on A^T & take transpose in the end.

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