Assignment 1 | Probability and Statistics

Deadline : 23:55 27th September 2020

Instructions:

- There are a total of 3 sections.
- You may attempt **any 7** problems in each section.(Total 3*7 = 21)
- Each question in section 1 is worth 4 points, section 2 is worth 5 points and section 3 is worth 6 points.
- You may attempt more than 7 problems in each section, **best 7** will be considered for grading.

Section 1

1. Tharun being poor at shooting takes 10 shots at a target and has probability 0.2 of hitting the target with each shot, independently of all other shots. Let X be the number of hits.

    1. Calculate the PMF of X.
    2. Find the expectation and the variance of X.
    3. Suppose Tharun has to pay \$3 to enter the shooting range and he gets \$2 dollars for each hit. Let Y be his profit. Find the expectation and the variance of Y .
    4. Now let's assume that Tharun enters the shooting range for free and gets the number of dollars that is equal to the square of the number of hits. Let Z be his profit. Find the expectation of Z.

2. Kesav has two bags with him. Bag 1 has 5 white balls and 3 black balls and Bag 2 has 3 white balls and 5 black balls. Kesav forgets things easily and his friends also scold him for the same. The problem is that Kesav had withdrawn 2 balls from 1st bag and put them into 2nd bag in past and now he forgot the colors of the balls. Now if Kesav withdraws 2 balls from 2nd bag. What is the probability that balls drawn are white and black in color (i.e. he picks 1 white and 1 black ball)?

3. What is the probability that the equation $x^2 + 2kx + m$ has at least one real root where k and m are randomly picked from the real number line?

    Hint: What constraints do k and m have? Starting with a constraint $-L <= k, m <= L$ and then tending L to infinity might help.

4. Atsi is one of the 2 children of his/her parents. Find the probability that he/she has a sister given that:

   1. Atsi is a girl
   2. Atsi is a boy

5. You can type blindfold at your keyboard, but that comes at a disadvantage of making typos. Let us say that you are required to type the word FLASH. You might insert an extra letter (all 26 letters equally likely) with probability 0.1, skip any required letter with probability 0.1, or write the correct required letter with probability 0.8. What is the probability that, when required to type the word FLASH, you instead end up writing FAST?

6. In a room full of 30 people, what is the probability that at least two people have the same birthday? Assume birthdays are uniformly distributed and there is no leap year complication. (Hint: what is the probability that they all have different birthdays?)

7. Anuja is at the bottom left $(0, 0)$ of a coordinate grid of size $n * m$, and she can only move right and up unit distance along the edges (She cannot move towards the right and above of the point (n, m)). She chooses a path randomly and every path is equally likely. What is the probability that she passes through a point (x, y) in her path? (x>=0 and x<=n and y>=0 and y<=m)

8. If E and F are two independent events, such that $P(E \cap F) = 1/6$, $P(E' \cap F') = 1/3$ and $(P(E) - P(F))(1 - P(F)) > 0$, then derive the relation between P(E) and P(F).

9. There are n identical containers filled with n identical balls, what is probability that all containers have at most 2 balls?

10. There are 3 boxes, Box 1 has cards 1 to 3, Box 2 has 1 to 5, and Box 3 has 1 to 7. We pick random card from each box. What is the prob that they form an AP in any order?

Section 2

11. Let $C_1, C_2, \ldots . C_M$ be a disjoint partition of the sample space S, and A and B be two events. Suppose we know that

- A and B are conditionally independent given $C_i \forall i \in 1, 2, \ldots .., M$;
- B is independent of all C_i's.

Prove that A and B are independent.

12. On a fine Sunday morning, Koushik and his friend Neeraj are pondering what to do as they don't have any assignments to do. So, finally they thought to play a game. The game is that Koushik throws a die and reports a number and Neeraj has to guess whether Koushik is saying truth or false. Koushik is known to speak truth 2 out of 5 times. If Neeraj guess correctly then Koushik loses the game. Neeraj is equally probable in selecting either of the responses (i.e to say truth or false). Being known of Koushik's honesty what is the probability that Neeraj wins the game?

13. Due to current pandemic we are having online classes, Harish is also having online classes. He is having a quiz this weekend. In the quiz, Harish either guesses (or) copies from net(or) knows the answer to a multiple-choice question with 4 choices. Probability that he makes a guess is 1/3. He copies the answer is 1/6. The probability that his

answer is correct given that he copied it is 1/8. Find the probability that he knew the answer to the question. Given that he correctly answered it.

14. You have a fair dice with six faces, each representing a chess piece: pawn, king, queen, knight, bishop, rook.

    You roll three dice simultaneously. You can only move a chess piece if it appears on at least one dice. Find the probability that you will be able to move a pawn after rolling the three dice.

15. A biased coin (with probability of obtaining a Head equal to p > 0) is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss using total probability theorem.

16. The ArtSociety is making a modern art sculpture with large number of identical glass rods marked with a green dot at one end and a purple dot at another. One of the freshers accidentally drops a box of these rods:

    1. Assuming that the rods get broken into two, what is the expected length of the smaller piece?
    2. What is the average ratio of the smaller length to the larger length?
    3. Assuming that the rods get broken into 3, what is the expected length of the part with the green dot?

    Note: Assume that the point of breakage in each case is uniformly distributed

17. Suppose there are three desks, each with two drawers. One desk contains a gold medal in each drawer, one contains a silver medal in each drawer, and one contains one of each, but you don't know which desk is which. The question is this: If you open a drawer and find a gold medal, what are the chances that the other drawer in that desk also contains gold?

18. Anuja is shooting at a target with probability of hitting p. She keeps shooting till she hits α times (may be non-consecutive) (and is confident) or misses β times (may be non-consecutive) (and is disappointed). What is the probability that she comes out of practice confident?

19. You are given a circular piece of wire with radius r. You cut it into 3 pieces randomly. Find the probability that the 3 pieces form a non-degenerate triangle now.

20. Atsi arrives at a point with three roads. The first road brings him back to the same point after two hours of walk. The second road brings him back to the same point after four hours of travel. The last road leads to the city after three hours of walk. If every time, Atsi chooses one of three roads with equal probability, what is the expected time (in hours) until he arrives to the city?

Section 3

21. You are trapped in a grid with N rows and M columns ($N, M > 1$). You can make random jumps along a row or along a column. The decision for each jump is independent from other jumps, and is made as follows:

1. First you choose whether to do a row jump or a column jump with equal probability $= 0.5$.
2. Suppose you decided

- to do a row jump, then you can jump to any of the other $(M-1)$ cells in that row with equal probability.
- to do a column jump, then you can jump to any of the other $(N-1)$ cells in that column with equal probability.

Let us say that you are currently present at point (x, y). Find the probability that after making K random jumps, you will be present at (x, y) again. Give a general answer in terms of N, M, K. The expected answer is in the form of recurrences (you do not need to solve the recurrences).

22. Mr Nariyal and Mr Samso are contesting for the role of Felicity Coordinator by an election. Let us say they have N and S number of votes respectively. The ballot box is shaken, and the votes are tallied in a truly random order.

    1. Given N>S, what is the probability that there exists a point while counting the votes one by one, that they are tied at the number of votes?
    2. Given N=S, what is the probability that N never gets ahead of S at any point in the counting procedure?

23. A person's birthday occurs on a day i with probability P_i, where $i = 1, ..., n$. (Of course, $P_1 + ... + P_2 = 1$.) Assume independent assignment of birthdays among different people. In a room with k people, let $P_k = P_k(p_1, ..., p_n)$ be the probability that no two persons share a birthday. Show that this probability is maximized when all birthdays are equally likely: $p_i = 1/n \ \forall i$.

24. Anuja is shooting at a target with probability of hitting p. She keeps shooting (infinitely) till she hits α consecutive times (and is confident) or misses β consecutive times (and is disappointed). What is the probability that she comes out of practice confident?

25. There are x-girls and y-boys in a school line. They are arranged randomly. Find the probability that the number of boys in front of each girl is at least 1 more than the number of girls in front of her.

    Solve for:

    1. X = 2, Y = 3
    2. X = Y

26. Iron man before meeting Thanos did an attempt to create Iron avengers (bots which look and has same powers as that of avengers). He was trying to make a bot for Hawkeye and found a mathematical problem. Ever since he was stuck trying to solve it. Could you help him? The bot has 100 arrows. First time he hits the target and the second time, he misses it. For the next following shots, the probability of him hitting the target is equal to the number of times he hit the target before this try divided by the number of arrows he used before this try. Now what is the probability that he hits exactly half the targets.

27. There are **x** students working on a project. The students will put all the project's documents in an almirah that has a **locks**, and each student will take a set of b **unique**

keys, such that each key can open only a single lock. The almirah can be opened **if and only if** y or more of the students are present. Each of the almirah's lock has an infinite number of keys that can open it, and students may have similar or different sets of keys. In order to open the almirah, y or more students must be presented such that these students have at least one key for each almirah's lock.

1. What is the minimum number of locks needed?
2. What is the minimum number of keys to the locks each student must carry?

such that the almirah can be opened **iff** y or more of the students are present.

28. A bag initially contains one red ball and two blue balls. An experiment consists of selecting a ball at random, noting its color and replacing it together with an additional ball of the same color. If three such trials are made, then find probability for following cases

    1. exactly one blue ball is drawn.
    2. All drawn balls are red given that all the drawn balls are of the same color.
    3. At least one blue ball is drawn.
    4. At least one red ball is drawn.

29. A robot is stuck in a circle. The circle had numbers from 0 to n-1 arranged on it. The robot starts from 0 and at every step it randomly moves to its nearest neighbor. For each i from 1 to n-1 find probability that when it is on i for the first time, all other points have been previously visited.
