Points: 35 MA6.101
Due: at 11:55PM on 01/12 # Assignment 5 Fall 2020

Problem 1

There is a bag containing 40 blue marbles and 60 red marbles. You choose 10 marbles (without replacement) at random. Let X be the number of blue marbles and y be the number of red marbles. Find the joint PMF of X and Y. [2 points]

Problem 2

Consider two random variables X and Y with the range

$$R_{XY} = \{(i,j) \in \mathbb{Z}^2 \mid i,j \geq 0,\ |i-j| \leq 1\},$$

and joint PMF given by

$$P_{XY}(i,j) = \frac{1}{6 \cdot 2^{\min(i,j)}}, \quad \text{for } (i,j) \in R_{XY}.$$

1. Pictorially show R_{XY} in the xy plane.
2. Find the marginal PMFs $P_X(i)$, $P_Y(j)$.
3. Find $P(X = Y \mid X < 2)$.
4. Find $P(1 \leq X^2 + Y^2 \leq 5)$.
5. Find $P(X = Y)$.
6. Find $E[X \mid Y = 2]$.
7. Find $\mathrm{Var}(X \mid Y = 2)$.

[7 points]

Problem 3

If X and Y are bivariate normal and uncorrelated, then they are independent. [2 points]

Problem 4

Let X be an n-dimensional random vector and the random vector Y be defined as

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}, \tag{1}$$

where A is a fixed m by n matrix and b is a fixed m-dimensional vector. Show that

$$\mathbf{C_Y} = \mathbf{A C_X A^T}. \tag{2}$$

[2 points]

Problem 5

For a normal random vector X with mean m and covariance matrix C, the PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{C}}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right\} \tag{3}$$

Let X and Y be two jointly normal random variables with $X \sim N(\mu_X, \sigma_X), Y \sim N(\mu_Y, \sigma_Y)$ and $\rho(X,Y) = \rho$. Show that the above PDF formula for PDF of $\mathbf{X} = \begin{bmatrix} X \\ Y \end{bmatrix}$ is the same as $f_{X,Y}(x,y)$ as follows

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \tag{4}$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}. \tag{5}$$

[3 points]

Problem 6

Consider a random variable X with $E[X] = 10$, and X being positive. Estimate $E[\ln\sqrt{X}]$ using Jensen's inequality. [1 point]

Problem 7

Prove the Generalized union bound on slide 3 of lecture 38. Also, solve the example on slide 4 of **Lecture 38.** [2+2 points]

Problem 8

As an application of Chebychev inequality, solve the Example on slide 10 of **Lecture 38**.

[2 points]

Problem 9

Answer the Example on slide 16 of **Lecture 38**. [2 points]

Problem 10

There are N people sitting around a round table, where $N > 5$. Each person tosses a coin. Anyone whose outcome is different from his/her two neighbors will receive a present. Let X be the number of people who receive presents. Find $E[X]$ and $\mathrm{Var}(X)$. [2 points]

Problem 11

We have a rod of length l which. We break it at 1 random point to get 2 pieces. We again break the piece of the rod to the left at a random point. What is the probability that these points form a valid (non-zero area, planar) triangle? [2 points]

Problem 12

A surface has infinite parallel lines at distance d from adjacent lines. We have a needle of length l which we throw randomly on the surface. What is the probability that the needle intersects a line? Assume $l < d$. [4 points]