Assignment 4

October 26, 2020

General Instructions:

- Read carefully each question.
- Write clean and legible solutions.
- Don't assume any step as trivial unless it's fundamental. Please state your assumptions.
- Each question is worth 4 points.

1 Problems

1. X follows a gamma distribution with PDF $f(x) = 4xe^{-2x}$, where $X > 0$
   (a) Find $E(X), E(X^2)$, and $E(X^3)$.
   (b) Derive $E(X^n)$.

2. Answer the following questions
   (a) Find $\Gamma(7/2)$
   (b) Find the value of the following integral

$$I = \int_0^\infty x^6 e^{-5x} dx$$

3. Random variables X and Y have the joint PMF

$$f_{X,Y}(x,y) = \begin{cases} c(x^2+y^2), & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\} \\ 0, & \text{otherwise} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $P(Y < X)$?
   (c) What is $P(Y > X)$?
   (d) What is $P(Y = X)$?
   (e) What is $P(Y = 3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (g) Find the expectations $E[X], E[Y]$ and $E[XY]$.
   (h) Find the variances $\mathrm{Var}(X), \mathrm{Var}(Y)$ and $\mathrm{Var}(X+Y)$.
   (i) Let A denote the event $X \geq Y$. Find $E[X \mid A]$ and $\mathrm{Var}(X \mid A)$

4. Let Q be a continuous random variable with PDF

$$f_Q(q) = \begin{cases} 6q(1-q) & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$P(X = 1 \mid Q = q) = q.$$

Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all q.

5. Ben throws a dart at a circular target of radius r. We assume that he always hits the target, and that all points of impact (x, y) are equally likely. Compute the joint PDF $f_{X,Y}(x, y)$ of the random variables X and Y and compute the conditional PDF $f_{X|Y}(x|y)$.

6. Suppose 4 balls are randomly assigned to 4 bins (both labelled 1-4). Let A be the event that 1st ball is sent to the same labelled bin. Let X be number of balls in 3rd bin, N be number of non-empty bins. Find p_X, p_N, $p_{X,N|A}(x, n)$(conditioned on A) for $x = 1, 2, 3$ and $n = 2, 3$. Need to explain your solution

7. A drunken man at any time takes a step forward with pr. 1/2, 2 steps backward with pr. 1/4 and stays in place otherwise. He starts at $x = 0$ at time $t = 0$ and goes on till time $t = T$ when he passes out. The time T at which he passes out is 1 with probability 1/3, 2 with probability 1/3 and 3 with probability 1/3.
(a) Find the exp. of the total number of steps (in any direction) he has taken.
(b) Let S be the random variable corresponding to his position when he passes out. Find $E[S^5]$.

8. Homelander and Queen Maeve went on a rescue mission where a plane is hijacked by terrorists, Homelander want to save people by playing a game, Maeve tosses a biased coin (probability of head is p and tails is $1 - p$). Whenever a tail is followed by a head, Homelander save one person from the plane. Assume plane has infinite number of persons and the plane does not crash, If the total number of persons saved by Homelander is N, Find $E(N)$, $\text{Var}(N)$.

9. Buu has 6 different types (named 1 to 6) of chocolates (in large quantity). Now he plays a game to eat those chocolates. He throws a die (of six sides, non-biased, if the die shows 1, he eats chocolate 1. If the die shows 2, he eats chocolate 2 and so on. Now let X be the number of chocolates of type 1 are eaten and Y be the number of chocolates of type 2 are eaten. What is joint PMF of X and Y?

10. Let N be the number of customers that visits a certain shop in a given day. Suppose that $E[N] = e$, and $\text{Var}(N) = v$. Let X_i be the amount that the i^{th} customer spends on average. Assume that X_i's are independent of each other and independent of N. Further assume that that they have same mean and variance i.e., $E[X_i] = k, \text{Var}(X_i) = m$. Let Y be the stores total sales i.e., $Y = \sum_i X_i$. Find $E[Y]$ and $\text{Var}(Y)$.

11. A city's temperature is modelled as a normal random variable with mean and standard deviation both equal to 10 degrees Celsius. What is the probability that the temperature at a randomly chosen time will be less than or equal to 59 degrees Fahrenheit?

12. Biryani is one of the most loved meals served in Kadamba mess. Let the amount of rice served is normally distributed with mean of 370g and standard deviation of 24g. The amount of curry served is also a normal distribution of 170g and a standard deviation of 7g.
Assuming the amount of rice and curry served is independent of each other, Find the probability that the weight of the plate (consists of rice and curry only) is less than 575g.

13. If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent, then prove

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

14. Let the joint density of X and Y is given as

$$f(x, y) = \begin{cases} 4y(x - y)e^{-(x+y)} & 0 < x < \infty, 0 \leq \text{y} \leq \text{x} \\ 0 & \text{otherwise} \end{cases}$$

Find E$[X \mid Y = y]$ and $\text{Var}(X \mid Y = y)$.

15. Let X_1 be a normal random variable with mean 2 and variance 3 and let X_2 be a normal random variable with mean 1 and variance 4. Assume that X_1 and X_2 are independent.
(a) What is the distribution of the linear combination $Y = 2X_1 + 3X_2$?
(b) What is the distribution of the linear combination $Y = X_1 - X_2$?