Image Denoising

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Bayesian Image Denoising

- Optimal noiseless image is the one that maximizes the posterior PDF

$$P(\text{NoiselessImage}|\text{NoisyImage}) = \frac{P(\text{NoisyImage}|\text{NoiselessImage})P(\text{NoiselessImage})}{P(\text{NoisyImage})}$$

  - **Likelihood** PDF = **noise** model = probability of generating the data given the noiseless image
  - **Prior** PDF = our prior beliefs about the noiseless image **before** observing the data
  - **Posterior** PDF: product of likelihood and prior
    - What we get “post” / after observing the data

Optimization for Denoising

- Noiseless image $X = x$
  - X is a MRF
- Observed image data $Y = y$
- Noise model for intensities given noiseless intensities (i.i.d) $P(Y \mid X) := \Pi_i \; P(Y_i \mid X_i)$
  - e.g., If noise is additive i.i.d. zero-mean Gaussian, $P(Y_i \mid X_i) = G(y_i \mid x_i, \sigma^2)$
- Let θ = parameters underlying noise model and MRF model (in general)

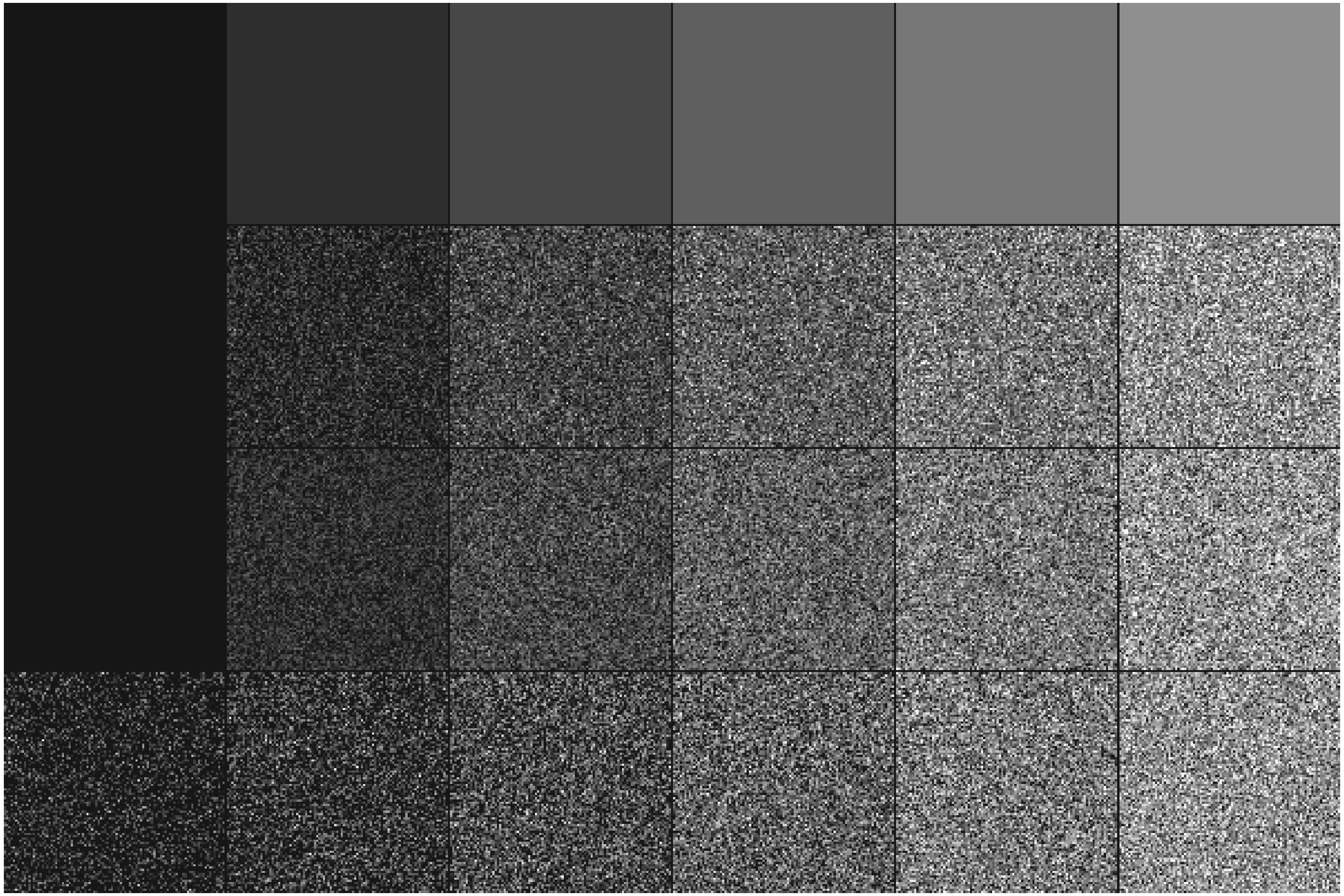
Noise Models

- Which of the noise models are “additive” in nature ?
  - Gaussian ?
  - Poisson ?
  - Rician ?

  - Gaussian noise is independent of pixel intensity
  - Poisson, Rician noise depends on pixel intensity

Noise Models

- Which of the noise models are "additive" in nature ?
  - What if approximate
    Poisson (mu) = Gaussian (mean = mu, variance = mu) ?
    - Row 1 true
      - Mu values
    - Row 2 Poisson
    - Row 3
      Gauss. approx.
    - Row 4
      Gauss. Approx.
      with
      Fixed Variance (for all mu)

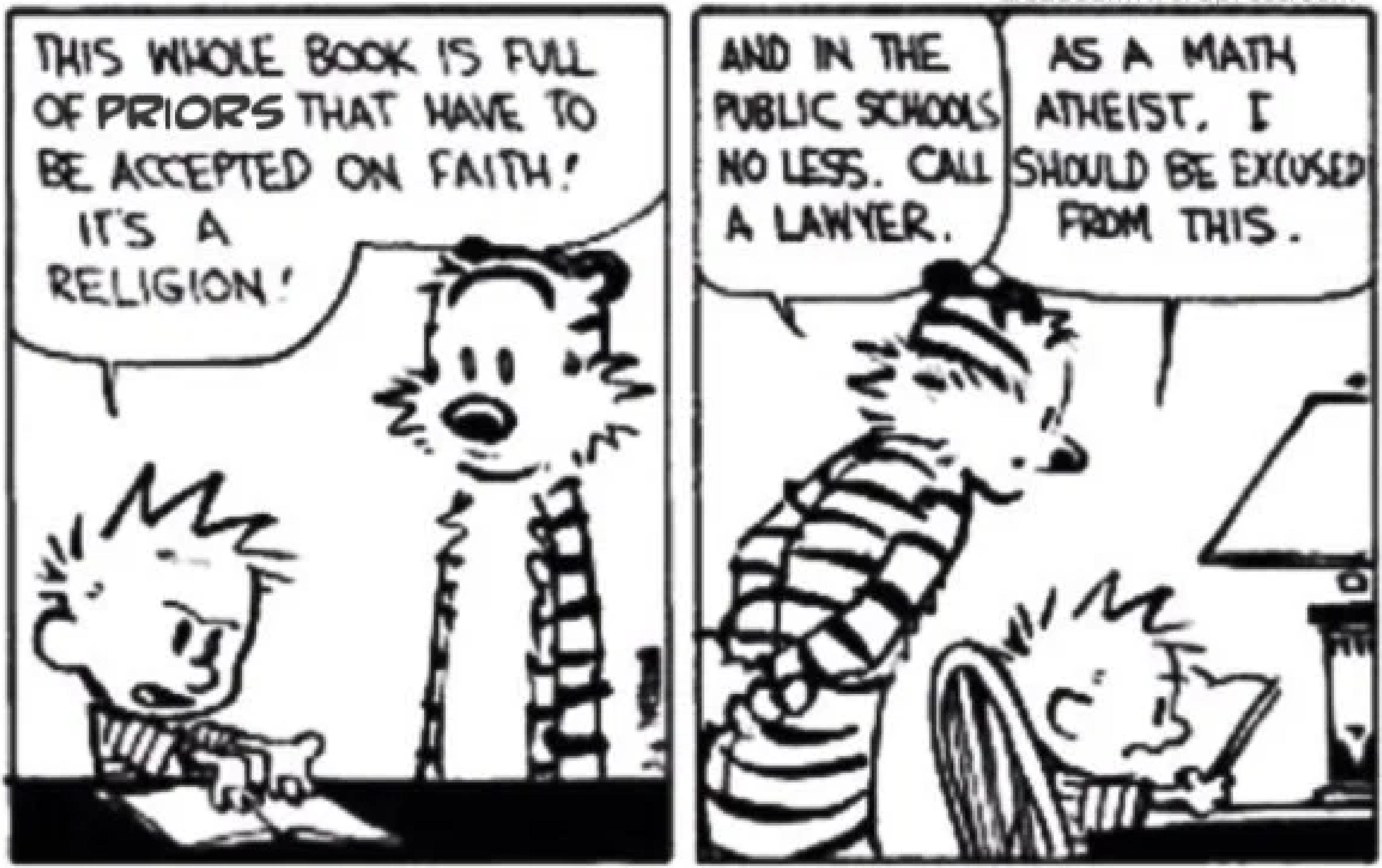


Noise Models

- Variance stabilization
  - Assume variance of X depends on its mean
  - Search for transformation Y = f(X), such that variance of Y doesn't depend on mean of Y
  - If X is Poisson: f(X) = \sqrt (X) makes variance nearly constant
    - Anscombe transform: f(X) = 2 \sqrt (X + 3/8) transforms
      - Poisson data 'x' with mean = 'm' to →

        Approx. Gaussian data with
        variance = 1,
        mean = 2 \sqrt (m + 3/8) – 1 / (4 \sqrt(m))

BAYESIAN INFERENCE



Optimization for Denoising

- Optimization Problem and Strategy

  (1) Assume: MRF parameters are user controlled

  - No need to optimize

  (2) Assume: noise level is already known

  - e.g., using the ML estimate in the background region, where signal is known to be zero

  (3) Get MAP estimate for noiseless image 'x' :

  $\max_x P(x \mid y, \theta)$

Optimization for Denoising

- Lets see **what happens at voxel i ?**
- Rewrite the objective function
  - P (X | y, theta)

$= P(X_i, X_{\sim i} | y, \theta)$

$= P(X_i | X_{\sim i}, y, \theta) P(X_{\sim i} | y, \theta)$ Conditional Probability

$= P(X_i | X_{N_i}, y, \theta) P(X_{\sim i} | y, \theta)$ Markov assumption on X

$= P(X_i | X_{N_i}, y_i, \theta) P(X_{\sim i} | y, \theta)$ Conditional independence assumption in noise model

Optimization for Denoising

- Optimization Algorithm 1
  - Iterated Conditional Mode (ICM)
  - **Consider optimization over a single voxel i**
    - Perform $\max_{x_i} P(X \mid y, \theta)$

$$= \max_{x_i} P(X_i|X_{N_i}, y_i, \theta) P(X_{\sim i}|y, \theta)$$

$$= \max_{x_i} P(X_i|X_{N_i}, y_i, \theta)$$ Second term doesn't depend on x_i

$$= \max_{x_i} \frac{P(y_i|X_i, X_{N_i}, \theta) P(X_i|X_{N_i}, \theta)}{P(y_i|X_{N_i}, \theta)}$$ Bayes Rule

$$= \max_{x_i} P(y_i|X_i, X_{N_i}, \theta) P(X_i|X_{N_i}, \theta)$$ Denominator doesn't depend on x_i

$$= \max_{x_i} P(y_i|X_i, \theta) P(X_i|X_{N_i}, \theta)$$ Conditional independence assumption in noise model

Optimization for Denoising

- Optimization Algorithm 1
  - Iterated Conditional Mode (ICM)
  - $\max_{x_i} P(y_i|X_i,\theta)P(X_i|X_{N_i},\theta)$
  - 1st term $P(y_i | X_i, \theta)$ = likelihood function
    - Noise model
  - 2nd term $P(X_i | X_{Ni}, \theta)$ = local / conditional prior on noiseless image
    - Image-regularity / smoothness model
  - ICM seeks mode of local / conditional posterior

Optimization for Denoising

- Various Optimization Algorithms
  - Order of Intensity Updates:
    - We want every update to increase the posterior P (x|y, θ)

      (1) Sequentially: Column by column, and then row by row
      - May lead to artifacts

      (2) Sequentially: Randomized order each iteration
      - Need to generate random sequence each iteration
      - Are artifacts eliminated ?

      (3) In Parallel: If seeking mode, doesn't guarantee increase in posterior probability (INVALID) unless …

      (4) In Parallel: Go towards the mode and monitor objective function
      - Gradient ascent : Dynamic step size + Objective-function monitoring
      - Guarantees increase in posterior probability

Optimization for Denoising

- Note on gradient ascent
  - Dynamic step sizing at each iteration :
    - Increase step size by (say) 10% when initial step size increases probability
      - Prevents very slow convergence when far away from optimum
    - Decrease step size by (say) 50% when initial step size decreases probability
      - Adapts step size as you get close to optimum
    - This dual strategy also prevents over-sensitivity to initial step size
  - Termination criteria :
    - Allowable step size becomes very small, e.g., 1e-8
    - Step doesn't increase posterior image probability by much, e.g., 0.01% of probability at current solution

Optimization for Denoising

- Various Optimization Algorithms
  - Gradient ascent needs :

    (1) Derivative of local conditional PDF w.r.t. xi

    $$P(X_i|X_{N_i},\theta) = \frac{1}{Z_i}\exp\left(-\sum_{a\in A} V_a(x_a)\right)$$

    where A is the set of cliques that contain site i

    (2) Derivative of noise model w.r.t. xi

MRI (Complex) (Noise: Gaussian)

- Noise Model
  - Circularly-symmetric univariate Gaussian (complex)

$$P(y|x) = \Pi_i P(y_i|x_i) = \Pi_i G_{\mathbb{C}}(y_i|x_i, \sigma^2) = \Pi_i \frac{1}{\sigma^2 \pi} \exp\left(-\frac{|y_i - x_i|^2}{\sigma^2}\right)$$

MRI (Complex) (Noise: Gaussian)

- For ICM optimization, at chosen voxel i, perform:

$$\begin{aligned}
\max_{x_i} P(x|y,\theta) &= \max_{x_i} P(y_i|x_i,\theta)P(x_i|x_{N_i},\theta) \\
&= \max_{x_i} \left(\log P(y_i|x_i,\theta) + \log P(x_i|x_{N_i},\theta) \right) \\
&= \max_{x_i} \left(\frac{-|y_i - x_i|^2}{\sigma^2} + \sum_{a \in A_i} (-V_a(x_a)) \right) \\
&= \min_{x_i} \left(\frac{|y_i - x_i|^2}{\sigma^2} + \sum_{a \in A_i} V_a(x_a) \right)
\end{aligned}$$

  - 1st term = Fidelity term = penalizes deviation (infidelity) of estimate x from data y
  - 2nd term = Regularity term: penalizes irregularity of x

MRI (Complex) (Noise: Gaussian)

- For gradient-descent optimization, at voxel i, derivative is:

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = \frac{1}{\sigma^2} 2(x_i - y_i) + \frac{\partial}{\partial x_i} \sum_{a \in A_i} V_a(x_a)$$

- For entire image x,
gradient (column vector) is:

$$g_2(x) := \left(\cdots, \frac{\partial P(x|y,\theta)}{\partial x_i}, \cdots \right)$$

- Current solution x^n at iteration n. Stepsize τ .
Updated solution is: $x^{n+1} = x^n - \tau g(x)$

Ultrasound Mag. (Noise: Speckle)

- For gradient-descent optimization, at voxel i, the derivative is :

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = \frac{1}{2x_i} + \frac{1}{2\sigma^2}\left(\frac{x_i^2 - y_i^2}{x_i^2}\right) + \frac{\partial}{\partial x_i}\sum_{a \in A_i} V_a(x_a)$$

Weighting Likelihood & MRF Prior

- Motivation for weighting
  - (1) We want to control the strength of the prior model based on certain criteria (e.g., noise level)
  - (2) We want to balance the enforcement of fidelity and regularity based on certain criteria (e.g., noise level)
  - Depends on data; which distribution it came from
  - Depends on task at hand

Weighting Likelihood & MRF Prior

- How to do it ?
  - Introduce a user-controlled parameter $\beta \in [0, 1]$ that specifies the balance between:

    (1) strength of the prior model and

    (2) strength of the likelihood model

  - A more straightforward way of thinking about β
    - If we don't know the true prior model,
      then β could be a parameter of the potential function $V_c(.)$ itself.
      This parameter is unknown, so we need to tune it.

Weighting Likelihood & MRF Prior

- Weighting MRF Prior
  - In the **prior PDF**, introduce a parameter **β** ∈ [0, 1] s.t.

$$P(x) := \frac{1}{Z(\beta)} \exp\left(-\beta \frac{1}{T} U(x)\right) \text{ where}$$

$$U(x) := \sum_{c \in C} V_c(x_c) \text{ where}$$

$$Z(\beta) := \sum_{x} \exp\left(-\beta \frac{1}{T} U(x)\right)$$

  - This changes the local conditional prior to

$$P(x_i | x_{N_i}, \theta) = \frac{1}{Z_i(\beta)} \exp\left(-\beta \frac{1}{T} \sum_{a \in A} V_a(x_a)\right)$$

  - Introducing β scales potential function $V_a(.)$ to give a new potential function

Weighting Likelihood & MRF Prior

- Weighting Likelihood (Complex-Gaussian Noise)
  - Introduce a parameter α ∈ [0, 1], where **α := 1 − β**

$$P(y|x) := \Pi_i P(y_i|x_i) := \Pi_i G_\alpha(y_i|x_i,\sigma^2)$$

$$G_\alpha(y_i|x_i,\sigma^2) := \frac{1}{Z(\sigma,\alpha)} \exp\left(-\alpha \frac{|y_i - x_i|^2}{\sigma^2}\right),$$

where $Z(\sigma, \alpha) = 1 / ((\sigma/\alpha)^2 \pi)$

  - Interpretation
    - Introducing α is similar to changing the "specified" noise level / standard deviation σ
    - This is the Complex-Gaussian PDF with parameters (x_i, σ^2/α)

Weighting Likelihood & MRF Prior

- Modified Optimization Problem (Complex-Gaussian Noise)
  - For ICM optimization, at a chosen voxel i, perform:

$$\max_{x_i} P(x|y,\theta) = \min_{x_i} \left((1-\beta)\frac{|y_i - x_i|^2}{\sigma^2} + \beta \sum_{a \in A_i} V_a(x_a) \right)$$

  - For gradient-descent optimization, at a chosen voxel i, the derivative is:

$$\frac{\partial P(x|y,\theta)}{\partial x_i} = (1-\beta)\frac{1}{\sigma^2}2(x_i - y_i) + \beta \frac{\partial}{\partial x_i} \sum_{a \in A_i} V_a(x_a)$$

  - $\beta=0$ ($T \to \infty$) ignores the prior; we get the ML estimate
  - $\beta=1$ ($\alpha = 0$; $\sigma \to \infty$) makes the likelihood a uniform PDF