MTH100 **Math 1 (Linear Algebra)** **Monsoon Semester – 2016-17**

Tutorial exercise for the week commencing Monday 15th August 2016.

1. Find the solution set in vector form for the homogeneous system $A\mathbf{x} = \mathbf{0}$ given A below. NB: A must be row-reduced to an RREF matrix in order to give the solution in standard form.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

2. a) Row reduce the augmented matrix of the system given below to an RREF matrix:

$$\begin{aligned} 3x + 2y + 7z + 9w &= 7 \\ 6x + 14y + 22z + 15w &= 13 \\ x + 4y + 5z + 2w &= 2 \end{aligned}$$

b) Express the solution (if the system is consistent) in the form of a vector **u** which is a particular solution plus scalar multiples of vector(s) which are solutions of the associated homogeneous system.

3. Determine the inverse of the given matrix A using row reduction.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

4. Is it possible for a non-homogeneous system $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$, to be inconsistent when the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution (i.e. only the trivial solution) ? Answer YES or NO, and justify your answer. If YES, construct an example and verify. If NO, explain with reference to suitable propositions and theorems.

5. Recall Proposition 5: if e is an elementary row-operation and E is the corresponding elementary matrix, then e(A) = E(A). Illustrate with one example each for scaling and interchange operations (the minimum size of the matrices in your examples should be 3×3).

6. Prove Proposition 5 in the general case, i.e. for any row operation e and any matrix A. (NB: the three cases of scaling, replacement and interchange require separate proofs.)

SOLUTIONS BELOW

Q1.

Find the solution set in vector form for the homogeneous system $A\bar{x} = \bar{0}$,

where $A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$

Ans: Step 1: Row-reduce A to get its RREF

matrix: $R = \begin{bmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix}$

Step 2: Write down the linear system corresponding to $R\bar{x} = \bar{0}$ and transfer free variables to RHS.

$$x_1 + \tfrac{15}{7}x_4 = 0 \quad \text{or} \quad x_1 = -\tfrac{15}{7}x_4$$

$$x_2 - \tfrac{4}{7}x_4 = 0 \quad \text{or} \quad x_2 = \tfrac{4}{7}x_4$$

$$x_3 - \tfrac{10}{7}x_4 = 0 \quad \text{or} \quad x_3 = \tfrac{10}{7}x_4$$

$$x_4 = x_4 \quad \text{or} \quad x_4 = 1\,x_4.$$

Step 3: Vectorial form of solution is

$x_4\begin{bmatrix} -\frac{15}{7} \\ \frac{4}{7} \\ \frac{10}{7} \\ 1 \end{bmatrix}$, i.e. all scalar multiples of the vector $\bar{u} = \begin{bmatrix} -\frac{15}{7} \\ \frac{4}{7} \\ \frac{10}{7} \\ 1 \end{bmatrix}$

Q2

(a) Row reduce the augmented matrix of the given system:

$$
\begin{aligned}
3x + 2y + 7z + 9w &= 7 \\
6x + 14y + 22z + 15w &= 13 \\
x + 4y + 5z + 2w &= 2
\end{aligned}
$$

Solution: the RREF matrix is:-

$$
\left[\begin{array}{cccc:c}
1 & 0 & \frac{9}{5} & \frac{16}{5} & \frac{12}{5} \\
0 & 1 & 4/5 & -3/10 & -1/10 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

(b) The system is consistent and the general solution can be expressed in the form:

$$
\bar{x} = \underbrace{\begin{bmatrix} \frac{12}{5} \\ -1/10 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} -\frac{9}{5} \\ -4/5 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_4 \underbrace{\begin{bmatrix} -16/5 \\ 3/10 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}}
$$

or $\bar{x} = \bar{u} + t\,\bar{v} + s\,\bar{w}$

where $\bar{u}$ is a particular solution and $\bar{v}$ and $\bar{w}$ are solutions of the corresponding homogeneous system.

Q3. Find the inverse of the matrix A by row reduction:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 3 \end{bmatrix}$$

Answer: $$A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

Q4. (Answer) YES, it is possible for a non-homogeneous system $A\bar{x} = \bar{b}$ to be inconsistent, when the associated homogeneous system $A\bar{x} = \bar{0}$ has a unique solution.

Consider: if the system ~~A~~ has more variables than ~~rows~~ equations, then there have to be free variables, i.e. infinitely many solutions. So this case isn't useful.

OTOH, if A is square, then by VIT, if $A\bar{x} = \bar{0}$ has only the unique solution, then A is invertible, and so by (d) of VIT, $A\bar{x} = \bar{b}$ has a solution for every $\bar{b}$. So this case also isn't useful.

Hence, we must take a system with more equations than variable, i.e. A has more rows than columns.

After reasoning as above, it is easy to construct an example; many examples ~~are~~ can ~~possible~~ be constructed.

For example;

$$\begin{aligned} 2x + 3y &= 13 \\ 3x + 5y &= 21 \\ x + y &= 6 \end{aligned}$$

We row-reduce the augmented matrix $[A : \bar{b}]$:-

$$\left[\begin{array}{cc|c} 2 & 3 & 13 \\ 3 & 5 & 21 \\ 1 & 1 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 3 & 5 & 21 \\ 2 & 3 & 13 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{array}\right] \xrightarrow[R_3 - 2R_2]{R_3 \to} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 6R_3} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

This has a row of the form $[0 \cdots 0\ b]$, $b \neq 0$, so is inconsistent. OTOH, the RREF of A is I_2, so $A\bar{x} = \bar{0}$ has a unique solution.

Q 5. Examples for Proposition 5.

Let us take $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

(i) Example for interchange.

Let $e_1 = R_1 \longleftrightarrow R_3$

$\therefore \quad E_1 = e_1(I_3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$e_1(A) \Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 3 \end{bmatrix} \xrightarrow{e_1} \begin{bmatrix} 5 & 2 & -3 \\ 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad (1)$$

$$E_1 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -3 \\ 0 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \quad (2)$$

(1) and (2) are equal.

(ii) Example for scaling.

Let $e_2 = R_1 \longrightarrow 2R_1$

$\therefore \quad E_2 = e_2(I_3) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$e_2(A) = \begin{bmatrix} 4 & 2 & -2 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \quad (3)$$

$$E_2 A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \quad (4)$$

(3) and (4) are equal.

§6. Proof of Proposition.

We need to show $e(A) = EA$.

Let $A = [a_{ij}]$ be a general $m \times n$ matrix, and let I be the $m \times m$ identity matrix.

We consider the three cases one by one.

(i) Scaling:- ~~we~~ ~~assume~~ suppose the ~~Let~~ ~~E_1~~ k-th row is to be scaled by the real number $r \neq 0$.

Then, in $e(A)$, all ~~elements are~~ entries are unchanged except in the k-th row, we now have entries $r a_{kj}$, $j = 1, \dots, n$. (1)

In ~~E~~ $E = e(I)$, all entries are unchanged except in the k-th row, k-th column, we have r instead of 1.

Now, in the product EA, only the k-th row will change, all other rows are unchanged (since all rows of E except k-th are same as I).

Consider:

$$k\text{-th} \leftarrow \underset{E}{\begin{array}{c} \\ \begin{bmatrix} 1 & 0 & \cdots & \overset{k\text{-th}}{0} & 0 \\ & & & \vdots & \\ 0 & & & r & \\ 0 & & & 0 & \\ 0 & \cdots & & \vdots & 1 \end{bmatrix} \end{array}} \underset{A}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{k1} & \cdots & a_{kn} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}$$

In the k-th row of EA, the first element is $r a_{k1}$; the 2nd element is $r a_{k2}$; and so on, i.e. the j-th element of the row is $r a_{kj}$. (2)

Hence, in the k-th row, each element is $r a_{kj}$, $j = 1, 2, \dots, n$. Comparing (2) with the statement (1) above, the result follows.

Q6 (cont'd).

(ii) Replacement: suppose the k-th row is to be replaced by the k-th row plus r times the p-th row, $r \neq 0$.

Then, in $e(A)$, all entries are unchanged, except in the k-th row, the typical entry is $a_{kj} + r a_{pj}$, $j = 1, \ldots, n$. (3)

In $E = e(I)$, all entries are unchanged, except that in the k-th row, the p-th entry is now r instead of being 0.

As before, in EA, all rows are unchanged except the k-th row.

Consider:

$$\begin{array}{c c} & \begin{array}{c} \quad\quad k\text{-th} \quad p\text{-th} \end{array} \\ k\text{-th} \leftarrow & \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 \cdots & r & \cdots & 0 \\ 0 & \cdots & & & & \cdots & 1 \end{bmatrix}}_{E} \end{array} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

In the k-th row of EA, the 1st element is $a_{k1} + r a_{p1}$, the second element is $a_{k2} + r a_{p2}$; and so on, i.e. the general element is $a_{kj} + r a_{pj}$, $j = 1, 2, \ldots, n$ (4).

Comparing statements (3) and (4), the result follows.

(iii) Interchange: suppose the k-th rows and p-th rows are to be interchanged, $k < p$.

Then, in $e(A)$, the typical entry in k-th row is a_{pj}, $j = 1, 2, \ldots, n$

and the typical entry in p-th row is a_{kj}, $j = 1, 2, \ldots, n$ (5)

Q6 (cont'd)

In $E = e(I)$, the k-th row is all zeroes, except for the p-th entry which is 1, and the p-th row is all zeroes except for the k-th entry which is 1.

Consider:

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} & k\text{-th} & & p\text{-th} & \end{array} \\
\begin{array}{c} \\ k\text{-th} \rightarrow \\ p\text{-th} \rightarrow \\ \\ \end{array} &
\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \\ 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}
\end{array}
\qquad
\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}
$$

E A

The only rows of $e(A)$ and EA which are different from A are the k-th and p-th rows; ~~as we saw~~ other rows have been multiplied by corresponding rows of I, so are unchanged.

In the k-th row of EA the first entry is a_{p1}; ~~[illegible]~~ the second entry is a_{p2}; and so on, i.e the typical element is a_{pj}, $j = 1, \ldots, n$. (6a)

Similarly, in the p-th row of EA, the typical element is a_{kj}, $j = 1, \ldots, n$ (6b)

Comparing the statement (5) with the statements (6a) and (6b), the result follows.