MTH100 MATHS I Monsoon Semester – 2016-17

Tutorial exercise for week commencing Monday 5th September 2016.

1. Given the following vectors in $\mathbb{R}^3$: $\mathbf{u} = (1,3,5)$, $\mathbf{v} = (1,4,6)$, $\mathbf{w} = (2, -1, 3)$ and $\mathbf{b} = (6,5,17)$. Does $\mathbf{b} \in W = \text{span}\ \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. If the answer to b) is yes, express $\mathbf{b}$ as a linear combination of $\mathbf{u},\mathbf{v},\mathbf{w}$.

2. Let U and W be two subspaces of the vector space V. Show that $U \cap W$ is also a subspace of V.

3. We define $U + W = \{\mathbf{u} + \mathbf{w}: \mathbf{u} \in U, \mathbf{w} \in W\}$. Show that U + W is a subspace of V, and moreover, U + W is the smallest subspace of V which contains both U and W.

4. Verify that the set $F = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a,b \in \mathbb{Q}\}$, with the usual addition and multiplication is a field. (NB: *You have to verify closure, as well as the zero, additive inverse, unity and multiplicative inverse properties. Since F is a subset of* $\mathbb{R}$, *associative, commutative and distributive properties naturally hold.*)

5. Find the multiplicative inverses of all the non-zero elements in $< \mathbb{Z}_7, \oplus, \otimes >$. (NB: *This verifies that* $\mathbb{Z}_p$ *is a field, in the particular case* $p = 7$.)

6. A sequence $<a_n>$ is said to be bounded if there exists a real number $M > 0$ such that $| a_n | \leq M$ for all n. M is said to be an upper bound for the sequence. Note that different sequences may have different upper bounds. We use the notation l_∞ for the set of all bounded sequences in $\mathbb{R}^\infty$.

   a) Show that l_∞ is a subspace of $\mathbb{R}^\infty$.

   b) Is every sequence in l_∞ convergent (YES/NO) ? Justify your answer.

SOLUTIONS FOLLOW

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Q 1.

~~Q3.~~ Given the following vectors in $\mathbb{R}^3$: $\bar{u} = (1, 3, 5)$, $\bar{v} = (1, 4, 6)$,

$\bar{w} = (2, -1, 3)$ and $\bar{b} = (6, 5, 17)$

To answer both the parts, we first row-reduce the augmented matrix for the system $A\bar{x} = \bar{b}$ where

$A = [\bar{u}, \bar{v}, \bar{w}]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 3 & 4 & -1 & 5 \\ 5 & 6 & 3 & 17 \end{array}\right] \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 1 & -7 & -13 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - R_2]{R_1 \to} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 19 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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1 is the RREF matrix of the augmented matrix.

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(a) Does $\bar{b} \in W = \text{span}\{\bar{u}, \bar{v}, \bar{w}\}$?

Answer: YES.

This reduces to solving the system

$$x\bar{u} + y\bar{v} + z\bar{w} = \bar{b}$$

i.e. the non-homogeneous system

$A\bar{x} = \bar{b}$ where A is as in the previous page.

From the RREF matrix ①, we see that the system is consistent, hence it has at least one solution, i.e. $\bar{b}$ can be expressed as a lin. comb. of $\bar{u}, \bar{v}, \bar{w}$.

1 (b) Any solution of the non-homogeneous system would work. Expressing ① in equation form, we have:

$$\begin{aligned} x &= 19 - 9z \\ y &= -13 + 7z \\ z &= 0 + z \end{aligned} \qquad \text{or } \bar{x} = \begin{bmatrix} 19 \\ -13 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 7 \\ 1 \end{bmatrix}$$

Obvious solution is $t = 0$.

Check: $19\bar{u} + (-13)\bar{v} + 0\bar{w} = 19\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + -13\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 6 \\ 7 \\ 17 \end{bmatrix} = \bar{b}, \text{ as desired}$$

Q2. Let U and W be subspaces of the vector space V. Show that $U \cap W$ is also a subspace of V.

Ans: We apply Proposition 8.

1. $\bar{0} \in U$, since U is a subspace
$\bar{0} \in W$, since W is a subspace
$\therefore \bar{0} \in U \cap W$

2. Closure under addition.
Suppose $\bar{v}_1, v_2 \in U \cap W$.
Then, $\bar{v}_1 \in U$ and $\bar{v}_2 \in U$, and so
$\bar{v}_1 + \bar{v}_2 \in U$, since U is a subspace.
Similarly, $\bar{v}_1 + \bar{v}_2 \in W$.
$\therefore \bar{v}_1 + v_2 \in U \cap W$.

3. Closure under scalar multiplication:
Suppose c is any scalar, and
$\bar{v} \in U \cap W$.
Then, $\bar{v} \in U \Rightarrow c\bar{v} \in U$, as U is a subspace.
Also, $\bar{v} \in W \Rightarrow c\bar{v} \in W$, since W is a subspace.
$\therefore c\bar{v} \in U \cap W$, as required.

Q3.

~~Q~~ :- U and W are subspaces of V.

Show that $U+W = \{ \bar{u}+\bar{w} : \bar{u} \in U, \bar{w} \in W \}$ is again a subspace of V.

Furthermore, show that $U+W$ is the smallest subspace of V containing both U and W.

Ans: ~~We verify the three properties~~

We verify the three properties given in Prop. 8 (test for subspace).

1. $\bar{0} = \bar{0} + \bar{0}$, where $\bar{0} \in U$, $\bar{0} \in W$, so $\bar{0} \in U+W$

2. Suppose $\bar{v}_1, \bar{v}_2 \in U+W$

Then
$$\left.\begin{array}{l} \bar{v}_1 = \bar{u}_1 + \bar{w}_1 \\ \bar{v}_2 = \bar{u}_2 + \bar{w}_2 \end{array}\right\} \begin{array}{l} \bar{u}_i \in U \\ \bar{w}_i \in W. \end{array}$$

$$\therefore \bar{v}_1 + \bar{v}_2 = (\bar{u}_1 + \bar{w}_1) + (\bar{u}_2 + \bar{w}_2)$$
$$= (\bar{u}_1 + \bar{u}_2) + (\bar{w}_1 + \bar{w}_2)$$
$$= \bar{u}_3 + \bar{w}_3, \text{ where } \bar{u}_3 \in U, \bar{w}_3 \in W.$$

$\therefore \bar{v}_1 + \bar{v}_2 \in U+W$

3. Finally, if $c \in F$, then $c\bar{v}_1 = c(\bar{u}_1 + \bar{w}_1) = c\bar{u}_1 + c\bar{w}_1$
$= \bar{u}_4 + \bar{w}_4 \in U+W$.

Now, suppose X is a subspace of V s.t. $U \subseteq X$ and $W \subseteq X$. Let $\bar{v} = \bar{u} + \bar{w}$ be any element of $U+W$. Then, $\bar{u} \in X$ and $\bar{w} \in X$. Hence, $\bar{u} + \bar{w} \in X$ by additive closure.

$\therefore U+W \subseteq X$ as desired.

4. Verify that $\mathbb{Q}(\sqrt{2}) = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

Answer: Since $\mathbb{Q}(\sqrt{2})$ is a subset of $\mathbb{R}$, we don't have to prove all the properties. The remaining ones are:

(i) additive closure: if $a+b\sqrt{2}$ and $c+d\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, then
$$(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c)+(b+d)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

(ii) additive identity: $0 \in \mathbb{Q}(\sqrt{2})$

(iii) additive inverse: $(-a) + (-b)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ is the additive inverse of $a+b\sqrt{2}$

(iv) multiplicative closure: $(a+b\sqrt{2})(c+d\sqrt{2})$
$$= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd$$
$$= (ac+2bd) + (ad+bc)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

(v) multiplicative identity: $1 \in \mathbb{Q}(\sqrt{2})$

(vi) multiplicative inverse: if $0 \neq a+b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ the inverse (in $\mathbb{R}$) is $\dfrac{1}{a+b\sqrt{2}} = x$, say.

We need to show $x \in \mathbb{Q}(\sqrt{2})$

But
$$x = \frac{1}{a+b\sqrt{2}} = \frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}}$$
$$= \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

Also: $a^2-2b^2 \neq 0$, since $a^2-2b^2=0 \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow$ ~~$2 \in \mathbb{Q}$~~ $\Rightarrow \sqrt{2} \in \mathbb{Q} \Rightarrow \Leftarrow$

Remark:- Fields like $\mathbb{Q}(\sqrt{2})$ are known as <u>algebraic number fields</u> and are used extensively in number theory.

Q 5. Recall that $\mathbb{Z}_7 = \{0, 1, 2, \ldots, 6\}$ with multiplication defined by
$a \otimes b = (a \times b) \pmod 7$.

Clearly, the multiplicative inverse of 1 is 1. For the others:

$2 \otimes 4 = 8 \pmod 7 = 1$ and so

$2^{-1} = 4$, so $4^{-1} = 2$.

Also, $3 \otimes 5 = 15 \pmod 7 = 1$ and so

$3^{-1} = 5$, $5^{-1} = 3$

Finally, $6 \otimes 6 = 36 \pmod 7 = 1$ and

so $6^{-1} = 6$.

We get the following table:

x	x^{-1}
1	1
2	4
3	5
4	2
5	3
6	6

Q6. Before commencing the answer, we note that if $\langle a_n \rangle$ is bounded, and M is an bound for $\langle a_n \rangle$, then $|a_n| < M$ for all n. If now $M_1 > M$, then M_1 is also an upper bound for $\langle a_n \rangle$

(a) To show l_∞ is a subspace, we use Prop. 8

1. Clearly, the zero sequence, $\langle 0 \rangle \in l_\infty$.

2. <u>Additive closure</u>: Suppose $\langle a_n \rangle, \langle b_n \rangle \in l_\infty$.
Then, $|a_n| \leq M_1$ for some $M_1 > 0$ for all n
and $|b_n| \leq M_2$ for some $M_2 > 0$ for all n,
Then, for the sum sequence $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$, we have:

$|a_n + b_n| \leq |a_n| + |b_n|$ — property of absolute values, known as triangle inequality

$\leq M_1 + M_2$, for all n.

Hence, $M_3 = M_1 + M_2$ is an upper bound for $\langle a_n \rangle + \langle b_n \rangle$, i.e.

$\langle a_n \rangle + \langle b_n \rangle \in l_\infty$

3. <u>Closure under scalar multiplication</u>:
If $\langle a_n \rangle \in l_\infty$ and $c \in \mathbb{R}$, then ~~$c\langle a_n \rangle = \langle c a_n \rangle$~~. $|a_n| \leq M_1$ for some M_1, for all n

But then $|c a_n| \leq |c||a_n| \leq c M_1$ for all n,
i.e. $c\langle a_n \rangle = \langle c a_n \rangle$ is a bounded sequence, ~~i.e.~~

Q 6, (cont'd)

i.e. $c \langle a_n \rangle \in l_\infty$.

(h) Is every sequence in l_∞ convergent?

<u>No</u>

Easy to find counter-examples,

$s = \langle 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots \rangle$

or

$\langle a_n \rangle$ where $a_n = (-1)^n$.

Actually, it can be shown that every convergent sequence is bounded, or in other words, c is a subspace of l_∞, or $c \subsetneqq l_\infty \subsetneqq \mathbb{R}^\infty$.

This is difficult, interested students may try to do it.