Proof of Prop. 29 :-

Let $\bar{x} \in V$; since B is a basis for B, we can write $\bar{x} = b_1 \bar{u}_1 + \cdots + b_n \bar{u}_n$ (1).

Then $[\bar{x}]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ (2)

Since C is also a basis for V, we can write:

$$\left.\begin{aligned} \bar{u}_1 &= A_{11}\bar{v}_1 + A_{21}\bar{v}_2 + \cdots + A_{n1}\bar{v}_n \\ &\vdots \\ \bar{u}_n &= A_{1n}\bar{v}_1 + A_{2n}\bar{v}_2 + \cdots + A_{nn}\bar{v}_n \end{aligned}\right\} \quad (3)$$

From (3), we get that

$$[\bar{u}_i]_C = \begin{bmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{bmatrix} \text{ for } i = 1, 2, \cdots, n \quad (4)$$

Substituting from (3) in (1), we get:-

$$\bar{x} = b_1(A_{11}\bar{v}_1 + \cdots + A_{n1}\bar{v}_n) + b_2(A_{12}\bar{v}_1 + \cdots + A_{n2}\bar{v}_n) + \cdots + b_n(A_{1n}\bar{v}_1 + \cdots + A_{nn}\bar{v}_n) \quad (5)$$

Re-arranging and collecting the coefficients of $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n$, we get:

$$\bar{x} = (A_{11}b_1 + A_{12}b_2 + \cdots + A_{1n}b_n)\bar{v}_1 + (A_{21}b_1 + A_{22}b_2 + \cdots + A_{2n}b_n)\bar{v}_2 + \cdots + (A_{n1}b_1 + A_{n2}b_2 + \cdots + A_{nn}b_n)\bar{v}_n \quad (6)$$

Proof of Prop. 29 – cont'd.

Hence, $$[\bar{x}]_C = \begin{bmatrix} A_{11}b_1 + A_{12}b_2 + \cdots + A_{1n}b_n \\ A_{21}b_1 + A_{22}b_2 + \cdots + A_{2n}b_n \\ \vdots \\ \vdots \\ A_{n1}b_1 + A_{n2}b_2 + \cdots + A_{nn}b_n \end{bmatrix}$$

$$= \begin{bmatrix} A_{ij} \end{bmatrix}_{n\times n} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = P\,[\bar{x}]_B \qquad (7)$$

As noted already ~~in~~ from (4), the columns of ~~the~~ the matrix P are nothing but the coordinate vectors of the old ~~vector~~ basis, i.e. the $\bar{v}_i$'s, in terms of the new basis, i.e. the $\bar{w}_j$'s.

Finally, we note that P must be invertible for the following reason :– the coordinate mapping is an isomorphism from V to ~~the~~ F^n. Since B is a basis of V, it goes to a basis of F^n under the coordinate mapping with regard to basis C (Prop. 27 (a)). Since the columns of P ~~are~~ form a basis of F^n, ~~the~~ P is invertible by VIT (g).

Proof of Prop. 30: Similarity is an equivalence relation on $F^{n\times n}$.

(i) Reflexive: If $A \in F^{n\times n}$, then $A = \cancel{PP} \; I A I^{-1}$, so A is similar to A

(ii) Symmetric: Suppose B is similar to A.

Then, $\exists P$ s.t. $B = PAP^{-1}$.

Put $Q = P^{-1}$

$$\therefore \quad QBQ^{-1} = P^{-1}(PAP^{-1})(P^{-1})^{-1}$$
$$= A$$

$\therefore$ A is similar to B.

(iii) Transitive: Suppose B is similar to A and C is similar to B.

Then $B = PAP^{-1}$
and $C = QBQ^{-1}$

i.e. $$C = Q(PAP^{-1})Q^{-1}$$
$$= (QP)A(P^{-1}Q^{-1})$$
$$= (QP)A(QP)^{-1}$$

Utility of idea of similarity in matrix computations:-

Suppose $B = PAP^{-1}$

Then $B^k = \underbrace{PAP^{-1}.\,PAP^{-1}\,.....\,PAP^{-1}}_{k \text{ times}}$

$= PA^kP^{-1}$

Hence, if it is easy to find the powers of A, then it's easy to find power of B.

Easiest case: A is diagonal

$A = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$

Then $A^k = \mathrm{diag}\{\lambda_1^k, \ldots, \lambda_n^k\}$

Unfortunately, not every matrix is ~~diagonal to a~~ similar to a diagonal matrix. But in many important applns, they are.

Example for Change of Basis

We consider the old basis $\alpha = \{ \bar{e}_1, \bar{e}_2 \}$

The new basis $\beta = \{ \bar{u}_1, \bar{u}_2 \}$

where $\bar{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Construct the matrix Q which has the vectors of β as its columns.

$$\therefore \quad Q = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

new basis in terms of old

Then, the change of basis matrix

$$P_{\alpha \to \beta} = Q^{-1}$$

$$\therefore \quad P = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Let us check with a specific vector, say $\bar{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha$

Then, $[\bar{u}]_\beta = P \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}_{\alpha}$

$= \begin{bmatrix} -26 \\ 11 \end{bmatrix}_\beta$

Check: $-26\,\bar{u}_1 + 11\,\bar{u}_2 = -26\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 11\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ ✓ Confirmed

Verification of the remark that columns of P are the coordinate vectors of old basis in terms of new basis.

Now $\begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta = 3\,\bar{u}_1 + (-1)\,\bar{u}_2 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \bar{e}_1$, i.e. $[\bar{e}_1]_\beta = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta$

and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}_\beta = (-5)\,\bar{u}_1 + 2(\bar{u}_2) = (-5)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \bar{e}_2$, i.e. $[\bar{e}_2]_\beta = \begin{bmatrix} -5 \\ 2 \end{bmatrix}_\beta$

Now, let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation 7

given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}$

Let us first determine $A = [T]_\alpha$ = matrix of T relative to standard basis.

Now $T\bar{e}_1 = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\bar{e}_1 + 3\bar{e}_2$

and $T\bar{e}_2 = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2e_1 + 4e_2$

$\therefore [T]_\alpha = A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Applying Prop. 31, the matrix relative to the new basis β would be $B = PAP^{-1}$, where $P = P_{\alpha\to\beta}$, the change of basis matrix, i.e.

$$B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}$$

Let us verify our calculations with an example, say $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta$

$$\left[\text{since } [\bar{u}]_\beta = P_{\alpha\to\beta}[\bar{u}]_\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta\right]$$

$$\therefore [T\bar{u}]_\alpha = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha$$

and $[T\bar{u}]_\beta = \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}_\beta \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta = \begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta$

But $\begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta = -40\bar{u}_1 + 17\bar{u}_2$

$$= -40\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 17\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha$$

✓

The Idea behind Prop. 31

Think of a matrix as a system.

The input is a vector $\bar{x}$ given as coordinate vector with regard to a basis α, and the ~~out~~ output is again a vector $\bar{y}$, also given in terms of α.

Diagram:

$$[\bar{x}]_\alpha \longrightarrow \boxed{A} \longrightarrow [\bar{y}]_\alpha .$$

However, now suppose that the input is given as a coordinate with regard to basis β, and the output is also desired in this form.

So, we have to proceed as follows (Recall that if the change of basis matrix from α to β is P, the change of basis matrix from β to α is P^{-1}):—

$$[\bar{x}]_\beta \xrightarrow{P^{-1}} [\bar{x}]_\alpha \longrightarrow \boxed{A} \longrightarrow [\bar{y}]_\alpha \xrightarrow{P} [\bar{y}]_\beta .$$

(PTO)

Prop. 31 (cont'd) :-

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We now express the above system diagram in matrix terms. Recall that when a product of matrices is to ~~be~~ operate (multiply) ~~on~~ a vector, ~~the p.~~ we proceed from right to left.

So $[T]_\beta [\bar{x}]_\beta = \cancel{P^{-1}} (PAP^{-1}) [\bar{x}]_\beta$,

i.e. $[T]_\beta = B = PAP^{-1}$

About Prop. 32

$$W^V = \{ f : V \longrightarrow W \}$$

→ why this notation?

$|X| = n$ elements

$|Y| = m$ elements in it

Then, what is the no. of fns. from X to Y?

$$= m^n = |Y|^{|X|}$$

$$|Y^X| = |Y|^{|X|}$$ when

X and Y are finite

[NB: In our case, W and V are both infinite, considered as sets. But this notation has been adopted for the set of functions from ~~W~~ V to W.]