Proof of Multiplicative Property (Prop. 36)

Suppose that $T\bar{v}_i = a_{1i}\bar{v}_1 + a_{2i}\bar{v}_2 + \cdots + a_{ni}\bar{v}_n$ ①
for $i = 1_2$ to n.

Recall that this is the i-th column of the matrix $[T]_\beta$.

Similarly, $U\bar{v}_i = b_{1i}\bar{v}_1 + \cdots + b_{ni}\bar{v}_n$ ② for $i = 1, \ldots, n$
is the i-th column of the matrix $[U]_\beta$.

$$\therefore (UT)\bar{v}_i = U(a_{1i}\bar{v}_1 + \cdots + a_{ni}\bar{v}_n) \text{ - from ①}$$

$$= a_{1i}U(\bar{v}_1) + a_{2i}U\bar{v}_2 + \cdots + a_{ni}U\bar{v}_n$$

$$= a_{1i}(b_{11}\bar{v}_1 + b_{21}\bar{v}_2 + \cdots + b_{n1}\bar{v}_n) + \cdots + a_{ni}(b_{1n}\bar{v}_1 + \cdots + b_{nn}\bar{v}_n)$$

from ②

$$= (a_{1i}b_{11} + a_{2i}b_{12} + \cdots + a_{ni}b_{1n})\bar{v}_1 + \cdots + (a_{1i}b_{n1} + \cdots + a_{ni}b_{nn})\bar{v}_n$$

In other words, the i-th column of $[UT]_\beta$ is:

$$\begin{bmatrix} b_{11}a_{1i} + b_{12}a_{2i} + \cdots + b_{1n}a_{ni} \\ \vdots \\ b_{n1}a_{1i} + \cdots + b_{nn}a_{ni} \end{bmatrix} = i\text{-th column of } [U]_\beta [T]_\beta$$

as we wanted

PROOF OF PROP. 37

Consider the function $T^{-1}: W \longrightarrow V$

(i) Consider $\bar{w}_1, \bar{w}_2 \in W$

We have to show: $T^{-1}(\bar{w}_1 + \bar{w}_2)$

$$= T^{-1}(\bar{w}_1) + T^{-1}(\bar{w}_2) \qquad (1)$$

Let us apply T to both sides of (1) :- (LHS first)

$$T\left(T^{-1}(\bar{w}_1 + \bar{w}_2)\right) = (TT^{-1})(\bar{w}_1 + \bar{w}_2)$$

$$= I(\bar{w}_1 + \bar{w}_2) = \bar{w}_1 + \bar{w}_2 \qquad (2)$$

Let us apply T to RHS of (1)

$$T\left(T^{-1}(\bar{w}_1) + T^{-1}(\bar{w}_2)\right)$$

$$= T(T^{-1}(\bar{w}_1)) + T(T^{-1}\,\bar{w}_2))$$

Since T is given to be linear

$$= (TT^{-1})(\bar{w}_1) + (TT^{-1})(\bar{w}_2)$$

$$= \bar{w}_1 + \bar{w}_2 \qquad (3)$$

Since T is <u>injective</u>, from (2) and (3) we get that:

LHS of (1) = RHS of (1), as reqd.

(PTO)

We also need to show

$$T^{-1}(c\bar{w}_1) = cT^{-1}(\bar{w}_1) \quad 4$$

→ Apply the same strategy, apply T to both sides of 4

From LHS: $T(T^{-1}(c\bar{w}_1)) = (TT^{-1})(c\bar{w}_1)$

$= I(c\bar{w}_1) = c\bar{w}_1$ 5

From RHS: $T(cT^{-1}\bar{w}_1) = c\,T(T^{-1}\bar{w}_1)$

(since T is given to be linear

$= c\,(TT^{-1})(\bar{w}_1) = c\bar{w}_1$. 6

Since T is injective, from 5 and 6 we get that

LHS of 4 = RHS of 4,

as required.

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