Wed-Thurs 05-06 Oct 2016

Notes for [redacted]

The matrix linear transformation

$T_A : F^n \longrightarrow F^m$ given by $T_A(\bar{x}) = A\bar{x}$

where A is an $m \times n$ matrix (i.e. $A \in F^{m \times n}$).

For any vectors $\bar{x}, \bar{y} \in F^n$,

$$A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y}$$

and $A(c\bar{x}) = cA\bar{x}$

—x—x—x—

what is $\ker(T_A)$?

$$\ker T_A = \{\bar{x} \in F^n : A\bar{x} = \bar{0}\}$$

$$= \text{Nul}(A)$$

$\therefore$ nullity $(T_A) =$ nullity A

What is Range (T_A)?

$$\text{Range } T_A = \{\bar{b} \in F^m : \bar{b} = A\bar{x} \text{ for some } \bar{x} \in F^n\}$$

$$= \text{Col}(A)$$

$\therefore$ rank $(T_A) = \dim \text{Col } A = \text{rank } A$

→ In future, we need not distinguish between T_A and A

Example to illustrate coordinate systems.

In $\mathbb{R}^2$, given a vector $\bar{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, its coordinate vector with regard to the standard ordered basis S is nothing but $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_S$.

However, suppose we take a different ordered basis, say $B = \{ \bar{u}_1 , \bar{u}_2 \}$ where $\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \bar{u}$

We see by inspection that

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore [\bar{u}]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_B$$

Let's take another vector,

say $\bar{w} = \begin{bmatrix} 15 \\ 24 \end{bmatrix}$.

To find $[\bar{w}]_B$, we need to find scalars x_1 and x_2 such that

$$x_1 \bar{u}_1 + x_2 \bar{u}_2 = \bar{w}$$

i.e. $$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 24 \end{bmatrix}$$

i.e. $$\begin{aligned} x_1 + x_2 &= 15 \\ x_2 &= 24 \end{aligned}$$

i.e. $$A \bar{x} = \bar{w} \qquad (1)$$

where A is the matrix with $\bar{u}_1$ and $\bar{u}_2$ as its columns.

But now, since the columns of A form a basis (i.e. B), A is invertible.

$\therefore$ the solution of (1) is given by $\bar{x} = A^{-1} \bar{w}$.

So if we find A^{-1} we can find out the ~~the~~ coordinate vector relative to B for any arbitrary vector $\bar{y} \in \mathbb{R}^2$.

Now: $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\therefore [\bar{v}]_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 24 \end{bmatrix} = \begin{bmatrix} -9 \\ 24 \end{bmatrix}_B$

Check: $-9\,\bar{u}_1 + 24\,\bar{u}_2 = -9\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 24\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 15 \\ 24 \end{bmatrix}$ ✓

Outcome of the above discussion:- To find the coordinate vector of any $\bar{x}$ wrt a new ~~basis~~ ordered basis $B = \{\bar{v}_1, \ldots, \bar{v}_k\}$, set up the matrix $A = [\bar{v}_1 \ \bar{v}_2 \ldots \bar{v}_k]$. A is invertible by VIT, so we can find A^{-1}.

Then $[\bar{x}]_B = A^{-1}\bar{x}$ ($\bar{x}$ is given as an k-tuple in F^k, i.e. its standard coordinate vector is given)