MTH100 **Monsoon Semester – 2016-17** **Section A**

Submission for Tuesday 30th August 2016. Time: 15 minutes. Max Marks: 5

Show that the set C of complex numbers is a vector space over the field R of real numbers. The usual addition and multiplication operations of C should be used. You are required to prove only the following properties (axioms): A, i.e. closure; B, only b) identity or zero property and c) additive inverse property; c) only a) distributive property.
Recall that C = {x + y*i*: x,y ∈ R }. *(5 marks)*

Solution

A. Closure under addition:-

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are in $\mathbb{C}$, then $z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i)$

$= (x_1 + x_2) + (y_1 + y_2) i = x_3 + y_3 i$,

where $\left.\begin{array}{l} x_3 = x_1 + x_2 \\ \text{and } y_3 = y_1 + y_2 \end{array}\right\} \in \mathbb{R}$, ~~as defined~~

i.e. $z_1 + z_2 \in \mathbb{C}$, as defined

Closure under scalar multiplication.

If $r \in \mathbb{R}$, then $r z_1 = r(x_1 + y_1 i)$

$= r x_1 + r y_1 i = x_4 + y_4 i$ where

$\left.\begin{array}{l} x_4 = r x_1 \\ y_4 = r y_1 \end{array}\right\} \in \mathbb{R}$. $\therefore$ $r z_1 \in \mathbb{C}$, as defined.

B. b) Identity or zero property: The complex number $0 = 0 + 0i$ satisfies $z + 0 = 0 + z = z$ for all $z \in \mathbb{C}$

Inverse property

& c) If $z = x + yi$, then putting $-z = -x + (-y)i$,

we get $z + (-z) = (x + yi) + (-x + (-y)i)$

$= (x + (-x)) + (y + (-y))i = 0 + 0i = 0$, as desired.

(PTO)

C. (a) If $r \in \mathbb{R}$, and $z_1, z_2 \in \mathbb{C}$, as before,

then $r(z_1+z_2) = r[(x_1+y_1 i)+(x_2+y_2)i]$

$= r[\cancel{x_1+y_1 i}\ (x_1+x_2)+(y_1+y_2)i]$

$= r(x_1+x_2) + r(y_1+y_2)i$ (1)

OTOH, $rz_1 + rz_2 = r[x_1+y_1 i] + r[x_2+y_2 i]$

$= r(x_1) + r(y_1)i + rx_2 + ry_2 i$

$= r(x_1+x_2) + r(y_1+y_2)i$ (2)

Result follows from (1) and (2), comparing the RHS of each.

Remark :- This example (and also the similar example ~~give~~ done by Section B – see below) shows that F and K are fields, with $F \subseteq K$, and using the same definition for additions and multiplication in F as holds in K, then K is a vector space over F.

This is often useful; for example, we can use results of linear algebra in field theory

Note: A field K is a vector space over its subfield F; however, every vector space need not be a field.

MTH100 **Monsoon Semester – 2016-17** **Section B**

Submission for Friday 2nd September 2016. Time: 15 minutes. Max Marks: 5

Show that the set R of real numbers is a vector space over the field Q of rational numbers. The usual addition and multiplication operations of R should be used. You are required to prove only the following properties (axioms): A, i.e. closure; B, only b) identity or zero property and c) additive inverse property; C, only a) distributive property.

(5 marks)

SOLUTION

~~A.~~ Let r_1, r_2 be arbitrary real numbers and let q be any rational number, i.e. $r_1, r_2 \in \mathbb{R}, q \in \mathbb{Q}$.

A. Closure under addition.

- $r_1 + r_2$ is a real number, i.e. $r_1 + r_2 \in \mathbb{R}$

Closure under scalar multiplication:-

- qr_1 is a real number, i.e. $qr_1 \in \mathbb{R}$

B ~~Add~~ b) Identity or zero property.

~~$0 \in \mathbb{R}$~~ is ~~actually~~ ~~a rational~~ ~~number~~, ~~i.e. 0~~

$0 \in \mathbb{R}$ ~~satisfies~~ $0 + r_1 = r_1 + 0 = r_1$ for all $r_1 \in \mathbb{R}$

c) Additive inverse property

If $r_1 \in \mathbb{R}$, then ~~$(-1)r_1 \in \mathbb{R}$ and~~ $-r_1 \in \mathbb{R}$, and $r + (-r_1) = 0$, the zero element.

C. (a) Distributive Property

$q(r_1 + r_2) = qr_1 + qr_2$, using the distributive property for real numbers.

Note: Please read the remark under the solution for Section A.