MTH100 - 20160914 - Notes

An Application of Proposition 18

Is $C[a,b]$, i.e. the space of continuous real-valued functions defined on an arbitrary closed interval, finite-dimensional or infinite-dimensional?

Answer: Infinite-dimensional.

Proof: Suppose BWOC that $C[a,b]$ is finite-dimensional.

Now, consider $P[a,b]$ = set of ~~all~~ polynomial functions with domain $[a,b]$.
Clearly, $P[a,b] \subseteq C[a,b]$ since all polynomial functions are continuous.
Furthermore, it is easy to see that $P[a,b]$ is actually a subspace of $C[a,b]$.
At this stage, we recall the result that $\mathbb{R}[t]$ is infinite-dimensional. $\mathbb{R}[t]$ and $P[a,b]$ are not exactly the same space, but they are very similar. So the proof that $\mathbb{R}[t]$ is infinite-dimensional can be applied to $P[a,b]$ with minor modifications to show that $P[a,b]$ is infinite-dimensional.

But, by Prop. 18, any subspace of a finite-dimensional subspace must be finite-dimensional. This is a contradiction!

Hence, $C[a,b]$ must be ~~fi~~ infinite-dimensional.

—x—x—x—x—

The importance of $P[a,b]$ follows from the following deep result: Weierstrass Approximation Theorem: Let $f \in C[a,b]$ and let any positive tolerance ϵ be given. Then, there is a ~~p~~ polynomial function $p(x) \in P[a,b]$ such that the distance between $p(x)$ and $f(x)$ is less than ϵ, i.e.

$$|p(x) - f(x)| < \epsilon \quad \text{for all } x \in [a,b]$$

Remarks: The above is very useful both in the theory and in practical applications since polynomials are easy to calculate with. Recall ~~the~~ Taylor's Theorem:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a)$$
$$+ \text{ Remainder Term.}$$

This can be used to determine the approximating polynomial provided two conditions hold:

(i) $f(x)$ is not just continuous but also has derivatives of all orders, i.e. ~~$f(x)$~~ $f \in C^\infty[a,b]$

(ii) The Remainder Term converges to 0 as $n \to \infty$ (for small h).

Though we can find functions $f \in C^\infty[a,b]$ which do not satisfy (ii) above, many of the functions found in practice, such as exponential, trigonometric, etc., do satisfy (ii). Of course, if $f \notin C^\infty[a,b]$ but $f \in C[a,b]$, we can use Weierstrass Theorem directly.

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NB:- notation is slightly different from class:-

PROOF OF PROP. 20:-

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

PROOF: Put $X = U \cap W$ for convenience.

Let $\{\bar{x}_1, \dots, \bar{x}_k\}$ be a basis for X.

Expand it to a basis for U by adjoining $\{\bar{u}_1, \dots, \bar{u}_m\}$, say B_1.

~~At~~ Similarly, expand it to a basis of W by adjoining $\{\bar{w}_1, \dots, \bar{w}_n\}$, say B_2.

Let $B = \{\bar{x}_1, \dots, \bar{x}_k, \bar{u}_1, \dots, \bar{u}_m, \bar{w}_1, \dots, \bar{w}_n\}$.

We claim that B is a basis for $U+W$. Clearly, B is a spanning set for $U+W$. It remains to show that B is linearly independent.

Suppose:

$$a_1\bar{x}_1 + \cdots + a_k\bar{x}_k + b_1\bar{u}_1 + \cdots + b_m\bar{u}_m + c_1\bar{w}_1 + \cdots c_n\bar{w}_n = \bar{0} \quad (1)$$

$$\therefore \; a_1\bar{x}_1 + \cdots + b_m\bar{u}_m = -c_1 w_1 - \cdots - c_n\bar{w}_n \quad (2)$$

LHS $\in U$, RHS $\in W$, $\therefore$ this vector belongs to $U \cap W$.

Hence we can write

$$-c_1\bar{w}_1 - \cdots - c_n\bar{w}_n = d_1\bar{x}_1 + \cdots + d_k x_k \quad (3)$$

Or $\quad d_1\bar{x}_1 + \cdots + d_k\bar{x}_k + c_1\bar{w}_1 + \cdots + c_n\bar{w}_n = \bar{0} \quad (4)$

Since B_2 is l.i., being a basis for W, we must have $d_1 = \cdots = d_k = c_1 = \cdots = c_n = 0 \quad (5)$

$\therefore$ (1) becomes:-

$$a_1\bar{x}_1 + \cdots + a_k\bar{x}_k + b_1\bar{u}_1 + \cdots + b_m\bar{u}_m = \bar{0} \quad (6)$$

Now, since B_1 is a basis for U, we must have $a_1 = \cdots = a_k = b_1 = \cdots = b_m = 0$.

~~Result follows.~~

~~Clai~~ Hence, our claim is justified.

Finally, dim X ~~$\cancel{}$~~ = no. of elements in a basis for ~~Z~~ $U+W$

$= |B|$

$= k+m+n$

$= (k+m)+(k+n)-k$

$= \dim U + \dim W - \dim X$

$= \dim U + \dim W - \dim(U \cap W)$

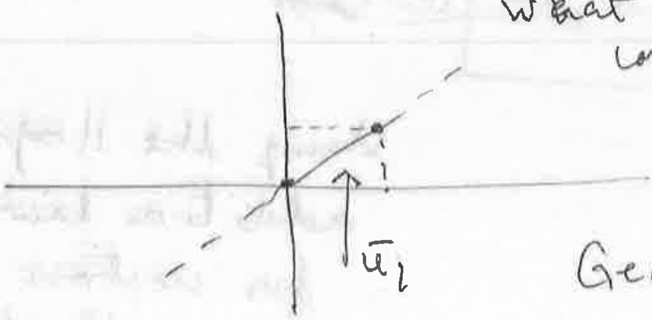
Additional Notes

→ Subspaces of $\mathbb{R}^2$, say U, W

- only possibility for a proper subspace U is $\dim U = 1$, i.e. it has a basis consisting of one (non-zero vector), $\bar{u}_1$

Geometrically, $\bar{u}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ corresponds to the ~~line~~ point (x_1, y_1) in the plane, or more specifically, the directed line segment joining $(0,0)$ to (x_1, y_1).

Taking a particular case, say $\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



What does U correspond to:

Algebraically,

$U = \{ k\bar{u}_1 : k \in \mathbb{R} \}$.

Geometrically, U = line through the origin and $(1, 1)$.

→ Any ~~subsp~~ 1-dimensional subspace corresponds to a line through the origin, and conversely.

Now, suppose $\bar{u}_2$ is a vector which is not a scalar multiple of $\bar{u}_1$, say $\bar{u}_2$.

$\therefore$ $\bar{u}_1, \bar{u}_2$ are lin. indep.

$\therefore$ $\{ \bar{u}_1, \bar{u}_2 \}$ is necessarily a basis, i.e. any vector $\bar{u}$ can be uniquely expressed as a linear combination of $\bar{u}_1$ and $\bar{u}_2$.

If $U = \text{span}\{\bar{u}_1\}$, $W = \text{span}\{\bar{u}_2\}$ (2)

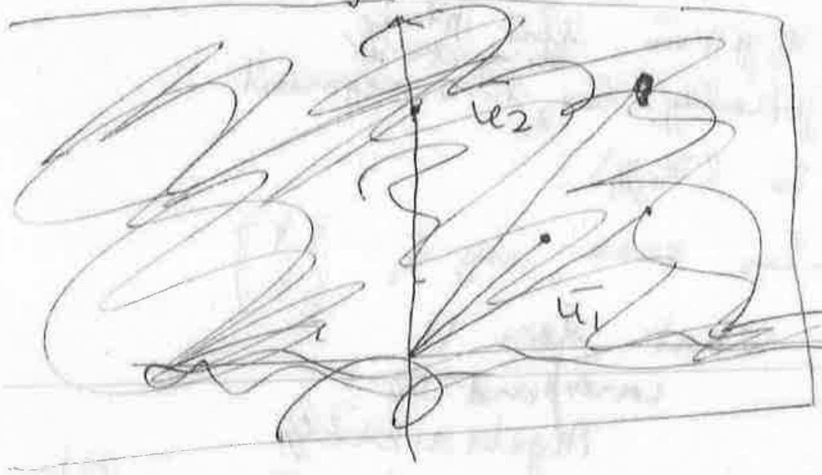
then ~~#~~ $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

i.e. $2 = 1 + 1 - \dim(U \cap W)$

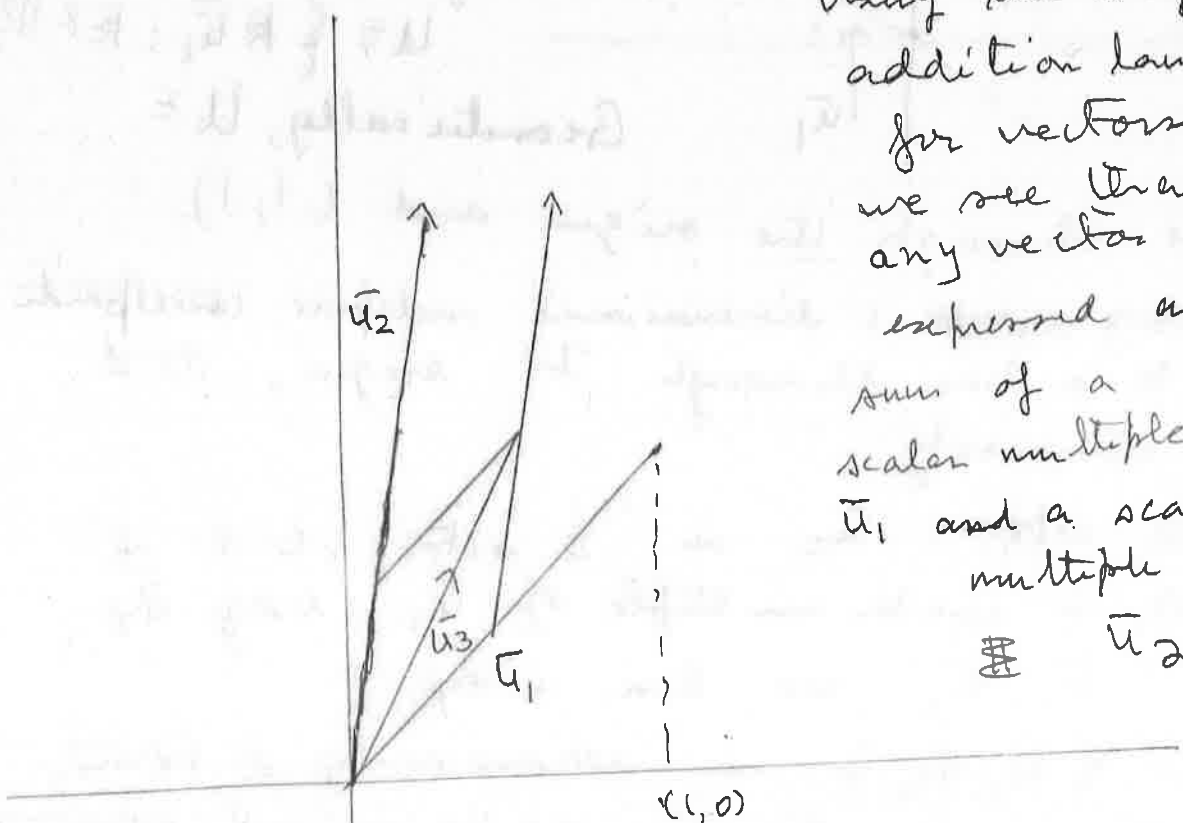
and $\therefore$ $\dim(U \cap W) = 0$, i.e. $U \cap W = \{\bar{0}\}$.

This can be seen geometrically, also.

Take any ~~particular~~ other ~~case~~ vector, say $\bar{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Take any arbitrary vector $\bar{u}_3$



Using the ‖ogram addition law for vectors, we see that any vector is expressed as the sum of a scalar multiple of $\bar{u}_1$ and a scalar multiple of ~~#~~ $\bar{u}_2$

(2a)

(Optional) – $\mathbb{R}^2$ – continued

Algebraically, we can see something from ~~more~~ a different perspective.

Since $\bar{u}_1$ and $\bar{u}_2$ are linearly independent, the equation

$$c_1 \bar{u}_1 + c_2 \bar{u}_2 = \bar{0}$$ holds only ~~only the trivial solution~~ for $c_1 = c_2 = 0$.

Hence, the homogeneous system

$A\bar{x} = \bar{0}$ where

$A = [\bar{u}_1\ \bar{u}_2]$, the matrix with $\bar{u}_1$ and $\bar{u}_2$ as its columns,

has only the trivial soln.

Hence, by IMT, A is invertible!!

Also, the system $A\bar{x} = \bar{b}$ has a solution (actually unique) for every $\bar{b} \in \mathbb{R}^2$, i.e.

every $\bar{b} \in \operatorname{Span}\{\bar{u}_1, \bar{u}_2\}$,

i.e. $\{\bar{u}_1, \bar{u}_2\}$ is a basis

So now let us consider subspaces of $\mathbb{R}^3$,

say U, W

Case 1. $\dim U = 1$

Then U has a basis consisting of a single vector, say $\bar{u}_1$. As before, $\bar{u}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ corresponds to the directed line segment joining $(0,0,0)$ to (x_1, y_1, z_1), and $U = \text{span}\{\bar{u}_1\}$ corresponds to the line through the origin $(0,0,0)$ and (x_1, y_1, z_1).

Case 2. $\dim U = 2$; then U has a basis consisting of two linearly independent vectors $\bar{u}_1$ and $\bar{u}_2$, i.e.

$$U = \text{span}\{\bar{u}_1, \bar{u}_2\}$$

$\rightarrow$ Then U corresponds to a plane through the origin — corresponds to the geometrical fact that two intersecting lines in $\mathbb{R}^3$ determine a plane + ~~$\bar{u}_1$ and $\bar{u}_2$ are~~ obviously span $\{\bar{u}_1\}$ and span $\{\bar{u}_2\}$ intersect at the origin.

We can see this using coordinate geometry also.

Solving for c_1 and c_2 in terms of x and y, i.e. solving the system

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 2 & 3 & y \end{array}\right] \xrightarrow[R_2-2R_1]{R_2\to} \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y-2x \end{array}\right]$$

$$\xrightarrow{R_1\to R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & 3x-y \\ 0 & 1 & y-2x \end{array}\right],$$

i.e. $c_1 = 3x-y$; $c_2 = y-2x$.

Substituting in the 3rd row of system (1),

we get $z = 3c_1 + 5c_2$

$= 3(3x-y) + 5(y-2x)$

$= -x + 2y \rightarrow$ eqn. of a plane through the origin.

What about the intersection of two ~~planes~~ planes through the origin, i.e. two 2-dim. subspaces U and W.

We have $\dim(U+W) = \dim U + \dim W - \dim(U\cap W)$

$= 4 - \dim(U\cap W)$. (2)

Now, $U+W$ has U and W as subspaces, so

$\dim(U+W) = 3$ or 2.

$\dim(U+W) = 3$ makes (2) into $3 = 4 - \dim(U\cap W)$

$\Rightarrow \dim(U\cap W) = 1$, i.e. $U\cap W$ is a line (1-dim. subspace through the origin).

NB: if $\dim(U+W) = 2$, then $U+W = U = W \Rightarrow$ U and W were actually the same subspace, and (2) becomes $2 = 2+2-2$, which is obviously true.

→ Let us take a plane through the origin, say $x + y + z = 0$

This corresponds to the linear system

$$A\bar{x} = \bar{0}$$

where $A = [1 \quad 1 \quad 1]$.

Since A is already an RREF matrix, we solve the corresponding homog. system:

$$x_1 = -x_2 - x_3$$
$$x_2 = x_2$$
$$x_3 = x_3$$

to get $\bar{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

i.e. the plane corresponds to the span of the two linearly independent vectors

$\bar{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\bar{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

→ Conversely, suppose we take any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in the span of the linearly independent vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad (1)$$

(PTO)