NOTES FOR ~~MATH~~ MTH100 - SYMMETRIC MATRICES - done on 20161109_10 (WED-THU)

PRELIMINARY NOTES FOR PROP. 54.

Another way to interpret matrix multiplication:

Suppose A is an $m\times n$ matrix and B is an $n\times k$ matrix, so that AB is well-defined and AB is $m\times k$. Let $AB = C = [c_{ij}]$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_m \end{bmatrix}$, where the $\bar{r}_i$ are rows of A.

Let $B = \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \\ b_{n1} & \cdots & b_{nk} \end{bmatrix} = [\bar{v}_1 \ \cdots \ \bar{v}_k]$, where the $\bar{v}_i$ are columns of B.

Now, $c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}$

$= [a_{11} \ \cdots \ a_{1n}] \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix}$, it is a product of a $1\times n$-matrix and an $n\times 1$-matrix, hence, a 1×1-matrix or scalar

$= \bar{r}_1 \bar{v}_1$ (matrix product)

We can also interpret it as the dot-product of the 1st row of A with 1st column of B.

Hence, we can write:

$$AB = \begin{bmatrix} \bar{r}_1\bar{v}_1 & \bar{r}_1\bar{v}_2 & \cdots & \bar{r}_1\bar{v}_k \\ \vdots & & & \\ \bar{r}_m\bar{v}_1 & \bar{r}_m\bar{v}_2 & \cdots & \bar{r}_m\bar{v}_k \end{bmatrix} = [\bar{r}_i \bar{v}_j],$$

i.e. $c_{ij} = \bar{r}_i \bar{v}_j$

Proof of Prop. 53 | A – symmetric

Let λ_1, λ_2 be two distinct eigenvalues.

Let $\bar{v}_1$ and $\bar{v}_2$ be their corresp. eigenvectors.

$$\lambda_1 (\bar{v}_1 \circ \bar{v}_2)$$

$$= (\lambda_1 \bar{v}_1 \circ \bar{v}_2)$$

$$= (A\bar{v}_1) \circ \bar{v}_2 \quad (\because \bar{v}_1 \text{ is an eigenvector for } \lambda_1)$$

$$= (A\bar{v}_1)^T \bar{v}_2 \quad (\text{by defn.})$$

$$= \bar{v}_1^T A^T \bar{v}_2$$

$$= \bar{v}_1^T (A\bar{v}_2) \quad \text{since } A^T = A$$

$$= \bar{v}_1^T (\lambda_2 \bar{v}_2) \quad - \because \bar{v}_2 \text{ is an eigenvector for } \lambda_2$$

$$= \bar{v}_1 \circ \lambda_2 \bar{v}_2$$

$$\lambda_1 (\bar{v}_1 \circ \bar{v}_2) = \lambda_2 (\bar{v}_1 \circ \bar{v}_2)$$

$$\therefore \bar{v}_1 \circ \bar{v}_2 = 0 \quad \text{— as we wanted}$$

NUMERICAL EXAMPLE FOR SYMMETRIC MATRICES

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = -(\lambda-1)^2(\lambda-4)$$

Putting $\lambda = 4$: $A - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{RREF}$$

Solving the system, we get $\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as an eigenvector, normalizing: $\bar{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Putting $\lambda = 1$, $A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{RREF}$$

or $\bar{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$

(PTO)

We thus get $\bar{u}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\bar{u}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

as eigenvectors. We see that $\bar{u}_1 \circ \bar{u}_2 = \bar{u}_1 \circ \bar{u}_3 = 0$, but $\bar{u}_2 \cdot \bar{u}_3 = 1 \neq 0$ -

So now we have to apply Gram-Schmidt process to W = eigenspace of $\lambda = 1$.

$$\bar{v}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\bar{v}_3 = \bar{u}_3 - \frac{\bar{u}_3 \circ \bar{v}_2}{v_2 \circ \bar{v}_2}\,\bar{v}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -1\end{bmatrix}$$

Now, normalize to get $\bar{v}_2' = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$

$$\bar{v}_3' = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

Check: we must have $A = PDP^{-1} = PDP^T$ ✓

or $AP = PD$, where P has $\bar{v}_1'$, $\bar{v}_2'$, $\bar{v}_3'$ as columns

$$\cancel{= P \begin{bmatrix}4 & & 0\\ 0 & 1 & \\ & & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3}\\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0\\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{2}{\sqrt6}\end{bmatrix} =}$$

$$\cancel{P \begin{bmatrix}\frac{4}{\sqrt3} & \frac{4}{\sqrt3} & \frac{4}{\sqrt3}\\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0\\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{2}{\sqrt6}\end{bmatrix}}$$

(See ~~[redacted]~~ next page)

Check for numerical example

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -2/\sqrt{6} \end{bmatrix} \to P$$

$$= \begin{bmatrix} \frac{4}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$ ✓

OTOH, $\cancel{DP}$ PD $= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -1/\sqrt{2} & \frac{1}{\sqrt{6}} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{4}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 4/\sqrt{3} & -1/\sqrt{2} & \frac{1}{\sqrt{6}} \\ 4/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$ ✓

Proof of Theorem 9 (b): Every eigenvalue of a real symmetric matrix is real, i.e. if A is an $n \times n$ real symmetric matrix, then its characteristic poly. has only real roots (no complex roots).

Step 1 (Lay 23/341): Let A be an $n \times n$ real symmetric matrix, and let $x \in \mathbb{C}^n$. Put $q = \bar{x}^T A x$, where $\bar{x}$ = complex conjugate of x. Then q is real.

Proof: We have: $\bar{q} = \overline{\bar{x}^T A x}$ (by definition)

$= x^T \overline{Ax}$ (since $\bar{\bar{y}} = y$ for any $y \in \mathbb{C}^n$, and conjugate of a product is = product of conjugates – this was a tutorial problem)

$= x^T A \bar{x}$ – since $\bar{A} = A$, as A is real

$= (x^T A \bar{x})^T$ – since the quantity in brackets is actually a scalar, hence its transpose is equal to itself

$= \bar{x}^T A^T x$ → since $(AB)^T = B^T A^T$

$= \bar{x}^T A x$ – ~~again~~ since A is symmetric, $A^T = A$

$= q$.

Since $\bar{q} = q$, q is real.

[This is one of the ways to show a complex number z is real – i.e. show that $\bar{z} = z$]

Proof (cont'd):-

Step 2 (Lay 24|341) Show that if $AX = \lambda X$ for some non-zero vector X in $\mathbb{C}^n$, then λ is in fact real, and the real part of X is in fact an eigenvector of A (in $\mathbb{R}^n$).

Proof: Consider $q = \bar{X}^T A X$.

By Step 1, q is known to be real.

Now, $q = \bar{X}^T (\lambda X)$ since X is an eigenvector

$$= \lambda (\bar{X}^T X). \quad (1)$$

Now, $\bar{X}^T X$ is real and > 0 :-

Put $X = \begin{bmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, where each $z_i \in \mathbb{C}$.

Then $\bar{X}^T X = [\bar{z}_1 \cdots \bar{z}_n] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, where $\bar{z}_i$ = complex conjugate of z_i

$$= z_1\bar{z}_1 + \cdots + z_n\bar{z}_n > 0$$ since X is non-zero.

Putting $\bar{X}^T X = r \in \mathbb{R}$, $r > 0$,

we get $q = \lambda r$ from (1), whence $\lambda = \frac{q}{r}$ is again real, as desired.

Finally, if $X = U + iV$, where $U, V \in \mathbb{R}^n$

Then $AX = A(U + iV) = AU + iAV$

$$= \lambda(X) = \lambda(U + iV) = \lambda U + i\lambda V.$$

Equating real and imaginary parts, $AU = \lambda U$, and we are done.