PROOF OF VIT.

WED: 20160817
MTH 100

We will proceed as follows:

(a) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a)

[(a) $\Longleftrightarrow$ (d), i.e. (a) $\Longrightarrow$ (d) $\Longrightarrow$ (a)
will be done after doing some of
the corollaries.]

(a) $\Longrightarrow$ (c)

Given that A is invertible.

Need to show that the system

$A\bar{x} = \bar{0}$ has only the

trivial soln.

Suppose $\bar{y}$ is any soln. of the
homog. system.

$\therefore \quad A\bar{y} = \bar{0}$

Multiply on left by A^{-1}

$\therefore \quad (A^{-1}A)\bar{y} = A^{-1}\bar{0}$

or $\quad I\bar{y} = \bar{0}$

or $\quad \bar{y} = \bar{0}$.

(c) $\Longrightarrow$ (a) — NB: this is actually Prop. 3 (but we will now give a proof)

~~Suppose~~

Suppose $A\bar{x} = \bar{0}$ has only the trivial soln. $\bar{x} = \bar{0}$

If R is the RREF matrix of A, then R has no free variables.

$\therefore$ all variables of R are basic variables.

Since no. of variables = no. of columns

= no. of rows (since A is square)

there must be a basic variable in each row and in each column.

$\therefore$ R = I, as required.

(h) $\Rightarrow$ (a)

Given A is row-equivalent to I, need to show A is invertible.

$\therefore$ there are elementary row operations ~~not~~ such that

$$e_1(e_2(e_3)(\cdots\cdots(e_n A) = I$$

$$\therefore \quad (E_1 E_2 \cdots E_n)A = I$$

Put $B = E_1 E_2 \cdots E_n$

Then, since each E_i is invertible, so is B.

$$\therefore \quad BA = I$$

$$(B^{-1}B)A = B^{-1}I$$

$$A = B^{-1}$$

So A being the inverse of an invertible matrix is itself invertible.