MTH100 MATHS I Monsoon Semester – 2016-17 ①

Tutorial exercise for the week of Monday 29th August 2016.

1. Prove Proposition 7: Let V be a vector space. Then:
   a) The zero vector is unique.
   b) The additive inverse vector of any vector **u** is unique; we use the notation **−u** for the inverse vector
   c) $0\mathbf{u} = \mathbf{0}$ for every vector **u**
   d) $c\mathbf{0} = \mathbf{0}$ for every scalar c
   e) $-\mathbf{u} = (-1)\mathbf{u}$ for every vector **u**

2. Given any vector space V, show that if $c\mathbf{v} = \mathbf{0}$, where **v** is a non-zero vector, then the scalar $c = 0$.

3. a) Show that every vector space V satisfies the (additive) **cancellation law**, i.e. show that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{v} = \mathbf{w}$.
   b) Give an example of a set X and an operation involving elements of X, which does not satisfy the cancellation law. Briefly justify your answer.

4. In the following is W a subspace of V? Base field is taken as R in all. Justify your answer.
   a. $V = R[t]$ = vector space of all polynomials with real coefficients, W = set of all polynomials with integer coefficients.
   b. $V = R^2$, $W = \{(x,y): x + y \geq 0\}$.
   c. $V = R^2$, $W = \{(x,y): x^2 + y^2 \geq 0\}$.

5. Consider the space V of all 2×2 matrices over R, i.e. $V = R^{2\times 2}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.
   - All upper triangular matrices (i.e. matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$)
   - All A such that AB = BA where B is some fixed matrix in V
   - All A such that BA = 0 where B is some fixed matrix in V
   - Would the above results hold for all n×n matrices where n is a general positive integer ($n \geq 2$) ?

SOLUTIONS FOLLOW

1. ~~Pf~~ To prove Proposition 7.

(a) Suppose $\bar{u}$ and $\bar{v}$ are two zero vectors.

Then $\bar{u} + \bar{v} = \bar{u}$, since $\bar{v}$ is a zero vector. ①

Also $\bar{u} + \bar{v} = \bar{v}$, since $\bar{u}$ is a zero vector ②

From ① and ②, $\bar{u} = \bar{v}$

(b) Suppose $\bar{u}_1$ and $\bar{u}_2$ are two additive inverses for $\bar{u}$.

Then $\bar{u} + \bar{u}_1 = \bar{0}$

Adding $\bar{u}_2$ to both sides,

$$\bar{u}_2 + (\bar{u} + \bar{u}_1) = \cancel{\bar{0}+\bar{u}_2}\ \bar{u}_2 + \bar{0}$$

or $(\bar{u}_2 + \bar{u}) + \bar{u}_1 = \bar{u}_2$

or $\bar{0} + \bar{u}_1 = \bar{u}_2$

or $\bar{u}_1 = \bar{u}_2$

(c) $0\bar{u} = \bar{0}$ for all vectors $\bar{u}$

Ans: $0\bar{u} = (0+0)\bar{u} = 0\bar{u} + 0\bar{u}$

Let ~~X~~ $\bar{v}$ be the additive inverse of $0\bar{u}$, and add $\bar{v}$ to both sides.

Then $\bar{v} + 0\bar{u} = (\bar{v} + 0\bar{u}) + 0\bar{u}$

or $\bar{0} = \bar{0} + 0\bar{u} = 0\bar{u}$

[We use the fact that adding the additive inverse gives the zero vector.]

(PTO)

(d) $c\bar{0} = \bar{0}$ for every scalar c.

We have $c\bar{0} = c(\bar{0}+\bar{0}) = c\bar{0} + c\bar{0}$

As in the previous case (proof of ~~…~~ (c)), we add the additive inverse, say $\bar{v}$, of $c\bar{0}$ to both sides:

$$\bar{v} + c\bar{0} = (\bar{v} + c\bar{0}) + c\bar{0}$$

or $$\bar{0} = \bar{0} + c\bar{0}$$

or $$\bar{0} = c\bar{0}$$

(e). $-\bar{u} = (-1)\,\bar{u}$ for all $\bar{u}$

Proof: We have: $\bar{0} = 0\bar{u} = [1+(-1)]\bar{u}$

$$= 1\,\bar{u} + (-1)\,\bar{u} = \bar{u} + (-1)\,\bar{u}$$

Now, add $-\bar{u}$ to both sides:

$$-\bar{u} + \bar{0} = (-\bar{u} + \bar{u}) + (-1)\,\bar{u}$$

or $$-\bar{u} = (-1)\,\bar{u}$$

[NB: in the above, we have only used the properties of a vector space. After doing ~~Q 6a~~ Q 6a – the Cancellation Law – these proofs become shorter.]

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SOLUTIONS

2\. Given $c\bar{v} = \bar{0}$, suppose by way of contradiction that ~~where~~ c is a non-zero scalar. To show that $\bar{v} = \bar{0}$.

Since scalars belong to a field, there is universal division (except for the zero scalar)

Since $c \neq 0$, we can divide by c, or to be more technically correct, multiply by c^{-1}.

$\therefore$ we get $c^{-1}(c\bar{v}) = c^{-1}\bar{0}$

or $(c^{-1}c)\bar{v} = \bar{0}$ (Prop. 7(d))

or $1 . \bar{v} = \bar{0}$

or $\bar{v} = \bar{0}$ (applying axiom C)

This gives a contradiction, proving the result.

3\. (a) To prove the Cancellation Law:

$$\bar{u} + \bar{v} = \bar{u} + \bar{w}$$

Adding the additive inverse of $\bar{u}$ to both sides:

$$(-\bar{u} + \bar{u}) + \bar{v} = (-\bar{u} + \bar{u}) + \bar{w}$$

(also using Assoc. property)

or $\bar{0} + \bar{v} = \bar{0} + \bar{w}$

or $\bar{v} = \bar{w}$, as required.

(b) The standard example in the set $\mathbb{R}^{2\times 2}$ with the multiplication operation

For example: $\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \underset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$

But $A \neq C$

Q4. [redacted] (a) $V = \mathbb{R}[t]$

W = set of all polynomials with integer coefficients.

Ans: W is not a subspace.

Note that W satisfies additive closure, zero vector property, etc.

However, it does not satisfy scalar multiplication closure.

e.g. $p(x) = 1 + x$ [redacted] $\in W$

but $\frac{1}{2} p(x) = \frac{1}{2} + \frac{1}{2} x \notin W$

(b) $V = \mathbb{R}^2$, $W = \{(x, y) : x + y \geq 0\}$

Ans: W is not a subspace.

Again fails to satisfy scalar multiplication closure, & also additive inverse property.

e.g. $\bar{u} = (1, 1) \in W$ but $-\bar{u} = (-1, -1) \notin W$

and $c\bar{u}$ if $c = -2$ becomes $(-2, -2) \notin W$

(c) $V = \mathbb{R}^2$, $W = \{(x, y) : x^2 + y^2 \geq 0\}$

Ans: YES.

$W = V$, which is obviously a subspace of itself.

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Q 5. 6

Are the following sets of matrices in $V = \mathbb{R}^{2\times 2}$ subspaces? Justify.

(a) All upper Δular matrices,

i.e. $W = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} ; \ a, b, d \in \mathbb{R} \right\}$

Answer: YES.

1. Clearly $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

2. If $A = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ are in W, so is $A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & d_1+d_2 \end{bmatrix}$

3. If $c \in \mathbb{R}$, then $cA = \begin{bmatrix} ca_1 & cb_1 \\ 0 & cd_1 \end{bmatrix} \in W$ also

(b) All A s.t. $AB = BA$ where B is fixed.

Answer: Yes. Put $W = \{ A \in \mathbb{R}^{2\times 2} ; \ AB = BA, \ B \text{ fixed} \}$

1. $[O].B = [O] = [O].B$ where $[O]$ is the zero matrix of all zeroes.

2. Suppose $A_1, A_2 \in W$.

Then $(A_1 + A_2) B = A_1 B + A_2 B$

$= B A_1 + B A_2 = B(A_1 + A_2)$ —

closure under addition.

3. If $c \in \mathbb{R}$, then

$(cA_1) B = c(A_1 B) = c(BA_1) = B(cA_1)$,

so $cA_1 \in W$

(PTO)

~~Q2~~ Q5 —

[redacted] continued

[redacted]

(c) All A such that $BA = 0$ where B is fixed.

~~Answer~~ : YES. Put $W = \{ A \in \mathbb{R}^{2\times 2} : BA = 0,\ B \text{ fixed} \}$

1. ~~Clearly~~ $B[0] = [0]$, i.e. $[0] \in W$.

2. If $A_1, A_2 \in W$, then
$$B(A_1 + A_2) = BA_1 + BA_2 = [0] + [0] = [0].$$

3. If $c \in \mathbb{R}$, then $B(cA_1) = c(BA_1) = c[0] = [0]$.

(d) Would the above results hold for $n\times n$ - matrices in general

Ans : YES. We did not use the fact that $n = 2$ in any of the above, simply general properties of matrix addition and multiplication.