Three Important Subspaces – Example

$$A = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 6 & 4 & -6 \\ 3 & 9 & 7 & -11 \\ 8 & 24 & 9 & -10 \end{bmatrix} = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 \end{bmatrix}.$$

$$R = \begin{array}{c} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & & \uparrow & \\ \text{Basic V} & & \text{Basic V} & \end{array} \end{array} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{0} \\ \bar{0} \end{bmatrix}$$

$\therefore$ Basis of Nul A $= \{\bar{u}_1, \bar{u}_2\}$

where $\bar{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

Basis of Col A $= \{\bar{v}_1, \bar{v}_3\}$ $\rightarrow$ cols. of A corresponding to basic variables

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Basis of Row A $= \{\bar{r}_1, \bar{r}_2\}$ $\rightarrow$ non-zero rows of R

Computations for Example

$-19-8(-3)$
$-10-8(-3)$
$= 14$

$$A = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 6 & 4 & -6 \\ 3 & 9 & 7 & -11 \\ 8 & 24 & 9 & -10 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 8R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -7 & 14 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & +14 \end{bmatrix} \xrightarrow[R_4 + 7R_2]{R_4 \to} \begin{bmatrix} 1 & 3 & 2 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Hence, the corresponding system $R\bar{x} = \bar{0}$ is

$$\begin{aligned} x_1 &= -3x_2 + (-1)x_4 \\ x_2 &= x_2 \\ x_3 &= + 2x_4 \\ x_4 &= x_4 \end{aligned}$$

or $$\bar{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = x_2 \bar{u}_1 + x_4 \bar{u}_2$$

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check: $$A\bar{u}_1 = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 6 & 4 & -6 \\ 3 & 9 & 7 & -11 \\ 8 & 24 & 9 & -10 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-3 \\ 6-6 \\ -9+9 \\ -24+24 \end{bmatrix} = \bar{0}$$

$$A\bar{u}_2 = \begin{bmatrix} 1 & 3 & 2 & -3 \\ 2 & 6 & 4 & -6 \\ 3 & 9 & 7 & -11 \\ 8 & 24 & 9 & -10 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+4-3 \\ -2+8-6 \\ -3+14-11 \\ -8+18-10 \end{bmatrix} = \bar{0}$$