2016-

MTH100 - Monsoon ~~2014-15~~

Notes for ~~Tuesday 28/10/2014~~ (1)

Example for eigenvalues and eigenvectors

Let $A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

Let $\bar{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Then $A\bar{v} = \begin{bmatrix} 20 \\ -11 \\ 38 \end{bmatrix}$

So $A\bar{v} \neq c\bar{v}$ for any scalar c, and hence $\bar{v}$ is not an eigenvector.

OTOH, let $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$,

$\bar{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Then:

$$\left.\begin{aligned} A\bar{v}_1 &= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = 1.\bar{v}_1 \\ A\bar{v}_2 &= \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} = 1.\bar{v}_2 \end{aligned}\right\}$$

Hence, we see that $\bar{v}_1$ and $\bar{v}_2$ are both eigenvectors of A corresponding to the eigenvalue $\lambda_1 = 1$.

Similarly, $A\bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 . \bar{v}_3$

$\therefore$ $\bar{v}_3$ ~~is~~ an eigenvector corresp. ~~to the~~ eigenvalue $\lambda_2 = 0$ (0 is allowed to be an eigenvalue).

Put $\bar{v}_4 = \bar{v}_1 + \bar{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}$

Then $A\bar{v}_4 = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}$,

again an eigen~~value~~ vector for λ_1.

→ So if we manage to find one eigenvector for an eigenvalue, we can find more by taking sums and scalar multiples.

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Proof of Prop. ~~3~~:

Suppose BWOC that $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p$ are linearly dependent. Let m be the smallest no. s.t. $\bar{v}_1, \dots, \bar{v}_m$ are lin. indep. but $\bar{v}_{m+1}$ is a lin. comb. of the preceding vector ??? (at the worst, m would be $p-1$).

$$\therefore \quad c_1\bar{v}_1 + \cdots + c_m\bar{v}_m = \bar{v}_{m+1} \qquad (1)$$

Apply A on the LHS and RHS of (1), we get:-

$$c_1A\bar{v}_1 + c_2A\bar{v}_2 + \cdots + c_m A\bar{v}_m = A\bar{v}_{m+1},$$

and using the fact that the $\bar{v}_i$ are eigenvectors, we get:

$$c_1\lambda_1\bar{v}_1 + c_2\lambda_2\bar{v}_2 + \cdots + c_m\lambda_m\bar{v}_m = \lambda_{m+1}\bar{v}_{m+1} \qquad (2)$$

Now, multiplying (1) by λ_{m+1}, we get

$$c_1\lambda_{m+1}\bar{v}_1 + c_2\lambda_{m+1}\bar{v}_2 + \cdots + c_m\lambda_{m+1}\bar{v}_m = \lambda_{m+1}\bar{v}_{m+1} \qquad (3)$$

(PTO)

Subtracting (3) from (2), we get:

$$c_1(\lambda_1-\lambda_{m+1})\bar{v}_1 + \cdots + c_m(\lambda_m-\lambda_{m+1})\bar{v}_m = \bar{0} \qquad (4)$$

But since $\bar{v}_1, \dots, \bar{v}_m$ are lin. indep., all the co-effs. in (4) must be 0,

i.e. $c_1(\lambda_1-\lambda_{m+1}) = 0$

But $\lambda_1 \neq \lambda_{m+1}$ (since these are *distinct* eigenvalues),

so $c_1 = 0$.

Similarly, $c_2 = c_3 = \cdots = c_m = 0$.

But then from (1), $\bar{v}_{m+1} = \bar{0}$, which is a contradiction, since all the $\bar{v}_i$ are *eigenvectors*.

Hence, our initial hypothesis is wrong and so $\bar{v}_1, \dots, \bar{v}_p$ are lin. indep.

MTH100

Notes for WED-THU 20161026-27

→ How to check whether a given no. λ is indeed an eigenvalue for a given matrix A?

Ans: The equation $A\bar{x} = \lambda \bar{x}$

or $(A - \lambda I)\bar{x} = \bar{0}$ should

have a non-trivial solution $\bar{x}$.

Example:- $A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

(We saw yesterday that 1 and 0 are eigenvalues).

How about $\lambda = 3$?

Lets try:-

$$A - \lambda I = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 1 \\ 6 & 4 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 + 3R_1}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -2 \\ 0 & -8 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 8R_2]{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -14 \end{bmatrix}$$

we omit remaining steps, but clearly

$A - \lambda I$ is row-equivalent to I_3, hence

only trivial solution. $\therefore$ 3 is not an eigenvalue

Example of a real matrix with no real eigenvalues :-

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2 + 1 = 1 - 2\lambda + \lambda^2 + 1$$

$$= 2 - 2\lambda + \lambda^2$$

Solution: $$\lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

only complex eigenvalues

~~Tuesday~~ 27/10

Notes for ~~Thursday~~ ~~29/10~~ ⑤

(Not done in class - notes only)

Proof of Prop. ~~[illegible]~~ 41

[$\Longrightarrow$] Suppose λ is an eigenvalue of A.

Then, there is a <u>non-zero</u> vector $\bar{v}$ s.t. $A\bar{v} = \lambda\bar{v}$ ($\bar{v}$ is an eigenvector)

i.e. $(A-\lambda I)\bar{v} = \bar{0}$

$\therefore$ the homog. system $(A-\lambda I)\bar{x} = \bar{0}$ has a non-zero solution,

$\therefore$ $(A-\lambda I)$ is not invertible

$\therefore$ $\det(A-\lambda I) = 0$

[$\Longleftarrow$] Conversely, suppose λ is a root of the characteristic eqn.

~~det~~

$\therefore$ $\det(A-\lambda I) = 0$

$\therefore$ the matrix $A-\lambda I$ is not invertible.

$\therefore$ by ~~[illegible]~~ VIT, the system $(A-\lambda I)\bar{x} = \bar{0}$ has a non-zero solution $\bar{v}$.

Since $(A-\lambda I)\bar{v} = \bar{0}$,

$A\bar{v} = \lambda\bar{v}$, and $\bar{v}$ is an eigenvector corresp. to λ ~~eigenvector~~.

Proof of Prop. ~~[scribble]~~ 42 :-

Suppose B is similar to A,
i.e. $B = PAP^{-1}$ for some invertible P.

$\therefore$ char. poly. of $B = \det(B - \lambda I)$

$$= \det(PAP^{-1} - \lambda I)$$

$$= \det(PAP^{-1} - P(\lambda I)P^{-1})$$

$$= \det(P(A - \lambda I)P^{-1})$$

$$= \det P \det(A - \lambda I) \det(P^{-1})$$

$$= \det P \det(A - \lambda I)(\det P)^{-1}$$

$$= \det(A - \lambda I).$$

Remark: However, note that this is only a nec. condition! We can find matrices A and B s.t.

char. poly. of A = char. poly. of B

but B is not similar to A !

[Exercise]