MTH100

→ Notes on Span of a set of vectors.

Let $S = \{\bar{u}, \bar{v}\}$ where

$$\bar{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ and } \bar{v} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Note that $\bar{u} \in \text{Span } S$, $\bar{v} \in \text{Span } S$,

$S \subseteq \text{Span } S$.

$$\bar{u} + \bar{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \in \text{Span } S$$

$$2\bar{u} + (-1)\bar{v} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \in \text{Span } S$$

Clearly, while S is finite, Span S is infinite (assuming the field F is infinite).

By the way, $\bar{0} = 0\,\bar{u} + 0\,\bar{v}$

$\in \text{Span } S$

→ Constructing vectors in Span S is easy.

What about the reverse question: given a vector $\bar{w}$, does $\bar{w} \in \text{Span}\, S$.

If $\bar{w} \in \text{Span}\, S$, then $\bar{w} = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \cdots + c_p \bar{v}_p$ for some scalars c_i.

So we have to solve a linear system!

As before, let $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$.

Put $\bar{w} = \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix}$ → solve $c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix}$

$$\left[\begin{array}{cc:c} 1 & 1 & 3 \\ 3 & 1 & 2 \\ 2 & 4 & 13 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & -2 & -7 \\ 0 & 2 & 7 \end{array}\right] \xrightarrow[R_3 + R_2]{R_3 \to}$$

$$\left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[\frac{1}{2} R_2]{R_2 \to} \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - R_2]{R_1 \to} \left[\begin{array}{cc:c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 \end{array}\right]$$

So, YES → $\bar{w} = \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

OTOH, consider $\bar{w}_1 = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$.

Then, $\left[\begin{array}{cc:c} 1 & 1 & -3 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc:c} 1 & 1 & -3 \\ 0 & -2 & 7 \\ 0 & 2 & 13 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2}$

$\left[\begin{array}{cc:c} 1 & 1 & -3 \\ 0 & -2 & 7 \\ 0 & 0 & 20 \end{array}\right]$ → inconsistent! So $\bar{w}_1 \notin \text{Span}\, S$