MTH100 - MONSOON SEMESTER 2016 – CLASS TEST – 22/08/2016

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TIME: 1 HOUR   MAXIMUM MARKS: 50   INSTRUCTIONS: Attempt all questions. All questions carry equal marks. Clearly identify any standard result or known theorem used in a proof.

For questions 1, 2, and 3, use the matrices A and B given below. You will not be awarded marks unless your calculations are shown in full.

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 4 & 13 \\ 2 & 5 & 17 \\ 3 & 6 & 21 \end{bmatrix}$$

1. Is A row-equivalent to B? Justify your answer with reference to a suitable result.
2. Given the non-homogeneous matrix equation $B\mathbf{x} = \mathbf{c}$, where $\mathbf{c}$ is the vector $[19 \quad 30 \quad 33]$ (regard $\mathbf{c}$ as a column vector), determine whether the system is consistent. If yes, find the general solution to the system in vector form. In case the system has more than one solution, exhibit three distinct solutions.
3. Is the matrix A invertible ? If yes, determine its inverse using row reduction.
4. a) Show that every invertible matrix A is a product of elementary matrices. *(6 marks)*

   b) Further, show that any sequence of row operations that reduces A to I also transforms I into A^{-1}. *(4 marks)*

5. Consider the relation $\approx$ on Z, i.e. the set of all integers, given by $m \approx n$ if $m^2 = n^2$.

   a) Prove or disprove: $\approx$ is an equivalence relation. *(5 marks)*

   b) In case it is an equivalence relation, describe the equivalence classes. Show at least two distinct equivalence classes. How many equivalence classes are there ? *(5 marks)*

——— X ——— X ——— X ———

SOLUTIONS FOLLOW

(NOT NECESSARILY IN SAME ORDER)

Q2 Given the non-homogeneous system $B\bar{x} = \bar{c}$. To answer all the questions, we row-reduce the augmented matrix $[B : \bar{c}]$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 13 & 19 \\ 2 & 5 & 17 & 30 \\ 3 & 6 & 21 & 33 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 4 & 13 & 19 \\ 0 & -3 & -9 & -8 \\ 0 & -6 & -18 & -24 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc:c} 1 & 4 & 13 & 19 \\ 0 & 1 & 3 & \frac{8}{3} \\ 0 & -6 & -18 & -24 \end{array}\right] \xrightarrow{R_3 \to R_3 + 6R_2} \left[\begin{array}{ccc:c} 1 & 4 & 13 & 19 \\ 0 & 1 & 3 & \frac{8}{3} \\ 0 & 0 & 0 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{8}(R_3)} \left[\begin{array}{ccc:c} 1 & 4 & 13 & 19 \\ 0 & 1 & 3 & \frac{8}{3} \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - \frac{8}{3}R_3]{R_1 \to R_1 - 19R_3} \left[\begin{array}{ccc:c} 1 & 4 & 13 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 4R_2} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \text{RREF matrix of augmented matrix}$$

→ The system is not consistent, since the ~~has a~~ RREF matrix of the augmented matrix has a row of the form $[0 \; 0 \; 0 \; b]$ with $b \neq 0$.

NB: In the above, we ~~had~~ have ~~the~~ obtained the RREF matrix of the original co-efficient matrix, B. It is:

$$R_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Q3. We proceed to row-reduce the enlarged matrix of A, i.e. $[A : I] = \left[\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 3 & 3 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2} R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & +4 & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow[\text{further steps}]{\cancel{\text{further}}}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right]$$

The above calculation show that:

A is invertible,

$$A^{-1} = \begin{bmatrix} +\frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{bmatrix}$$

and the RREF matrix of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

Q 3. While doing the calculations for ~~Q3~~ Q 2 and Q 3, we found that the RREF matrices for A and B are:

$$R_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$R_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $R_A \neq R_B$, A and B are not row-equivalent.

Suitable ~~to~~ result is ~~attached~~ shown on next page.

RESULT FOR Q 3

Row Equivalence - 2

- **Remark 2**: In fact, the RREF matrix of any given matrix is unique, i.e. a matrix cannot be row-equivalent to two distinct RREF matrices. Alternatively, two distinct RREF matrices cannot be row-equivalent to each other.
- We shall see a proof of this later.
- **Concluding Remark**: So, inside each equivalence class for this equivalence relation, there is a distinctive member, i.e. the one and only RREF matrix in it. This fact can be used to determine whether two matrices are row-equivalent to each other.

Q4.

Calculation of the Inverse Matrix - I

In order to calculate the inverse of a matrix, we use the following result:

- **Corollary 1.1:** An invertible matrix A is a product of elementary matrices. Any sequence of row operations that reduces A to I also transforms I into A^{-1}.
- NB: We are implicitly using Theorem 1(b) here.

Proof of Corollary 1.1

- **Proof:** If A is invertible, then by VIT, A is row equivalent to the identity, i.e. $I = (e_p\, e_{p-1} \ldots . e_1)A$ for some sequence of elementary row operations. If E_1 to E_p are the corresponding elementary matrices, then $I = (E_p \ldots . E_1)A$. Each E_i being invertible, we can write $A = (E_p \ldots . E_1)^{-1} I = E_1^{-1} \ldots . E_p^{-1}$

(a)

Hence A is a product of elementary matrices. Furthermore, $A^{-1} = (E_1^{-1} \ldots . E_p^{-1})^{-1}$

$$= (E_p \ldots . E_1) = (E_p \ldots . E_1)I = (e_p\, e_{p-1} \ldots . e_1)I$$

In other words, the same sequence of row operations that reduces A to I also reduces I to A^{-1}.

(b)

Q5. Given the relation $\sim$ on $\mathbb{Z}$ where $m \sim n$ if $m^2 = n^2$.

(a) $\sim$ ~~is~~ an equivalence relation.

(i) Reflexive property: For any $m \in \mathbb{Z}$, $m^2 = m^2$, so $m \sim m$.

(ii) Symmetric property: Suppose $m \sim n$. Then $m^2 = n^2$, so $n^2 = m^2$ and $n \sim m$ also holds.

(iii) Transitive property: Suppose $m \sim n$ and $n \sim k$. Then $m^2 = n^2$ and $n^2 = k^2$, i.e. $m^2 = k^2$, and so $m \sim k$ as required.

(b) If m is a positive integer, then $m^2 = (-m)^2$, and so the equivalence class of m, ~~is~~

$$[m] = \{m, -m\} \quad (1)$$

The equivalence class of $\boxed{0}$ 0, i.e.

$$[0] = \{0\}. \quad (2)$$

So all equivalence classes have two ~~ele~~ members as in (1) with only $[0]$ having a single member. We can easily select two distinct equivalence classes, for example,

$$[1] = \{1, -1\}, \quad [2] = \{2, -2\}, \quad [0] = \{0\}.$$

There are infinitely many equivalence classes, in fact, one for each member of $\mathbb{N}$ = set of natural numbers $= \{0, 1, 2, \ldots\}$.

This is often stated as: countably infinite.