Example for Matrix of a linear transformation:

Consider $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$T(x,y,z) = (x+y+z,\ x+2y+3z)$

We will take the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

So: $T\bar{e}_1 = T(1,0,0) = (1,1) = 1\bar{e}_1 + 1e_2$

$T\bar{e}_2 = T(0,1,0) = (1,2) = 1\bar{e}_1 + 2\bar{e}_2$

$T\bar{e}_3 = T(0,0,1) = (1,3) = 1\bar{e}_1 + 3\bar{e}_2$

$\therefore$ the matrix of T, say

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Take any vector in $\mathbb{R}^3$, say $\bar{u} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_{S_3}$

Then $A\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_{S_3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_{S_3}$

$= \begin{bmatrix} 15 \\ 32 \end{bmatrix}_{S_2}$

$= T\bar{u}$ (expressed in the standard basis S_2 for $\mathbb{R}^2$)

If we take any general vector in $\mathbb{R}^3$,

say $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Then $A\bar{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \begin{bmatrix} x + y + z \\ x + 2y + 3z \end{bmatrix}$$

$= T\bar{x}$ expressed as a coordinate vector with regard to the S_2-basis.

Another example: Consider the differentiation transformation

$$D : \mathbb{R}_3[t] \longrightarrow \mathbb{R}_2[t]$$

We will use the ordered basis

$$B = \{1, t, t^2, t^3\} \quad \text{for } \mathbb{R}_3[t]$$

and the ordered basis $C = \{1, t, t^2\}$ for $\mathbb{R}_2[t]$.

NB: D is certainly a linear transformation, since

$$D\,(p(t) + q(t)) = D\,p(t) + D\,q(t)$$

and

$$D\,(c\,p(t)) = c\,D\,p(t).$$

Now:

$$\begin{aligned}
D(1) &= 0.1 + 0.t + 0.t^2 \\
D(t) &= 1.1 + 0.t + 0.t^2 \\
D(t^2) &= \cancel{2t}\ 0.1 + 2.t + 0.t^2 \\
D(t^3) &= \cancel{3t^2}\ 0.1 + 0.t + 3t^2
\end{aligned}$$

$\therefore$ the matrix of D is :-

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \rightarrow 3 \times 4 \text{ (as expected)}$$

(PTO)

Suppose we want to find $D\,p(t)$ for any polynomial $p(t) \in \mathbb{R}_3[t]$,

e.g. $p(t) = 10 + 5t + 3t^2 - 7t^3$

We see that $[p(t)]_B = \begin{bmatrix} 10 \\ 5 \\ 3 \\ -7 \end{bmatrix}_B$

$$\therefore \; A\,[p(t)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 3 \\ -7 \end{bmatrix}_B$$

$$= \begin{bmatrix} 5 \\ 6 \\ -21 \end{bmatrix}_C$$

or $D(p(t)) = 5 + 6t - 21t^2$ ✓

In general, if $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$,

$$[p(t)]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_B$$

and $A\,[p(t)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_B$

$$= \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}_C = a_1 + 2a_2 t + 3a_3 t^2$$