

Proof of Prop. 29 :-

Let $\bar{x} \in V$; since B is a basis for B , we can write $\bar{x} = b_1 \bar{u}_1 + \dots + b_n \bar{u}_n$ (1)

Then $[\bar{x}]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ (2)

Since C is also a basis for V , we can write:

$$\left. \begin{aligned} \bar{u}_1 &= A_{11} \bar{v}_1 + A_{21} \bar{v}_2 + \dots + A_{n1} \bar{v}_n \\ &\vdots \\ \bar{u}_n &= A_{1n} \bar{v}_1 + A_{2n} \bar{v}_2 + \dots + A_{nn} \bar{v}_n \end{aligned} \right\} \quad (3)$$

From (3), we get that

$$[\bar{u}_i]_C = \begin{bmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{bmatrix} \text{ for } i = 1, 2, \dots, n \quad (4)$$

Substituting from (3) in (1), we get:-

$$\bar{x} = b_1 (A_{11} \bar{v}_1 + \dots + A_{n1} \bar{v}_n) + b_2 (A_{12} \bar{v}_1 + \dots + A_{n2} \bar{v}_n) + \dots + b_n (A_{1n} \bar{v}_1 + \dots + A_{nn} \bar{v}_n) \quad (5)$$

Re-arranging and collecting the coefficients of $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$, we get:

$$\bar{x} = (A_{11} b_1 + A_{12} b_2 + \dots + A_{1n} b_n) \bar{v}_1 + (A_{21} b_1 + A_{22} b_2 + \dots + A_{2n} b_n) \bar{v}_2 + \dots + (A_{n1} b_1 + A_{n2} b_2 + \dots + A_{nn} b_n) \bar{v}_n \quad (6)$$

(2)

Proof of Prop. 29 - cont'd.

$$\begin{aligned} \text{Hence, } [\bar{x}]_C &= \begin{bmatrix} A_{11}b_1 + A_{12}b_2 + \dots + A_{1n}b_n \\ A_{21}b_1 + A_{22}b_2 + \dots + A_{2n}b_n \\ \vdots \\ A_{n1}b_1 + A_{n2}b_2 + \dots + A_{nn}b_n \end{bmatrix} \\ &= \begin{bmatrix} A_{ij} \end{bmatrix}_{n \times n} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = P [\bar{x}]_B \quad (7) \end{aligned}$$

As noted already ~~is~~ from (4), the columns of ~~the~~ the matrix P are nothing but the coordinate vectors of the old ~~new~~ basis, i.e., the \bar{b}_i 's, in terms of the new basis, i.e., the \bar{e}_j 's.

Finally, we note that P must be invertible for the following reason :-

the coordinate mapping is an isomorphism from V to F^n . Since B is a basis of V , it goes to a basis of F^n under the coordinate mapping with regard to basis C (Prop. 27 (a)). Since the columns of P ~~are~~ form a basis of F^n , ~~the~~ P is invertible by VIT (g).

Proof of Prop. 30: Similarity is
an equivalence relation on $F^{n \times n}$.

3

(i) Reflexive: If $A \in F^{n \times n}$,
then $A = I A I^{-1}$,
so A is similar to A .

(ii) Symmetric: Suppose B is
similar to A .

Then, $\exists P$ s.t. $B = P A P^{-1}$.

Put $Q = P^{-1}$

$$\therefore Q B Q^{-1} = P^{-1} (P A P^{-1}) (P^{-1})^{-1} \\ = A$$

$\therefore A$ is similar to B .

(iii) Transitive: Suppose B is similar
to A and C is similar to B .

Then $B = P A P^{-1}$

and $C = Q B Q^{-1}$

$$\text{i.e. } C = Q (P A P^{-1}) Q^{-1} \\ = (Q P) A (P^{-1} Q^{-1}) \\ = (Q P) A (Q P)^{-1}$$

Utility of idea of similarity
in matrix computations:-

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Example for Change of Basis

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We consider the old basis $\alpha = \{ \bar{e}_1, \bar{e}_2 \}$

The new basis $\beta = \{ \bar{u}_1, \bar{u}_2 \}$

where $\bar{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Construct the matrix Q which has the vectors of β as its columns.

$$\therefore Q = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

↑ ↑
new basis in terms of old

Then, the change of basis matrix

$$P_{\alpha \rightarrow \beta} = Q^{-1}$$

$$\therefore P = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Let us check with a specific vector, say $\bar{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha$

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$$\text{Then, } [\bar{u}]_\beta = P \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}_{\cancel{\alpha}}$$

$$= \begin{bmatrix} -26 \\ 11 \end{bmatrix}_\beta$$

$$\text{Check: } -26 \bar{u}_1 + 11 \bar{u}_2 = -26 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 11 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \quad \checkmark \quad \underline{\text{Confirmed}}$$

Verification of the remark that columns of P are the coordinate vectors of old basis in terms of new basis.

$$\text{Now } \begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta = 3 \bar{u}_1 + (-1) \bar{u}_2 = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \bar{e}_1, \text{ i.e. } [\bar{e}_1]_\beta = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta$$

$$\text{and } \begin{bmatrix} -5 \\ 2 \end{bmatrix}_\beta = (-5) \bar{u}_1 + 2(\bar{u}_2) = (-5) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \bar{e}_2, \text{ i.e. } [\bar{e}_2]_\beta = \begin{bmatrix} -5 \\ 2 \end{bmatrix}_\beta$$

Now, let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation, given by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}$ (7)

let us first determine $A = [T]_{\alpha} =$ matrix of T relative to standard basis.

$$\text{Now } T\bar{e}_1 = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\bar{e}_1 + 3\bar{e}_2$$

$$\text{and } T\bar{e}_2 = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\bar{e}_1 + 4\bar{e}_2$$

$$\therefore [T]_{\alpha} = A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Applying Prop. 31, the matrix
relative to the new basis β
would be $B = P A P^{-1}$, where
 $P = P_{\alpha \rightarrow \beta}$, the change of
basis matrix, i.e.

$$B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}$$

let us verify our calculations with
an example, say $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\alpha} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_{\beta}$

$$[\text{since } [\bar{u}]_{\beta} = P_{\alpha \rightarrow \beta} [\bar{u}]_{\alpha} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\alpha} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_{\beta}]$$

$$\therefore [T\bar{u}]_{\alpha} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\alpha} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\alpha} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_{\alpha}$$

$$\text{and } [T\bar{u}]_{\beta} = \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}_{\beta} \begin{bmatrix} -7 \\ 3 \end{bmatrix}_{\beta} = \begin{bmatrix} -40 \\ 17 \end{bmatrix}_{\beta}$$

$$\text{But } \begin{bmatrix} -40 \\ 17 \end{bmatrix}_{\beta} = -40\bar{u}_1 + 17\bar{u}_2$$

$$= -40 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 17 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_{\alpha}$$

✓

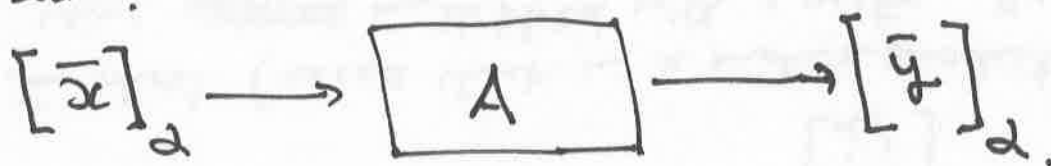
The Idea behind Prop. 31

9

Think of a matrix as a system.

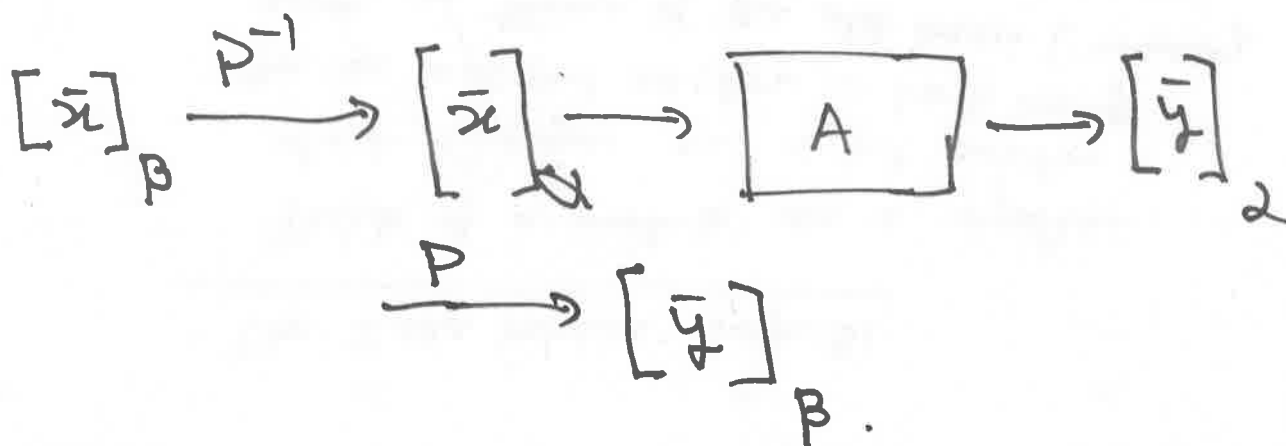
The input is a vector \vec{x} given as \mathbb{R}^n coordinate vector with regard to a basis α , and the ~~out~~ output is again a vector \vec{y} , also given in terms of α .

Diagram:



However, now suppose that the input is given as a coordinate with regard to basis β , and the output is also desired in this form.

So, we have to proceed as follows (Recall that if the change of basis matrix from α to β is P , the change of basis matrix ~~is~~ from β to α is P^{-1}):-



(PTO)

Prop. 31 (cont'd) :-

(10)

We now express the above system diagram in matrix terms.

Recall that when a product of matrices is to ~~be~~ operate (multiply) ~~on~~ a vector, ~~the~~ we proceed from right to left.

$$\text{So } [T]_{\beta} [\bar{x}]_{\beta} = \cancel{I} (PAP^{-1}) [\bar{x}]_{\beta},$$

$$\text{i.e. } [T]_{\beta} = B = PAP^{-1}$$

About Prop. 32

(11)

$$W^V = \{ f: V \rightarrow W \}$$

→ why this notation?

$|X| = n$ elements

$|Y| = m$ elements
in it

Then, what is the no. of
fun. from X to Y ?

$$= m^n = |Y|^{|X|}$$

$$|Y^X| = |Y|^{|X|} \quad \text{when}$$

X and Y are
finite

[NB: In our case, W and V are both infinite,
considered as sets. But this notation has been adopted
for the set of functions from V to W .]