1. Reduce the following matrix to an RREF matrix using elementary row operations:

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 1 \end{bmatrix}$$

2. Reduce the following matrix to an RREF matrix using elementary row operations:

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

3. Explicitly describe all non-zero 2×2 RREF matrices. You may also try to do this for 2×3 and 3×3 RREF matrices.

4. Define a relation T on the real number system **R** by xTy if y − x ∈ **Z**, the set of integers. Is T an equivalence relation? Justify your answer. If yes, can you find a special representative in each equivalence class, just as we could do for row-equivalence of matrices?

5. Prove that row-reduction is an equivalence relation on the set **R** $^{m \times n}$ of all m by n matrices with real entries.

6. Show that if E is an equivalence relation on a set X, then any two distinct equivalence classes must be disjoint. Also, show that every element of X has to belong to an equivalence class. NB: the equivalence class of any element a ∈ X is the set of all elements of X which are related to a, the formal definition is:

[a] = { x ∈ X: x E a, i.e. x is related to a under the relation E}

SOLUTION SET FOR TUTORIAL

WEEK COMMENCING MONDAY 08|08|2016

MTH 100 – MONSOON SEMESTER 2016

Solutions

~~Sol~~

Q 1. The RREF MATRIX IS:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q2. The RREF Matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{15}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{10}{7} \end{bmatrix}$$

Q3. For 2×2-matrices, there are only 3 forms:-

$$\begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(where x indicates any real number including 0)!

For 2×3-matrices, there are 6 forms.

As the size increases, the number of forms increases very rapidly.

(Hint: There has to be a 1 in the first row; you get various cases by putting it in 1st, 2nd, or 3rd column and proceeding.]

-NB: The zero matrix is also an RREF matrix. It has not been considered above.

Q4. YES, it is an equivalence relation. We have to verify the three properties: reflexive, symmetric, transitive.

(i) Reflexive: if $x \in \mathbb{R}$, then $x - x = 0 \in \mathbb{Z}$, hence $x T x$, as required.

(ii) Symmetric: Suppose $x T y$. Then $y - x = n \in \mathbb{Z}$. But then $x - y = -(y-x) = -n \in \mathbb{Z}$.
$\therefore$ $y T x$, as required.

(iii) Transitive: Suppose $x T y$ and $y T z$.
Then $y - x = n_1 \in \mathbb{Z}$
and $z - y = n_2 \in \mathbb{Z}$.
Hence, $z - x = (z-y) + (y-x)$
$= n_2 + n_1 \in \mathbb{Z}$
$\therefore$ $x T z$ as required.

Yes, we can find a special representative in each equivalence class.
If $x \in \mathbb{R}$, then $x T r$ for some $r \in [0, 1)$.
Specifically, $r = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer $\leq x$.
Clearly, $r \in [0, 1)$ and there cannot be two distinct real numbers from the interval $[0, 1)$ in the same equivalence class, since if $r_1, r_2 \in [0, 1)$, then $|r_1 - r_2| < 1$, i.e. the difference is not an integer.

Q5. We have to verify the three properties (reflexive, ~~trans~~ symmetric, transitive) for row-equivalence.

(i) Reflexive: if $A \in \mathbb{R}^{m \times n}$, then clearly A is row-equivalent to itself.

(ii) Symmetric: Suppose B is row-equivalent to A. Then, there are elementary row-operations e_1 to e_k such that applying them successively takes A to B, i.e.

$$A \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \longrightarrow \cdots \xrightarrow{e_k} A_k = B.$$

But as we noted earlier, for each elementary row operation e, there is a row-operation which reverses the effect of e. ~~taken~~

If we ~~do~~ denote these row-operations by $e_1^{-1}, e_2^{-1}, \ldots, e_k^{-1}$ respectively, we get:

$$B = A_k \xrightarrow{e_k^{-1}} A_{k-1} \longrightarrow \cdots \xrightarrow{} A_2 \xrightarrow{e_2^{-1}} A_1 \xrightarrow{e_1^{-1}} A.$$

$\therefore$ A is row-equivalent to B, as required.

(iii) Transitive: Suppose B is row-equivalent to A and C is row-equivalent to B. Using the same notation as part (ii), we can write:

$$A \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \longrightarrow \cdots \xrightarrow{e_k} A_k = B \qquad (1)$$

and $$B \xrightarrow{b_1} B_1 \xrightarrow{b_2} B_2 \longrightarrow \cdots \xrightarrow{b_m} B_m = C \qquad (2)$$

But, then we get a finite sequence of row operations taking A to C as follows:

$$A \xrightarrow{e_1} A_1 \longrightarrow \cdots \xrightarrow{e_k} \underbrace{A_k}_{\text{same as } B} \xrightarrow{b_1} B_1 \longrightarrow \cdots \xrightarrow{b_m} C,$$ as required

Q6. Suppose $[a]$ and $[b]$ are two distinct equivalence classes under the relation.

Suppose $[a] \cap [b]$ is non-empty, i.e. let $c \in [a] \cap [b]$.

Now, let $x \in [b]$.

By the transitive property, $c E x$.

Also, $a E c$, so by the transitive property, $a E x$, so that $x \in [a]$.

Hence $[b] \subseteq [a]$ ①

In a similar way, we can show that

$[a] \subseteq [b]$ ②

From ① and ②, it follows that $[a] = [b]$, but they were given to be distinct.

So we get a contradiction.

Hence, $[a] \cap [b] = \emptyset$, the empty set.

Clearly, for any $a \in X$, $a \in [a]$ by the reflexive property.