MTH100 MATHS I Monsoon Semester – 2016-17

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Tutorial exercise for the week of Monday 17th October 2016

1. a) Find the change of basis matrix $P_{S\to\beta}$ where S is the standard basis and β is the ordered basis given below:

   $\beta = \{ (1,1,1), (1,2,3), (1,3,6)\}$

   b) Find the coordinates of the vectors $\mathbf{v}_1 = (2,3,4)$ and $\mathbf{v}_2 = (1, -1, 2)$ with respect to the ordered basis β above.

   c) If $[\mathbf{v}]_\beta = (2,3,2)$, find $[\mathbf{v}]_S$ where S is the standard basis for R^3.

2. Determine all linear transformations T: $R^1 \to R^1$. (NB: R^1 is the vector space consisting of all 1-tuples with real entries; it is essentially the same as R , however regarded as only a vector space rather than a field.)

3. Consider the field C of complex numbers as a vector space over R .
   a) Show that the function ϕ: C $\to$ C given by $\phi(z) = \bar{z}$ is a linear transformation. Here $\bar{z}$ indicates the complex conjugate of z, i.e. if $z = a + bi$, then $\bar{z} = a - bi$.
   b) Show that complex conjugation is actually a multiplicative function, i.e. if w, z $\in$, then $\phi(wz) = \phi(w)\phi(z)$.
   c) Show that ϕ is the only multiplicative linear transformation on C to C, other than the zero and identity linear transformations.

4. *OMITTED*

5. Prove that there does not exist a linear transformation T: $R^5 \to R^2$ such that Ker T = $\{(x_1,x_2,x_3,x_4,x_5) \in R^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

6. Given any m×n matrices A and B, prove that rank (A + B) $\leq$ rank (A) + rank (B). Give a non-trivial example in which equality is achieved, and a non-trivial example in which strict inequality holds.

SOLUTIONS FOLLOW

Q 1 (a) To find the change of matrix $P_{S \to \beta}$, we have to take the inverse of the matrix Q whose columns are the vectors in β.

Hence, $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

$\therefore P = Q^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

(NB: P can be determined by row reduction or by the adjoint formula as you prefer.)

(b) $[\bar{v}_1]_\beta = P[\bar{v}_1]_S = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_\beta$

$[\bar{v}_2]_\beta = P[\bar{v}_2]_S = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}_\beta$

(c) Since $[\bar{v}_\beta] = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_\beta$,

$[\bar{v}]_S = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 23 \end{bmatrix}_S$

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Q2. What would be a linear transformation T: R → R, considering R as a vector space over itself ?

- Suppose we define a function by the rule T(x) = cx, where c is some real number.

We can easily see that T is a linear transformation.

- Conversely, suppose T: R → R is any linear transformation.

Now, T(1) = c for some real number c.

But then, for any real number x,

T(x) = T(x.1) = x T(1) = x.c = cx.

So, these are the only linear transformations from R to itself.

- However, the terminology "linear" is slightly different from the one used in calculus. In calculus, a function is called linear if its graph is a straight line. But a linear *transformation* would be a function whose graph is a straight line **through the origin**.

Q3 (a) If $z = a+bi$, $w = c+di \in \mathbb{C}$ and $r \in \mathbb{R}$ (i.e. a scalar), then

$$\phi(z+w) = \phi[(a+bi)+(c+di)] =$$

$$\phi[(a+c)+(b+d)i] = (a+c)-(b+d)i$$

$$= (a-bi)+(c-di) = \phi(w) + \phi(z)$$ (1)

Again, $\phi(rz) = \phi[r(a+bi)]$

$$= \phi(ra + rbi) = ra - rbi$$

$$= \cancel{r(a+} \; r(a-bi) = r\,\phi(z)$$ (2)

(1) and (2) show that ϕ is linear

(b) Show that complex conjugation is ~~linear~~ multiplicative.

Ans: We have $\varphi(zw) = \overline{zw} = \overline{(a+bi)(c+di)}$

$= \overline{(ac-bd) + (ad+bc)i} = (ac-bd) - (ad+bc)i$ (1)

OTOH, $\phi(z)\phi(w) = \bar{z}\,\bar{w}$

$= (a-bi)(c-di) = (ac-bd) + (-ad-bc)i$

$= (ac-bd) - (ad+bc)i$ (2)

Result follows from (1) and (2)

(c) Show that ϕ is the only multiplicative linear transformation on $\mathbb{C}$ to $\mathbb{C}$.

Ans: Suppose that ψ is another multiplicative linear transformation.

Since ψ is not the zero transformation, there is some complex number, say $z \neq 0$, s.t. $\psi(z) \neq 0$.

But then, $\psi(z) = \psi(1.z) = \psi(1)\psi(z)$ (1)

(since ψ is multiplicative).

Dividing (1) by $\psi(z)$ on both sides, we get

$\psi(1) = 1$.

$\therefore \psi(-1) = (-1)\psi(1)$ (since ψ is linear, and $(-1) \in \mathbb{R}$)

$= -1\,(1) = -1$.

Finally, $-1 = \psi(-1) = \psi(i^2) = \psi(i)\psi(i)$ (2)

$\therefore \psi(i) = i$ or $-i$.

If $\psi(i) = i$, then for any $z = a+bi$, we get:-

$\psi(a+bi) = a\psi(1) + b\psi(i) = a+bi \Rightarrow \psi =$ the identity transformation.

If $\psi(i) = -i$, then $\psi(z) = \psi(a+bi) = a\psi(1) + b\psi(i)$

$= a-bi = \bar{z} \Longrightarrow \psi = \phi$, as required.

Q5. ~~Q4~~ ~~Q10~~ Prove that there does not exist a linear map from $\mathbb{R}^5$ to $\mathbb{R}^2$ whose null space ~~equals~~ (kernel) ~~is~~ $= \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2$ and $x_3 = x_4 = x_5\}$

BWOC

Ans: Suppose that there exists such a linear map T. Then Null T is the solution space of the ~~syste~~ homogeneous system

$$\begin{aligned} x_1 - 3x_2 &= 0 \\ x_3 - x_4 &= 0 \\ x_3 - x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

The co-efficient matrix of this is

$$A = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Its RREF matrix is :-

$$R = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that R has $\underline{3}$ basic variables.

$\therefore$ $3 = \dim(\text{col } A) = \text{rank } A \Rightarrow \text{nullity } A = 2$

$\therefore$ Nullity $T = 2 \Rightarrow \text{Rank } T = 5 - 2 = 3$

$\Longrightarrow \Longleftarrow$ since Rank $T = \dim(\text{Range } T) \leq 2$

Q6.

Given any two $(m \times n)$-matrices A and B, show that $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$. Give a non-trivial example in which ~~exact~~ equality is achieved and a non-trivial example in which strict inequality holds.

(9)

Answer:- Recall Prop. 18: If U and W are finite-dimensional subspaces of V, then $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

In any case, $\dim(U+W) \leq \dim U + \dim W$ ①

So now suppose A, B are two $m \times n$-matrices,

$$A = [\bar{a}_1 \; \bar{a}_2 \cdots \bar{a}_n] \text{ and } B = [\bar{b}_1 \; \bar{b}_2 \cdots \bar{b}_n]$$

where the $\bar{a}_i, \bar{b}_i$ are column vectors in F^m.

Now, $\text{rank}(A+B) = \dim \text{Col}(A+B)$

But $\text{Col}(A+B) = \text{span}\{\bar{a}_1+\bar{b}_1, \bar{a}_2+\bar{b}_2, \ldots, \bar{a}_n+\bar{b}_n\}$

$$\subseteq \text{span}\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n\}$$

$$= \text{col}\, A + \text{col}\, B$$

$$\therefore \text{rank}(A+B) = \dim \text{Col}(A+B)$$
$$\leq \dim(\text{Col}\, A + \text{Col}\, B)$$
$$\leq \dim(\text{Col}\, A) + \dim(\text{Col}\, B), \text{ using ①}$$
$$= \text{rank}(A) + \text{rank}(B)$$

(i) Example in which equality is achieved:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) Example in which strict inequality holds

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A+B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

In (i), $\text{rank}(A+B) = 2 = \text{rank}\, A + \text{rank}\, B$

In (ii), $\text{rank}(A+B) = 1$ but $\text{rank}\, A + \text{rank}\, B = 1+1 = 2$

so $\text{rank}(A+B) < \text{rank}(A) + \text{rank}(B)$

(Of course, many other examples can be constructed)