MTH 100 – 2016 MONSOON [redacted] (1)

Section A → 20160808

Section B → 20160809

Example for solution of homogeneous system

$$x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 - 2x_3 = 0$$
$$3x_1 + 6x_2 - 4x_3 = 0$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -2 \\ 3 & 6 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

No. of pivot positions = 2

No. of variables = 3

∴ there ~~to~~ are free variables

This corresponds to the system:

$$x_1 + 2x_2 = 0$$
$$x_3 = 0$$

We re-write it as :-

$$x_1 = -2x_2$$
$$x_2 = x_2 \qquad = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$x_3 = 0$$

We can treat x_2 as a parameter; by setting it to different values, we get different soln. → infinitely many

An example to illustrate Proposition 3.

$A\bar{x} = \bar{0}$ where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Corresponds to the system $\left. \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array} \right| \to$

unique solution — necessarily the trivial solution.

To be done in both sections on Wednesday 20160810

2016 MONSOON

MTH 100 –

Examples for Non-Homogeneous System

Solve $A\bar{x} = \bar{b}$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 7 \\ 9 \\ 30 \end{bmatrix}$

Work with augmented matrix $[A : \bar{b}] =$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 2 & -1 & 3 & 9 \\ 4 & 1 & 8 & 30 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & -1 & -1 & -5 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow[R_3 \to (-1)R_3]{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Corresponds to the system : $x_1 = 1$

$x_2 = 2$

$x_3 = 3$

Unique Solution $\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

No free variables

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Example 2 :- $A\bar{x} = \bar{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 3 \\ 11 \\ 28 \end{bmatrix}$$

$$[A : \bar{b}] = \left[\begin{array}{ccc:c} 1 & 2 & 4 & 3 \\ 3 & 8 & 16 & 11 \\ 8 & 20 & 40 & 28 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{RREF}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{basic} & \text{basic} & \text{free} \end{array}$$

leads to the system

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \end{aligned}$$

i.e. $$\bar{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}} + x_3 \underbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}_{\bar{v}}$$

Every solution is of the form:

$u + x_3 \bar{v}$ where

$\bar{u}$ is a soln. of the given non-homog. system

$\bar{v}$ is a soln. of associated homog. system

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Check : $A\bar{u} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 28 \end{bmatrix} = \bar{b}$

OTOH, $A\bar{v} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 16 \\ 8 & 20 & 40 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

x — x — x

Another Example

$\bar{b} =$

Example 3:- $A\bar{x} = \bar{b}$ where $A = \left[\begin{array}{ccc:c} 1 & 2 & 4 & 4 \\ 3 & 8 & 16 & 11 \\ 8 & 20 & 40 & 28 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 8R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 4 \\ 0 & 2 & 4 & -1 \\ 0 & 4 & 8 & -4 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 4 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$ (*)

$\xrightarrow[R_3 \to -\frac{1}{2}R_3]{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 + \frac{1}{2}R_3]{R_1 \to R_1 - 4R_3} \left[\begin{array}{ccc:c} 1 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Here, the last row is of the form $[0 \cdots 0 \,\vdots\, b]$ with $b \neq 0$. It corresponds to the equation ~~then~~

$$0 \cdot x_1 + 0x_2 + 0x_3 = b \neq 0,$$ which is not possible. So this system is inconsistent.

Proof of Observation 6 : Suppose the non-homogeneous system $A\bar{x} = \bar{b}$ has at least one solution, say $\bar{u} \neq \bar{0}$.

A vector $\bar{y}$ is a solution of the system if and only if $\bar{y} = \bar{u} + \bar{v}$, where $\bar{v}$ is a solution of the associated homogeneous system $A\bar{x} = \bar{0}$.

[$\Longrightarrow$] Suppose $\bar{y}$ is a ~~so~~ solution of the system. Put $\bar{v} = \bar{y} - \bar{u}$.

Then $A\bar{v} = A(\bar{y} - \bar{u}) = A\bar{y} - A\bar{u}$

$= \bar{b} - \bar{b} = \bar{0}$.

$\therefore$ $\bar{v}$ is a solution of the homogeneous system, and $\bar{y} = \bar{u} + \bar{v}$.

[$\Longleftarrow$] Conversely, suppose $\bar{v}$ is any solution of the homogeneous system.

Then $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v}$

$= \bar{b} + \bar{0} = \bar{b}$.

$\therefore$ $\bar{u} + \bar{v}$ is a solution of the non-homogeneous system.

Summary for Non-Homogeneous System : $A\bar{x} = \bar{b}$

Associated Homogeneous System $A\bar{x}=\bar{0}$		Non-Homogeneous System
Case 1: Unique Solution (trivial) $\Updownarrow$ NO free variable	$\longleftrightarrow$	Inconsistent OR Unique Solution
Case 2: Infinitely many solution $\Updownarrow$ At least one free variable	$\longleftrightarrow$	Inconsistent OR Infinitely Many solutions