Proof of Theorem 5 (DT- V I T):-

(a) $[\Longrightarrow]$ Suppose A is diagonalizable. Then $A = PDP^{-1}$ for some diagonal matrix and some invertible matrix P, i.e. $AP = PD$ (1).

Let $P = [\bar{v}_1 \ \bar{v}_2 \ \cdots \ \bar{v}_n]$ and let $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where the λ_i's need not be distinct.

$\therefore$ (1) becomes

$$A[\bar{v}_1 \ \bar{v}_2 \cdots \bar{v}_n] = P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

or
$$[A\bar{v}_1 \ A\bar{v}_2 \cdots A\bar{v}_n] = [\bar{v}_1 \cdots \bar{v}_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= [\lambda_1\bar{v}_1 \ \lambda_2\bar{v}_2 \ \cdots \ \lambda_n\bar{v}_n]$$

Equating columns,

$$A\bar{v}_i = \lambda_i\bar{v}_i, \quad i = 1, 2, \cdots, n \qquad (2)$$

Now, the vectors $\bar{v}_1, \cdots, \bar{v}_n$ being columns of an invertible matrix are lin. indep. and (2) shows that they are eigenvectors.

Proof of (a) — cont'd.

[$\Leftarrow$] Conversely, suppose A has n lin. indep. eigenvectors so that

$$A\bar{v}_i = \lambda_i \bar{v}_i, \quad i = 1, 2, \ldots, n.$$

Form the matrix P with the $\bar{v}_i$'s as columns.

Then: $AP = A[\bar{v}_1\ \bar{v}_2 \cdots \bar{v}_n]$

$= [A\bar{v}_1\ A\bar{v}_2 \cdots A\bar{v}_n]$

$= [\lambda_1\bar{v}_1 \cdots \lambda_n\bar{v}_n]$

$= PD$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

But P is invertible, so

$A = PDP^{-1}$, and A is diagonalizable

(b) Part (b) has been proved en route to proving part (a).

~~Tuesday 29th Oct. 2013~~

~~Thursday 31st Oct. 2013~~

$-546 - 42\lambda$
$+13\lambda + \lambda^2$
$+ 726$

Example for Case 1.

$$A = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix}$$

Then: $\det(A - \lambda I) = \det \begin{bmatrix} 42-\lambda & -33 \\ 22 & -13-\lambda \end{bmatrix}$

$$= (42-\lambda)(-13-\lambda) + 22(33)$$

$$= 180 - 29\lambda + \lambda^2 = (20-\lambda)(9-\lambda)$$

Hence there are 2 distinct eigenvalues

$$\lambda_1 = 20, \quad \lambda_2 = 9$$

(i) For λ_1, $A - \lambda_1 I = \begin{bmatrix} 22 & -33 \\ 22 & -33 \end{bmatrix} \longrightarrow \begin{bmatrix} 22 & -33 \\ 0 & 0 \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}.$$

Let us take $\bar{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as an eigenvector

[Check: $A\bar{v}_1 = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 126 - 66 \\ 66 - 26 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix} = 20\,\bar{v}_1,$$

as reqd.]

(PTO)

Example (cont'd)

(ii) $\lambda_2 = 9$, $A - \lambda_2 I = \begin{bmatrix} 33 & -33 \\ 22 & -22 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so we take

$\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the eigenvector.

[Check: $\begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix} = 9 \bar{v}_2$]

Note: we should get $A = PDP^{-1}$
where $D = \text{diag}(20, 9)$ and $P = [\bar{v}_1 \ \bar{v}_2]$

Easier to check $AP = PD$

$$AP = \begin{bmatrix} 42 & -33 \\ 22 & -13 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 60 & 9 \\ 40 & 9 \end{bmatrix}$$

$$PD = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 60 & 9 \\ 40 & 9 \end{bmatrix},$$

as desired