A real poly can be factored as a product of ~~real~~ linear and quadratic factors:-

MTH 100 — 2016 MONSOON

$z = a + bi$

$\bar{z} = a - bi$

$z = \bar{z}$ iff z is a real no.

$z + \bar{z} = (a+bi) + (a-bi) = 2a \in \mathbb{R}$

$z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 \geq 0 \in \mathbb{R}$

$|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Basic Fact:

Any non-zero complex polynomial has n roots, where n is the degree, counting multiplicities.

← Fundamental Theorem of Algebra

$p(x) = a_n x^n + \cdots + a_0, \quad a_n \neq 0$

$a_i \in \mathbb{R}$

$= a_n (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k} \, q(x)$

We think of $q(x)$ as a complex poly, so by FTA it has a complex root, say $w = a + bi$

~~$p(w) = a_n x \quad a_n$~~

$q(w) = b_k w^k + b_{k-1} w^{k-1} + \cdots + b_0, \quad b_k \neq 0$

Conjugate this:

$\bar{b}_k \bar{w}^k + \cdots + \bar{b}_0 = \bar{0} = 0$

Each $\bar{b}_i$ is ~~real~~, $\therefore \bar{b}_i = b_i$

$q(\bar{w}) = 0$ also.

$\therefore$ $q(x)$ has a factor looking like $(x-w)(x-\bar{w})$

$x^2 - (w+\bar{w})x + w\bar{w}$

$\longrightarrow x^2 - 2ax + (a^2+b^2)$

$\uparrow$

real quadratic factor

$p(x) = (x-3)^3 (x-4)^2 (x^2+x+1)^5$. etc

(an example)

Example for Complex Eigenvalues

Suppose $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$

The char. poly. $\det(A - \lambda I) = 8 - 4\lambda + \lambda^2$

gives $\lambda = 2 \pm 2i$.

We can take either one of them,

so we take $\lambda = 2 + 2i$

$\therefore\ a = 2$ and $b = -2$

The corresponding eigenvector

$\bar{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, where x, y are complex nos.

is the solution of the homogeneous system $(A - \lambda I)\bar{z} = \bar{0}$, where

$$A - \lambda I = \begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix}$$

(PTO)

This leads to the homogeneous system :

$$(-2-2i)x + y = 0 \quad ①$$
$$(-8)x + (2-2i)y = 0 \quad ②$$

However, we know that the system has a non-trivial solution, i.e. the two equations represent the same relationship between x and y.

$\therefore$ we may give any value to one of them arbitrarily and obtain the second from either of the two equations.

So we put $x = 1$ in the first equation ① :-

$$(-2-2i) + y = 0$$

$$\therefore \quad y = 2+2i$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \bar{v} \quad \text{(say)}$$

Example (cont'd)

Thus the matrix

$$P = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} = [\operatorname{Re}\bar{v} \quad \operatorname{Im}\bar{v}]$$

and the matrix

$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$ (recall: $a = 2$, $b = -2$)

$$= \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

Let us verify that $PBP^{-1} = A$,

i.e. $PB = AP$

$$PB = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 0 & 8 \end{bmatrix} \quad ✓$$

$$AP = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 0 & 8 \end{bmatrix} \quad ✓$$

(PTO)

Example (cont'd) —

Note that

$$B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} = \sqrt{8} \begin{bmatrix} \frac{2}{\sqrt{8}} & \frac{2}{\sqrt{8}} \\ \frac{-2}{\sqrt{8}} & \frac{2}{\sqrt{8}} \end{bmatrix}$$

$$= \sqrt{8} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(Compare with the exercise we did in class to get the ~~rota~~ matrix for rotation thru' θ :-

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

→ we see that B represents a rotation through $\frac{7\pi}{4}$ followed by scaling by $\sqrt{8} = r = \sqrt{\lambda}$

→ The matrix P represents a change of coordinates (change of basis)

→ If we had taken $\lambda = 2 - 2i$, we would have got different P and B, but relationship would have been same.