MTH 100_ Notes for 20161003_ MONDAY

Proof of Prop. 27:

(a) Given $T: V \to W$ is an isomorphism.
Let $\{\bar{v}_1, \ldots, \bar{v}_n\}$ be any basis of V.
Consider now $\{T\bar{v}_1, \ldots, T\bar{v}_n\} = B$, say.
We need to show B is spanning & L.I.
So let $\bar{w} \in W$.
Then, $\bar{w} = T\bar{v}$ for some $\bar{v} \in V$
(by surjectivity of T).
So $\bar{v} = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$

$\therefore \bar{w} = T\bar{v} = T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n)$
$= c_1 T\bar{v}_1 + \cdots + c_n T\bar{v}_n$.

For l.i., suppose $c_1 T\bar{v}_1 + \cdots + c_n T\bar{v}_n = \bar{0}$
i.e. $T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n) = \bar{0}$
$\Rightarrow c_1\bar{v}_1 + \cdots + c_n\bar{v}_n = \bar{0}$ (by injectivity of T)
$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$, since the $\bar{v}_i$ are l.i.

ch

Suppose T takes at least one of ~~any~~ basis of V to a basis of W. Need to show T is surjective and injective.
Let $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be the basis of V,
then $\{T\bar{v}_1, \ldots, T\bar{v}_n\}$ is a basis of W, by hypothesis.
Suppose $\bar{w} \in W$. Then:
$\bar{w} = c_1 T\bar{v}_1 + \cdots + c_n T\bar{v}_n = T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n)$
$\Rightarrow \bar{w} \in \text{Range } T$.
Again, suppose $T\bar{v} = \bar{0}$ for some $\bar{v} \in V$.
Now, $\bar{v} = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n \Rightarrow T\bar{v} = T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n)$
$= c_1 T\bar{v}_1 + \cdots + c_n T\bar{v}_n = \bar{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0$
$\Rightarrow \bar{v} = \bar{0}$, as reqd.

Proof of Prop. 28:

[$\Rightarrow$] Suppose V and W are isomorphic, i.e. $\exists$ an isomorphism $T: V \to W$. If $\dim V = n$, V has a basis $\{\bar{v}_1, \dots, \bar{v}_n\}$. By Prop. 27, $\{T\bar{v}_1, \dots, T\bar{v}_n\}$ is a basis for W, hence $\dim W = n = \dim V$.

[$\Leftarrow$] Suppose V and W have the same dimension, say n.

Let $\{\bar{v}_1, \dots, \bar{v}_n\}$ be ~~any~~ a basis for V and let $\{\bar{w}_1, \dots, \bar{w}_n\}$ be a fixed basis for W.

Define ~~an isomorph~~ a linear transformation $T: V \to W$ by $T\bar{v}_i = \bar{w}_i$ for $i = 1, 2, \dots, n$.

This can be done in view of the remark after Prop. 26.

But now, from Prop. 27 (b), it follows that T is an isomorphism.