MTH100 **MATHS I** **Monsoon Semester – 2016-17**

Tutorial exercise for week commencing Monday 24th October 2016

1. a) Find the matrix relative to the standard basis of the linear operator T on R^3 given by: $T(x_1,x_2,x_3) = (x_1 + x_3,\ x_1 + 2x_2 + x_3,\ -x_1 + x_2)$.

   b) Find the matrix of the same linear operator T relative to the ordered basis $\beta = \{(1,1,1), (1,2,3), (1,3,6)\}$.
   [*NB: The change of basis matrix* $P_{S\to\beta}$ *for this basis was calculated in Q1 of last week's tutorial.*]

2. Let $T: R^3 \to R^2$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

   a) Find the matrix of T with respect to the standard bases for R^3 and R^2.

   b) Verify that $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$ is a basis for R^3.

   c) Now, determine the matrix of T with respect to the ordered bases β and $\beta' = \{(0,1), (1,0)\}$ for R^3 and R^2 respectively.

3. Let V be an n-dimensional space and let T be a linear operator on V such that Range (T) = Kernel (T). Show that n must be even. Give an example of such an operator.

4. a) Let $T: V \to W$ and $U: W \to Z$ be linear transformations, where V, W and Z are finite-dimensional vector spaces over F. Show that rank (UT) $\leq$ min {rank (T), rank (U)}.
   b) State an analogous result for matrices A and B, and comment briefly on its proof.
   c) For (b), give a non-trivial example (i.e. the matrices A, B should be non-zero and non-identity and should be of minimum size 2x2), in which equality is achieved, and a non-trivial example in which strict inequality holds.

5. Let $V = R^{2\times2}$ = vector space of 2x2 matrices with real entries, and consider the function $U: V \to V$ given by $U(A) = A + A^T$, for all $A \in V$, where A^T indicates the transpose of A.

   a) Show that U is a linear operator.
   b) Determine the matrix of U with regard to any suitable ordered basis β of V.
   c) Determine a basis for Ker U and determine a basis for Range U.
   d) Determine the dimension of $Sym_n(R)$, the space of symmetric nxn matrices with real entries. Briefly explain your answer.

6. Proposition 31 states that if A and B are the matrices of a linear operator T with regard to the ordered bases α and β respectively, then B is similar to A. Now prove the converse statement: if B is similar to A, then B is the matrix of the linear transformation T_A (i.e. left multiplication by the matrix A) with regard to a suitable basis β.

SOLUTIONS FOLLOW

Q 1.

a) Find the matrix relative to the standard basis S of the operator T given by:

$$T(x_1, x_2, x_3) = (x_1 + x_3,\ x_1 + 2x_2 + x_3,\ -x_1 + x_2).$$

Ans: We have $T\bar{e}_1 = (1, 1, -1) = 1\bar{e}_1 + 1\bar{e}_2 + (-1)\bar{e}_3$

$T\bar{e}_2 = (0, 2, 1) = 0.\bar{e}_1 + 2\bar{e}_2 + 1.\bar{e}_3$

$T\bar{e}_3 = (1, 1, 0) = 1.\bar{e}_1 + 1.\bar{e}_2 + 0.\bar{e}_3$

$$\therefore [T]_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) Find the matrix of T with respect to the basis β.

From Q 1, (last week) $P_{S \to \beta} = P = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

and $P^{-1} = Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

$$\therefore B = P A P^{-1} = P A Q$$

$$= \begin{bmatrix} -\text{[illegible]} & -11 & 16 \\ 14 & 26 & 40 \\ -6 & -11 & 17 \end{bmatrix}$$

Q2.
Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$$

(a) Find the matrix of T relative to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$

Ans: $T\bar{e}_1 = T(1,0,0) = (1,-1) = 1.\bar{e}_1 + (-1).\bar{e}_2$

$T\bar{e}_2 = T(0,1,0) = (1,0) = 1\bar{e}_1 + 0.\bar{e}_2$

$T\bar{e}_3 = T(0,0,1) = (0,2) = 0.\bar{e}_1 + 2.\bar{e}_2$

$$\therefore [T]_{S_3 \to S_2} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

(b) Verify that $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$ is a basis for $\mathbb{R}^3$

Ans: It suffices to show that the matrix A whose columns are the vectors in β is row-equivalent to I_3.

But $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 - R_2]{R_1 \to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) For convenience, let $\beta = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ and $\beta' = \{\bar{v}_1, \bar{v}_2\}$

$\therefore T\bar{u}_1 = T(1,0,-1) = (1,-2,-1) = (1,-3) = -3\bar{v}_1 + 1\bar{v}_2$

$T\bar{u}_2 = T(1,1,1) = (2,1) = 1.\bar{v}_1 + 2\bar{v}_2$

$T\bar{u}_3 = T(0,0,1) = (0,2) = 2\bar{v}_1 + 0.\bar{v}_2$

$$\therefore [T]_{\beta \to \beta'} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Q3. Given V f-d. and T a linear operator s.t. Range $(T) =$ Kernel (T). Show that $\dim V = n$, $n > 0$, is even. Give an example of such an operator.

Ans: Put Rank $T = m$. Then

$m = \dim(\text{Range } T) = \dim(\text{Ker } T) = \text{nullity } T.$

$\therefore$ by the Rank Theorem for Linear Transformations:

$m + m = n$ and so $n = 2m$

is even.

Example: Let us try with $n = 2$ (lowest possible). Then both Range T and Ker T should be 1-dimensional spaces. Simplest possibility for Ker T is x-axis, i.e. the subspace $U = \{(x, 0) : x \in \mathbb{R}\}$.

$\therefore$ for every $\bar{u} \in U$, $T\bar{u} = \bar{0}$. Since $\{\bar{e}_1\}$ is a basis for U, $T\bar{e}_1 = \bar{0}$ should hold. OTOH, vectors on the y-axis, i.e. the subspace $W = \{(0, y) : y \in \mathbb{R}\}$ should go to the x-axis only, since U is also the range of T. Recalling that a lin. transfn. is fully specified by its action on a basis, we can formulate the above ideas as follows:

$T\bar{e}_1 = \bar{0} = 0.\bar{e}_1 + 0.\bar{e}_2$; $T\bar{e}_2 = \bar{e}_1 = 1.\bar{e}_1 + 0.\bar{e}_2$

$\therefore$ Matrix of $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A$, say.

It is easily verified that Range $T = U =$ Kernel T

6

Q4 (a) Let $T: V \to W$ and $U: W \to Z$ be linear transformations, where V, W, Z are finite dimensional vector spaces over F. Show that $\text{Rank}(UT) \leq \min\{\text{rank}(T), \text{rank}(U)\}$

Answer: We see that rank (UT)

$= \dim \text{Range}(UT)$

But $\text{Range}(UT) = \{\bar{z} \in Z : \bar{z} = (UT)(\bar{v}), \bar{v} \in V\}$

$\subseteq \{\bar{z} \in Z : \bar{z} = U(\bar{w}), \bar{w} \in W\}$

$= \text{Range}\, U$

Hence, $\text{Rank}(UT) \leq \text{rank}(U)$. (1)

Again, put $W_1 = \text{Range}\, T$, so W_1 is a subspace of W and $\dim W_1 = \text{rank}\, T$ (2)

But now, if define the linear transformation

$U_1 : W_1 \to Z$ by $U_1(\bar{w}_1) = U(\bar{w}_1)$

for every $\bar{w}_1 \in W_1$ we see that

$\text{Range}\, U_1 = \text{Range}(UT)$

and so $\text{Rank}(UT) = \text{rank}\, U_1$ (3)

(U_1 is the restriction of U to the subspace W_1)

Applying the Rank Theorem to U_1,

$\text{rank}(U_1) + \text{nullity}(U_1) = \dim W_1 = \text{rank}(T)$ by (2)

and so $\text{rank}(U_1) \leq \text{rank}\, T$ (4)

Result follows from (1), (3), (4).

⑦

Q4 (b) The analogous result for matrices states that given two matrices A and B such that the product AB is defined, then Rank (AB) $\leq$ min{Rank A, Rank B}.

The proof follows directly from (a), by defining the linear transformations T and U to be left multiplication by B and A respectively.

Q4 (c) For equality, take A and B, to be $(n \times n)$- invertible matrices. Then AB is also invertible and

Rank (AB) = n = Rank A = Rank B.

For strict inequality, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so Rank A = Rank B = 1.

But AB = zero matrix, so

Rank (AB) = 0.

Q5. Let $V = \mathbb{R}^{2\times 2}$ and define

$U : V \longrightarrow V$ by $U(A) = A + A^T$,

where A^T is the transpose of A.

(a) Show that U is a linear operator.

Ans: We show additivity and homogeneity.

– For additivity: let $A, B \in V$

Then $U(A+B) = (A+B) + (A+B)^T$

$= A + B + A^T + B^T$

$= (A + A^T) + (B + B^T)$

$= U(A) + U(B)$

– For homogeneity, suppose $c \in \mathbb{R}$.

Then $U(cA) = (cA)^T = c\, A^T$.

Q5. Given $U: V \to V$ by $U(A) = A + A^T$.
Here $V = \mathbb{R}^{2\times 2}$

(d) Obtain the matrix of U with regard to any suitable ordered basis β of V.

Ans: The choice of basis is left to the student, but it is best to take the standard ordered basis

$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

Then $U(E_{11}) = E_{11} + E_{11}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$= 2E_{11}$ (1)

$U(E_{12}) = E_{12} + E_{12}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$= E_{12} + E_{21}$ (2)

$U(E_{21}) = E_{21} + E_{21}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= E_{12} + E_{21}$ (3)

$U(E_{22}) = E_{22} + E_{22}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= 2E_{22}$ (4)

Converting the equation (1) to (4) into columns, we get that

$$[U]_\beta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

As expected, since $\dim(V) = 4$, $[U]_\beta$ has to be a 4×4-matrix (this holds whatever the choice of ordered basis).

(c) Determine a basis for Ker $\mathcal{U}$ and determine a basis for Range $\mathcal{U}$.

Ans: Note that both of them are subspaces of V. ~~The Approach 1,~~ Two different approaches are given.

Approach 1: Direct Approach.

~~Sf~~ Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Ker } \mathcal{U}$

$$\therefore \mathcal{U}(A) = A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$= \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so we get $2a = 0 \implies a = 0$
$2d = 0 \implies d = 0$

and $b + c = 0 \implies c = -b$

$$\therefore A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

In short, every $A \in \text{Ker } \mathcal{U}$ is a scalar multiple of the matrix $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\therefore$ Ker $\mathcal{U}$ has dimension 1 and its basis is $\gamma = \{C\}$ 1

From the Rank Theorem, it follows that

$$\text{Rank}(\mathcal{U}) = \dim V - \text{nullity}(\mathcal{U}) = 4 - 1 = 3.$$ 2

(PTO)

Suppose now that $X \in$ Range $\mathcal{U}$

Then $X = A + A^T$ for some $A \in V$.

But then $X^T = (A + A^T)^T = A^T + (A^T)^T$

$= A^T + A = A + A^T = X$.

$\therefore X$ is a symmetric matrix, i.e.

$X \in Sym_2(\mathbb{R})$.

Now, we have Range $\mathcal{U}$ $\subseteq Sym_2(\mathbb{R}) \subseteq V$ (3)

$\therefore$ Rank $(\mathcal{U})$ ~~dim (X)~~ $\leq$ ~~Sym~~ dim $[Sym_2(\mathbb{R})]$

$\leq$ dim V

i.e. $3 \leq \dim[Sym_2(\mathbb{R})] \leq 4$ (4)

Since $Sym_2(\mathbb{R}) \neq V$, it follows from (4) that $\dim(Sym_2(\mathbb{R})) = 3$, i.e.

Range $\mathcal{U} = Sym_2(\mathbb{R})$. (5) $\ast$

Now, consider the matrices E_{11}, E_{22} and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Clearly, all three are symmetric, and they are linearly independent, since $xE_{11} + yE_{22} + zD = \begin{bmatrix} x & z \\ z & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Longrightarrow x = y = z = 0$.

$\therefore \mathcal{S} = \{E_{11}, E_{22}, D\}$ form a basis for Range $\mathcal{U} = Sym_2(\mathbb{R})$. (6)

NB: The matrix C we got earlier was skew-symmetric, i.e. it satisfies $A^T = -A$. Every ~~The only matrix which~~ skew-symmetric matrix has 0's on the diagonal, and $\bar{0}$ is the only matrix which is both symmetric and skew-symmetric.

(c) Continued – ~~M~~ Matrix Approach.

~~We ca~~ In this approach, we work with the matrix $[U]_\beta$ we obtained in part (a)

$\operatorname{Ker} U = \operatorname{Null}([U]_\beta)$ and

$\operatorname{Range} U = \operatorname{Col}([U]_\beta)$.

However, in this approach, we initially get the answers as coordinate vectors relative to the basis β, which have to be converted to matrices in V.

Now: $[U]_\beta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[R_4 \to \frac{1}{2} R_4]{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← RREF matrix. 7

This corresponds to the system:-

$$\begin{aligned} x_1 &= 0 \\ x_2 &= -x_3 \\ x_3 &= x_3 \\ x_4 &= 0 \end{aligned} \quad \text{or} \quad \bar{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}_\beta$$

~~∴ the Null Ker~~ But $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}_\beta = 0E_{11} + (-1)E_{12} + 1(E_{21}) + 0(E_{22})$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

∴ we get the basis for $\operatorname{Ker} U = \operatorname{Nul}([U]_\beta)$

$= \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} = \gamma'$, say. 8

From the RREF matrix ⑦ taking the pivot columns, we see that a basis for

$$\mathrm{Col}([U]_\beta) = \left\{ \begin{bmatrix} 2\\0\\0\\0 \end{bmatrix}_\beta, \begin{bmatrix} 0\\1\\1\\0 \end{bmatrix}_\beta, \begin{bmatrix} 0\\0\\0\\2 \end{bmatrix}_\beta \right\}$$

i.e. $\left\{ 2E_{11}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 2E_{22} \right\} =$

δ', say ⑨

Comparing γ and γ' (eqns ① and ⑧)

and δ and δ' (from ⑥ and ⑨)

we see that we get essentially the same answer whichever approach we follow.

Remark: We saw that

Ker U = space of skew-symmetric matrices

Range U = space of symmetric spaces,

with dimensions 1 and 3, respectively

Also, their intersection $= \{\bar{0}\}$

Hence $V = \mathrm{Ker}\, U \oplus \mathrm{Range}\, U$,

i.e. every matrix in $\mathbb{R}^{2\times 2}$ is uniquely expressible as the sum of a symmetric ~~and~~ matrix and a skew-symmetric matrix.

~~3~~ ~~4~~ - continued.

Above we showed that

Range $U = Sym_2(\mathbb{R})$ by using the dimension. However, this can be done directly as follows:

We already know

$$\text{Range } U \subseteq Sym_2(\mathbb{R}) \quad \text{from } (3)$$

Suppose now that $X \in Sym_2(\mathbb{R})$, so that $X = X^T$

Consider: $U\left(\frac{X}{2}\right) = \frac{X}{2} + \left(\frac{X}{2}\right)^T$

$$= \frac{X}{2} + \frac{X}{2} = X.$$

$\therefore X \in \text{Range}(U)$, i.e.

$$\cancel{\text{Range}} \quad Sym_2(\mathbb{R}) \subseteq \text{Range } U \qquad (10)$$

From (3) and (10), we get that

$$\text{Range } U = Sym_2(\mathbb{R})$$

Above proof is not so obvious, but has the advantage that it can be extended ~~to~~ for $n \geq 2$. We can define

$U : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ by $U(A) = A + A^T$.

Then, U is a linear operator and Range$(U) = Sym_n(\mathbb{R})$

(d) Determine $\dim(\mathrm{Sym}_n(\mathbb{R}))$ with brief explanation.

Ans: We generalize the approach we used for the case $n=2$. If $A=[a_{ij}]$ is symmetric, then $a_{ij}=a_{ji}$, hence entries symmetric with respect to the diagonal are equal. Such entries are $\therefore$ catered for by a term of the form $c(E_{ij}+E_{ji})$ for $i<j$, where c is any constant.

$\therefore$ we get basis ~~elements~~ matrices of the form $E_{ij}+E_{ji}$. There are $\binom{n}{2}=\frac{n(n-1)}{2}$ matrices of this type.

Also, since the diagonal elements can take any value, we get n additional basis matrices, say Λ_i, where Λ_i is a diagonal matrix with 1 in the i-th position on the diagonal and 0's elsewhere.

Hence, we get $\frac{n(n-1)}{2}+n=\boxed{\frac{n(n+1)}{2}}$ basis matrices. (11)

Remark: The space of skew-symmetric matrices has basis matrices of the form $E_{ij}-E_{ji}$, $i<j$. There $\boxed{\frac{n(n-1)}{2}}$ such matrices. (12)

Adding (11) and (12), we get :- $\frac{n(n+1)}{2}+\frac{n(n-1)}{2}=n^2$.

Hence, $\mathbb{R}^{n\times n}=\mathrm{Sym}_n(\mathbb{R})\oplus\text{Skew-Sym}_n(\mathbb{R})$ holds.

Q6. Suppose B is similar to A. To show that B is the matrix of T_A with regard to a suitable basis β.

For convenience, we write T for T_A.

Ans:- Since B is similar to A,

$B = PAP^{-1}$ for some invertible matrix. Let $Q = P^{-1} = [\bar{v}_1 \ \bar{v}_2 \cdots \bar{v}_n]$.

Since Q is invertible, the vectors $\beta = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ form a basis for V.

By Prop. 31, $[T]_\beta = P_{\alpha\to\beta}[T]_\alpha P^{-1}_{\alpha\to\beta}$

where $P_{\alpha\to\beta}$ is the change of basis matrix. Here, the old basis is the standard basis S. $\therefore$ $P_{\alpha\to\beta}$ is the inverse of the matrix whose columns are the vectors in β, i.e. $P_{\alpha\to\beta} = Q^{-1}$.

$\therefore$ $[T]_\beta = Q^{-1} A Q = PAP^{-1} = B$,

as desired.