Exercises

Remark: $L(V,W)$ = set of all linear transformations from V to W. You can easily show that $L(V,W)$ is itself a vector space (over the same base field F)

1. Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V,V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

2. Give an example of a function $f: \mathbf{R}^2 \to \mathbf{R}$ such that
$$f(av) = af(v)$$
for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but f is not linear.

Exercise 2 shows that homogeneity alone is not enough to imply that a function is a linear map. Additivity alone is also not enough to imply that a function is a linear map, although the proof of this involves advanced tools that are beyond the scope of this book.

3. Suppose that V is finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U,W)$, then there exists $T \in \mathcal{L}(V,W)$ such that $Tu = Su$ for all $u \in U$.

4. Suppose that T is a linear map from V to $\mathbf{F}$. Prove that if $u \in V$ is not in $\operatorname{null} T$, then
$$V = \operatorname{null} T \oplus \{au : a \in \mathbf{F}\}.$$

5. Suppose that $T \in \mathcal{L}(V,W)$ is injective and $(v_1, \dots, v_n)$ is linearly independent in V. Prove that $(Tv_1, \dots, Tv_n)$ is linearly independent in W.

6. Prove that if $S_1, \dots, S_n$ are injective linear maps such that $S_1 \dots S_n$ makes sense, then $S_1 \dots S_n$ is injective.

7. Prove that if $(v_1, \dots, v_n)$ spans V and $T \in \mathcal{L}(V,W)$ is surjective, then $(Tv_1, \dots, Tv_n)$ spans W.

8. Suppose that V is finite dimensional and that $T \in \mathcal{L}(V,W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.

9. Prove that if T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that
$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\},$$
then T is surjective.

10. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals
$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

11. Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

12. Suppose that V and W are both finite dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim W \le \dim V$.

13. Suppose that V and W are finite dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V,W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \ge \dim V - \dim W$.

14. Suppose that W is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W,V)$ such that ST is the identity map on V.

15. Suppose that V is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W,V)$ such that TS is the identity map on W.

4.3 Test

(1) A map $f : V \to W$ between vector spaces V and W over $\mathbb{F}$ is linear, if

☐ $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for all $x, y \in V$, $\lambda, \mu \in \mathbb{F}$.
☐ f satisfies the eight axioms for a vector space.
☐ $f : V \to W$ is bijective.

(2) By the kernel of a linear map $f : V \to W$ one understands

☐ $\{w \in W \mid f(0) = w\}$
☐ $\{f(v) \mid v = 0\}$
☐ $\{v \in V \mid f(v) = 0\}$

(3) Which of the following statements are correct? If $f : V \to W$ is a linear map, we have

☐ $f(0) = 0$.
☐ $f(-x) = -x$ for all $x \in V$.
☐ $f(\lambda v) = f(\lambda) + f(v)$ for all $\lambda \in \mathbb{F}$, $v \in V$.

(4) A linear map $f : V \to W$ is called an isomorphism if

☐ there exists a linear map $g : W \to V$ with $fg = \mathrm{Id}_W$ and $gf = \mathrm{Id}_V$.
☐ V and W are isomorphic.
☐ for each n-tuple $(v_1, \dots, v_n)$ in V, the n-tuple $(f(v_1), \dots, f(v_n))$ is a basis of W.

(5) By the rank $\mathrm{rk}(f)$ of a linear map $f : V \to W$, one understands

☐ $\dim \mathrm{Ker}\, f$ ☐ $\dim \mathrm{Im}\, f$ ☐ $\dim W$

(6) $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} =$

☐ $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ ☐ $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(7) The map $\mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x + y, x - y)$, is given by the following matrix ("The columns are the ..."):

☐ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

(8) Let V and W be two vector spaces with bases (v_1, v_2, v_3) and (w_1, w_2, w_3) and let $f : V \to W$ be the linear map with $f(v_i) = w_i$. Then the "associated" matrix is

☐ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ ☐ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ☐ $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(9) A linear map $f : V \to W$ is injective if and only if

☐ f is surjective. ☐ $\dim \mathrm{Ker}\, f = 0$. ☐ $\mathrm{rk}\, f = 0$.

(10) Let $f : V \longrightarrow W$ be a surjective linear map and $\dim V = 5$, $\dim W = 3$. Then

☐ $\dim \mathrm{Ker}\, f \geq 3$.
☐ $\dim \mathrm{Ker}\, f$ is 0, 1, or 2 and each of these cases can arise.
☐ $\dim \mathrm{Ker}\, f = 2$.