Example for Case 2(a)

$$A = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \longrightarrow$$ Compare with Tuesday's example

$$\det(A - \lambda I) = (-\lambda)(1-\lambda)^2$$

$\therefore$ the eigenvalues are :-

$\lambda_1 = 1$ (alg. multiplicity 2)

$\lambda_2 = 0$ (alg. multiplicity 1)

(i) Taking λ_1 : $A - \lambda_1 I = \begin{bmatrix} 3 & 2 & -1 \\ -3 & -2 & 1 \\ 6 & 4 & -2 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

$\uparrow$ YES !!

Putting $x_2 = -3$, $x_3 = 0$, we get

$\bar{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ as an eigenvector

Putting $x_2 = 0$, $x_3 = 3$, we get

$\bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ as an eigenvector.

Example (cont'd) :-

(ii) Taking $\lambda_2 = 0$, $A - \lambda_2 I = \begin{bmatrix} 4 & 2 & -1 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ -3 & -1 & 1 \\ 6 & 4 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$. Putting $x_3 = 2$,

we get $\bar{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ as an eigenvector corresp. to λ_2

In this case, A has turned out to be diagonalizable, because geometric multiplicity of $\lambda_1 = 2$ = algebraic multiplicity.

[problem comes only if alg. multiplicity > 1; if alg. mult. $= 1$, then geom. multiplicity has to be 1]

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Example for Case 2 (b)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda (1-\lambda)^2$$

so we get the same situation:-

$\lambda_1 = 1$ (multiplicity 2)

$\lambda_2 = 0$ (multiplicity 1)

(i) Taking $\lambda_1 = 1$, $A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix}$

$$\xrightarrow{\text{(interchange)}} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 - 3R_1]{R_2 \to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$ so :

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$; putting

$x_3 = 2$, we get $\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

(PTO)

$\bar{v}_1$ is certainly an eigenvector,

since $A\bar{v}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot \bar{v}_1$$

However, geometric multiplicity of $\lambda_1 = 1$ ~~where~~ $<$ alg. multiplicity $= 2$.

$\therefore$ A is not diagonalizable

[if geom. mult. $<$ alg. multn for any one ~~=~~ eigenvalue, the matrix is not diagonalizable]

What's happening in case 2 :- we have $\lambda_1, \dots, \lambda_p$ as distinct real eigenvalues, $p < n$, but alg. multiplicities add up to n.

$$\begin{array}{lcl} \text{geom. mult. } \lambda_1 & \leq & \text{alg. mult. } \lambda_1 \\ \text{geom. mult. } \lambda_2 & \leq & \text{alg. mult. } \lambda_2 \\ \vdots & & \vdots \\ \text{geom. mult. } \lambda_p & \leq & \text{alg. mult. } \lambda_p \\ \hline n & & n \end{array} +$$

$\Uparrow$ Sum on RHS $= n$

if A is diagonalizable, then LHS must also add to n (by D T - V I T).

So if (a) of Prop. 54 is proved, then (b) has to follow — i.e. all geom. multiplicities must equal the corresp. alg. multiplicities.
(c) follows without much trouble.
But (a) is advanced and beyond our scope.