MTH 100 - Notes

Fundamental Results - 1

- **Proposition 12 (Steinitz Exchange Lemma):** Suppose $v_1, v_2, \dots, v_n$ are linearly independent vectors in a vector space V, and suppose $\{w_1, w_2, \dots, w_m\}$ span V. Then:
  a) $n \leq m$
  b) $\{v_1, v_2, \dots, v_n, w_{n+1}, w_{n+2}, \dots, w_m\}$ span V, after re-ordering the w's if necessary.

Proof: So we have:

$\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ lin. indep.

$\bar{w}_1, \bar{w}_2, \dots, \bar{w}_m$ span V, i.e. $V = \text{Span}\{\bar{w}_1, \dots, \bar{w}_m\}$

Since $\bar{w}_1, \dots, \bar{w}_m$ span V, we must have

$$\bar{v}_1 = c_1\bar{w}_1 + c_2\bar{w}_2 + \dots + c_m\bar{w}_m \qquad (1)$$

for some scalar c_i.

If $c_i = 0$ for all i, then $\bar{v}_1 = \bar{0}$, which is not possible since any set containing the zero vector is l.d.

$\therefore$ $c_i \neq 0$ for at least one i, and re-numbering the w's if necessary, we can ~~as~~ assume that $c_1 \neq 0$. So we can re-write (1) as:

$$c_1\bar{w}_1 = \bar{v}_1 - c_2\bar{w}_2 - \dots - c_m\bar{w}_m,$$

and multiplying by c_1^{-1}, we get:

$$\bar{w}_1 = c_1^{-1}\bar{v}_1 - c_1^{-1}c_2\bar{w}_2 - \dots - c_1^{-1}c_m\bar{w}_m$$

Proof of Steinitz Exchange Lemma (2)

(cont'd)

or $\bar{w}_1 = d_1\bar{v}_1 + d_2\bar{w}_2 + \cdots + d_m\bar{w}_m$ (2)

where the d_i are scalars.

From (2), it follows that

$\text{Span}\{\bar{v}_1, \bar{w}_2, \ldots, \bar{w}_m\} = \text{Span}\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m\}$

$= V.$ (3)

Justification of (3): Suppose $\bar{x} \in V$,

~~i.e. Suppose~~ then $\bar{x} \in \text{Span}\{\bar{w}_1, \ldots, \bar{w}_m\}$, i.e.

$\bar{x} = f_1\bar{w}_1 + f_2\bar{w}_2 + \cdots + f_m\bar{w}_m$ (4)

Substituting for $\bar{w}_1$ in (4) from (2), we get:

$$\bar{x} = f_1(d_1\bar{v}_1 + d_2\bar{w}_2 + \cdots + d_m\bar{w}_m) + f_2\bar{w}_2 + \cdots + f_m\bar{w}_m$$

$$= f_1 d_1\bar{v}_1 + (f_1 d_2 + f_2)\bar{w}_2 + \cdots + (f_1 d_m + f_m)\bar{w}_m$$

$$= h_1\bar{v}_1 + h_2\bar{w}_2 + \cdots + h_m\bar{w}_m$$

$\therefore \bar{x} \in \text{Span}\{\bar{v}_1, \bar{w}_2, \ldots, \bar{w}_m\}$,

hence $V \subseteq \text{Span}\{\bar{v}_1, \bar{w}_2, \ldots, \bar{w}_m\}$

hence $V = \text{Span}\{\bar{v}_1, \bar{w}_2, \ldots, \bar{w}_m\}$ as claimed.

So, at the next step, we get that

$\bar{v}_2 = l_1\bar{v}_1 + l_2\bar{w}_2 + \cdots + l_m\bar{w}_m$ for some scalars l_i (5)

We see that at least one of $l_2, l_3, \ldots, l_m$ is not zero; if all are zero, then ~~again~~ $\bar{v}_2 = l_1 \bar{v}_1$ — contradicting lin. indep. of the $\bar{v}_i$'s.

By renumbering the w_j's, if necessary, we may assume $l_2 \neq 0$.

So then: $l_2 \bar{w}_2 = \cancel{l_1 \bar{v}_1} - l_1 \bar{v}_1 + \bar{v}_2$

$- l_3 \bar{w}_3 - \cdots - l_m \bar{w}_m$, and

arguing as before, we get that:

$\text{Span}\{\bar{v}_1, \bar{v}_2, \bar{w}_3, \ldots, \bar{w}_m\} = \text{Span}\{\bar{v}_1, \bar{w}_2, \ldots, \bar{w}_m\} = \text{Span}\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m\} = V.$

Proceeding in this way, we can step-by-step replace $\bar{w}_1$ by $\bar{v}_1$, $\bar{w}_2$ by $\bar{v}_2$, ..., etc. The process has to stop after the n-th step at most (since there are only n of the $\bar{v}$ vectors).

What is the situation when we ~~have t~~ come to the stop?

There are two possible cases:-

~~Case~~

Proof of Steinitz Exchange Lemma (cont'd).

Case 1 : $n \leq m$

In this case, we get the following situation:

$$\begin{array}{ccccccccc} \bar{v}_1, & \bar{v}_2, & \ldots, & \bar{v}_n & & & \\ \downarrow & \downarrow & & \downarrow & & & \\ \{\bar{w}_1, & \bar{w}_2, & \ldots, & \bar{w}_n, & \bar{w}_{n+1}, & \ldots, & \bar{w}_m\} \end{array}$$

We have replaced n of the $\bar{w}$ vectors, with re-numbering if necessary, and we get $V = \text{Span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n, \bar{w}_{n+1}, \ldots, \bar{w}_m\}$.

So in Case 1, the proposition is proved. [If $n = m$, then the vectors $\bar{w}_{n+1}$, etc. are not there in the original spanning set at all.]

Case 2 : $n > m$.

Then, we are only able to replace $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m$ and we are left with the vectors $\bar{v}_{m+1}, \ldots, \bar{v}_n$ of the original lin. indep. vectors.

Proof of Steinitz Exchange Lemma (completed):-

The situation looks like this:

$$\begin{array}{ccccccc} \bar{v}_1, & \bar{v}_2, & \dots, & \bar{v}_m, & \bar{v}_{m+1}, & \dots, & \bar{v}_n \\ \downarrow & \downarrow & & \downarrow & & & \\ \{\bar{w}_1, & \bar{w}_2, & \dots, & \bar{w}_m\}, & & & \end{array}$$

i.e. $\{\bar{v}_1, \dots, \bar{v}_m\}$ is now a spanning set for V.

~~Butts~~ But, then

$$\bar{v}_{m+1} \in \text{Span}\{\bar{v}_1, \dots, \bar{v}_m\}$$

or $\bar{v}_{m+1} = \cancel{k_1 w} \; k_1\bar{v}_1 + \dots + k_m\bar{v}_m$ for some scalars k_i.

But this contradicts linear indep. of the $\bar{v}_i$.

Hence, Case 2 <u>cannot happen.</u>

Only Case 1 can happen, and in this case, as we saw before, the Proposition 12 has been proved.

Prop 13: Any 2 bases of V have the same no. of elements.

[Assuming Prop. 12]

Suppose ~~D_1~~, B_1, ~~&~~ B_2 are two distinct bases of V, $|B_1| = m$

$|B_2| = n$

Then, $|B_1| \leq |B_2|$, since B_1 is l.i. and B_2 is a spanning set (Prop. 12(a))

i.e. $m \leq n$ (1)

In a similar way,

$|B_2| \leq |B_1|$ - again from Prop. 12

$n \leq m$ (2)

We get $m = n$