MTH100 **MATHS I** **Monsoon Semester – 2016-17**

Tutorial exercise for the week of Monday 3rd October 2016

1. V is a vector space with dim (V) = n. W_1 and W_2 are subspaces of V such that dim (W_1) = dim (W_2) = n – 1 and $W_1 \cap W_2 = \{0\}$. Find n.

2. Prove **Proposition 19**: If U and W are subspaces of the vector space V, then $V = U \oplus W$ if and only if $V = U + W$, and $U \cap W = \{0\}$.

3. Given the vector space R^3, let W_1 be the set of vectors of the form (x,y,0) and let W_2 be the set of vectors of the form (0,a,b).

   a) Show that W_1 and W_2 are subspaces of R^3.

   b) Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

   c) Find two distinct subspaces U_1 and U_2 of R^3 such that $R^3 = W_1 \oplus U_1 = W_1 \oplus U_2$, i.e. find two distinct complements of V. Justify your answer.

4. Given the matrix A below:
   a) Find a basis for each of the spaces Nul A, Col A and Row A.
   b) Find a basis for Row A consisting of rows of the given matrix A. This should be different from the one given in part a).
   c) Is A invertible? Justify your answer with reference to VIT.

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$$

5. Determine whether the following are linear transformations (yes or no). Justify your answers.

   a. T: $R^3 \to R^2$ given by T(x,y,z) = (x + y, x – z)

   b. T: $R^3 \to R^2$ given by T(x,y,z) = (x + y, z^2)

   c. U: $R^{n\times n} \to R^{n\times n}$ given by U(A) = A^T. Here A^T indicates the transpose of the matrix A.

   d. M: $R[t] \to R[t]$ given by M(p(t)) = tp(t) for all polynomials $p(t) \in R[t]$.

6. Consider the space V = C[R] and consider the mapping D_ε: V → V given by $D_\varepsilon(f) = f_\varepsilon$, where $f_\varepsilon(x) = f(x + \varepsilon)$ for all x. Here ε is an arbitrary but fixed real number. Is D_ε a linear transformation? Justify your answer.

SOLUTIONS FOLLOW

(NOT IN SAME ORDER)

Q2. [$\Rightarrow$] Suppose $V = U \oplus W$. Then, by definition every vector $\bar{v} \in V$ is <u>uniquely</u> expressible as $\bar{v} = \bar{u} + \bar{w}$, with $\bar{u} \in U$, $\bar{w} \in W$, i.e. every $\bar{v} \in V$ satisfies $\bar{v} \in U + W$, so $V = U + W$.

It only remains to show that $U \cap W = \{\bar{0}\}$.
So suppose $\bar{x} \in U \cap W$. Consider:

$$\bar{x} = \bar{x} + \bar{0}, \text{ where } \bar{x} \in U, \bar{0} \in W$$

and
$$\bar{x} = \bar{0} + \bar{x}, \text{ where } \bar{0} \in U, \bar{x} \in W.$$

By uniqueness of expression, $\bar{x} = \bar{0}$, as required.

[$\Leftarrow$] Suppose $V = U + W$ with $U \cap W = \{\bar{0}\}$.
We need to show that every $\bar{v} \in V$ is <u>uniquely</u> expressible as $\bar{v} = \bar{u} + \bar{w}$, with $\bar{u} \in U$, $\bar{w} \in W$.
Let $\bar{v} \in V$. Since $V = U + W$, $\bar{v}$ is expressible as $\bar{v} = \bar{u} + \bar{w}$, with $\bar{u} \in U$ and $\bar{w} \in W$.
Suppose the expression is not unique, i.e. we also have $\bar{v} = \bar{u}_1 + \bar{w}_1$, with $\bar{u}_1 \in U$, $\bar{w}_1 \in W$.

Subtracting, $\bar{0} = (\bar{u} - \bar{u}_1) + (\bar{w} - \bar{w}_1)$

or $\bar{u}_1 - \bar{u} = \bar{w} - \bar{w}_1$ (1)

The vector on LHS of (1) is in U, and vector on RHS of (1) is in W, i.e. each of them is in $U \cap W = \{\bar{0}\}$.

$\therefore$ $\bar{u}_1 - \bar{u} = \bar{0}$, i.e. $\bar{u} = \bar{u}_1$
and $\bar{w} - \bar{w}_1 = \bar{0}$, i.e. $w = \bar{w}_1$. This proves uniqueness.

Q 3 (a) $W_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$

$W_2 = \{(0, a, b) : a, b \in \mathbb{R}\}$

We show W_1 is a subspace of $\mathbb{R}^3$ using Prop 8. W_2 can be handled in the same way.

(i) Clearly, $\bar{0} \in W_1$

(ii) If $\bar{u}_1 = (x_1, y_1, 0)$ and $\bar{u}_2 = (x_2, y_2, 0) \in W_1$,

then $\bar{u}_1 + \bar{u}_2 = (x_1 + x_2, y_1 + y_2, 0) = (x_3, y_3, 0)$

where $x_3 = x_1 + x_2 \in \mathbb{R}$

and $y_3 = y_1 + y_2 \in \mathbb{R}$.

(iii) If $c \in \mathbb{R}$, then $c\bar{u}_1 = c(x_1, y_1, 0)$

$= (cx_1, cy_1, 0) = (x_4, y_4, 0)$ where

$x_4 = cx_1 \in \mathbb{R}$ and $y_4 = cy_1 \in \mathbb{R}$

(b) Find the dimensions of $W_1, W_2, W_1 + W_2$, $W_1 \cap W_2$.

We see that $\bar{e}_1 = (1, 0, 0)$ and $\bar{e}_2 = (0, 1, 0)$ are both in W_1 and are linearly independent. $\therefore \dim W_1 \geq 2$. OTOH, $\bar{e}_3 = (0, 0, 1) \notin W_1$, hence $W_1 \subsetneq \mathbb{R}^3$. $\therefore \dim W_1 < 3$ by Prop. 18, and so ~~dim $(W_1 + W_2)$ =~~ $\dim W_1 = 2$.

Similarly, $\dim W_2 = 2$.

Since $\bar{e}_3 \in W_2$, we get that $\bar{e}_1, \bar{e}_2, \bar{e}_3 \in W_1 + W_2$ and $\therefore \dim(W_1 + W_2) = 3$.

Now, applying Prop. 20, we get

$$3 = \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
$$= 2 + 2 - \dim(W_1 \cap W_2) = 4 - \dim(W_1 \cap W_2)$$

$\therefore \dim(W_1 \cap W_2) = 1$

Q3(c) Find two distinct complements of W_1 (4)

Answer:- Let $U_1 = \text{span}\{\bar{e}_3\}$

We easily see that $W_1 + U_1 = \mathbb{R}^3$.

Also, $\dim(W_1 \cap U_1) = \dim(W_1 + U_1) - \dim(W_1) - \dim(U_1)$

$= 3 - 2 - 1 = 0,$

$\therefore W_1 \cap U_1 = \{0\}$ and so $\mathbb{R}^3 = W_1 \oplus U_1$.

Again put $\bar{v} = (0, 1, 1)$. Then $\bar{v} \notin \text{Span}\{\bar{e}_1, \bar{e}_2\} = W_1$

$\therefore$ by Prop. 14, $\{\bar{e}_1, \bar{e}_2, \bar{v}\}$ is lin. indep., hence a basis for $\mathbb{R}^3$

$\therefore \mathbb{R}^3 = W_1 + U_2$ where $U_2 = \text{span}\{\bar{v}\}$

As above, we find that $\mathbb{R}^3 = W_1 \oplus U_2$.

$\therefore$ both U_1 and U_2 are complements of W_1, but $U_1 \neq U_2$ since $\bar{v} \in U_2$, $\bar{v} \notin U_1$.

NB: The above example illustrates that a subspace can have more than one complement. In fact, there was nothing special about our choices of $\bar{e}_1$ and $\bar{v}$ above. We simply had to select two vectors, $\bar{v}_1$ and $\bar{v}_2$ not in W_1, $\bar{v}_1, \bar{v}_2$ lin. indep.
Then, $\text{span}\{\bar{v}_1\}$ and $\text{span}\{\bar{v}_2\}$ would be distinct complements of W_1.

Q 5.

~~Q5~~ Determine whether the following are linear transformations or not, with justification.

(a) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T(x,y,z) = (x+y, x-z)$.

Answer: YES. If (x_1,y_1,z_1) and $(x_2,y_2,z_2) \in \mathbb{R}^3$,

then $T[(x_1,y_1,z_1)+(x_2,y_2,z_2)] = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (x_1+x_2+y_1+y_2,\ x_1+x_2-z_1-z_2)$ (1)

and $T(x_1,y_1,z_1) + T(x_2,y_2,z_2) = (x_1+y_1, x_1-z_1) + (x_2+y_2, x_2-z_2) = (x_1+y_1+x_2+y_2,\ x_1-z_1+x_2-z_2)$ (2)

(1) and (2) are equal, showing additivity.

Also $T(c(x_1,y_1,z_1)) = T(cx_1, cy_1, cz_1) = (cx_1+cy_1, cx_1-cz_1) = c(x_1+y_1, x_1-z_1) = cT(x_1,y_1,z_1)$ (3)

(3) shows homogeneity.

(b) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T(x,y,z) = (x+y, z^2)$.

Ans: NO. It is enough to give one counter-example for either additivity or homogeneity. We will give a counter-example for homogeneity (many others are possible).

Put $\bar{u} = (1,1,1)$ and $c = 2$.

Then $T(c\bar{u}) = T(2,2,2) = (4,4)$

OTOH, $cT(\bar{u}) = 2(2,1) = (4,2)$.

5 (c) $U : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ given by $U(A) = A^T$.

Ans. YES If $A, B \in \mathbb{R}^{n\times n}$, then we have $U(A+B) = (A+B)^T = A^T + B^T$

$= U(A) + U(B)$

Again, $U(cA) = (cA)^T = cA^T = cU(A)$.

(d) $M : \mathbb{R}[t] \to \mathbb{R}[t]$, given by $M(p(t)) = t\,p(t) \quad \forall\ p(t) \in \mathbb{R}[t]$.

Ans: YES. If $p(t), q(t) \in \mathbb{R}[t]$, then $M(p(t)+q(t)) = t(p(t)+q(t))$

$= t\,p(t) + t\,q(t) = M(p(t)) + M(q(t))$.

Also, if $c \in \mathbb{R}$, then $M(c\,p(t)) = t(c\,p(t)) = c(t\,p(t))$

$= c\,M(p(t))$

Q4. Given $A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$.
(a)

7

We first find the RREF matrix of A

$\xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 6 & 3 \\ 0 & 0 & -1 \\ 1 & 3 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & -1 \\ 2 & 6 & 3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_2 \to (-1)R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow R_A$ (: RREF matrix)

(a) Find bases for Nul A, Col A, Row A

For Nul A: solve the homog. system $A\bar{x} = \bar{0}$ equivalent to $R_A \bar{x} = \bar{0}$, i.e.

$x_1 = -3x_2$
$x_2 = x_2$ or $\bar{x} = t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$
$x_3 = 0x_2$

Basis = $\left\{ \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} \right\}$ → easily verified.

For Col A: take columns of A corresp. to pivot columns of R,

i.e. Basis = $\left\{ \begin{bmatrix} 2 \\ 4 \\ 13 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 17 \end{bmatrix} \right\}$

For Row A: Take non-zero rows of R, &

i.e. Basis = $\{ \underbrace{[1 \quad 3 \quad 0]}_{\bar{u}_1}, \underbrace{[0 \quad 0 \quad 1]}_{\bar{u}_2} \}$

(PTO)

Q4 (b) Find a basis for Row A consisting of rows of A.

Ans: We have to row-reduce $B = A^T$

$$= \begin{bmatrix} 2 & 4 & 13 \\ 6 & 12 & 39 \\ 3 & 5 & 17 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 2 & 4 & 13 \\ 0 & 0 & 0 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 4 \\ 2 & 4 & 13 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow R_B \quad \text{(RREF matrix)}$$

We must take columns corresponding to pivot columns of R_B.

$\therefore$ Basis $= \left\{ \underset{\bar{v}_1}{\begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} 4 \\ 12 \\ 5 \end{bmatrix}} \right\}$

We can easily see that $\bar{v}_1 = 2\bar{u}_1 + 3\bar{u}_2$
$\bar{v}_2 = 4u_1 + 5\bar{u}_2$

so span $\{\bar{u}_1, \bar{u}_2\}$ = span $\{\bar{v}_1, \bar{v}_2\}$.

(c) Is A invertible? Justify your answer with reference to IMT.

Ans: NO, A is not invertible.

By IMT, A is invertible iff the homog. system $A\bar{x} = \bar{0}$ has only the trivial soln.

However, in (a), we found a non-zero vector $\bar{v}$ s.t. $A\bar{v} = \bar{0}$

Q1. V is a vector space with $\dim V = n$, W_1 and W_2 are subspaces such that $\dim W_1 = \dim W_2 = n-1$, and $W_1 \cap W_2 = \{0\}$. Find n.

Ans: Since $W_1 + W_2$ is a subspace of V, we must have $\dim(W_1 + W_2) \leq \dim V = n$

But $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$, (by Proposition 20.)

$= (n-1) + (n-1) - 0 = 2n-2 \leq n$

or $n \leq 2$.

However, $n=0$ is not possible since $\dim(W_1) = \dim(W_2) = n-1$.

$\therefore$ $n = 1$ or 2.

10

Q6. Consider $D_\epsilon : C[\mathbb{R}] \to C[\mathbb{R}]$, where $D_\epsilon(f) = f_\epsilon$ such that $f_\epsilon(x) = f(x+\epsilon)$ for all $x \in \mathbb{R}$.
Is D_ϵ a linear transformation? Justify.

Ans. YES, (i) Suppose $f, g \in C[\mathbb{R}]$

Put $D_\epsilon(f+g) = h$.

Then
$$h(x) = (f+g)(x+\epsilon) \text{ by defn. of } D_\epsilon$$
$$= f(x+\epsilon) + g(x+\epsilon) \text{ by defn. of addition of function}$$
$$= f_\epsilon(x) + g_\epsilon(x), \text{ for all } x \in \mathbb{R}.$$

Hence, $h = f_\epsilon + g_\epsilon$

i.e. $D_\epsilon(f+g) = D_\epsilon(f) + D_\epsilon(g)$. — additivity

(ii) Again, if $c \in \mathbb{R}$, put

$D_\epsilon(cf) = h$

Again $h(x) = ~~c f(x+\epsilon) \text{ by defn of}~~$
$$= (cf)(x+\epsilon) \text{ by Defn. of } D_\epsilon$$
$$= c f(x+\epsilon)$$
$$= c f_\epsilon(x) \text{ for all } x \in \mathbb{R}.$$

Hence, $h = c f_\epsilon$

i.e. $D_\epsilon(cf) = c D_\epsilon(f)$, — homogeneity.

The linear transfns. D_ϵ are known as delay operators, which play a central role in signal processing.