MTH100 **MATHS I** **Monsoon Semester – 2016-17**

Tutorial exercise for Monday 12th September 2016

1. Prove Remark 6 related to linear dependence/independence : Any list which contains a linearly dependent list is linearly dependent.

2. Prove Remark 7 related to linear dependence/independence : Any subset of a linearly independent set is linearly independent .

3. Determine whether the given matrices in the vector space $R^{2\times 2}$ are linearly dependent or linearly independent.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

4. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e. functions) are linearly dependent or linearly independent :

   $f_1(x) = 1$, $f_2(x) = \sin(x)$, $f_3(x) = \sin(2x)$.

   (You must justify your answer.)

5. Given the standard basis $B = \{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$ of R^3 and the linearly independent vectors $\mathbf{v_1} = (0,1,1)$ and $\mathbf{v_2} = (1,1,1)$, apply the method of the Steinitz Exchange Lemma (Proposition 12) to exchange two of the vectors in B and obtain a basis C which includes $\mathbf{v_1}$ and $\mathbf{v_2}$. Show your calculations in detail.

6. Prove **Proposition 11**: The subset $B = \{\mathbf{v_1},\mathbf{v_2},\ldots,\mathbf{v_n}\}$ is a basis of the vector space V if and only if every vector $\mathbf{v} \in V$ is **uniquely** expressible as a linear combination of the elements of B.

SOLUTIONS FOLLOW

Q 1: Prove Remark 6: Any list which contains a linearly dependent list is linearly dependent.

~~Post~~ Proof: Let $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ be a linearly dependent list; we obtain a larger list by adjoining the vectors $\bar{w}_1, \dots, \bar{w}_m$ (not nec. distinct)

Since the $\bar{v}_i$ are ld, we have:

$$c_1\bar{v}_1 + \dots + c_k\bar{v}_k = \bar{0} \quad (1)$$ where not all the c_i are zero.

Now consider the ~~eq~~ relation:

$$c_1\bar{v}_1 + \dots + c_k\bar{v}_k + 0\cdot\bar{w}_1 + \dots + 0\cdot\bar{w}_m = \bar{0} \quad (2)$$

In (2), there is at least one non-zero c_i from (1).

But (2) shows that the list:

$\bar{v}_1, \dots, \bar{v}_k, \bar{w}_1, \dots, \bar{w}_m$ is lin. dep.

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(2) Prove Remark 7: Any subset of a lin. indep. set is lin. indep.

Proof: Suppose $\{\bar{v}_1, \dots, \bar{v}_n\}$ is lin. indep.

Suppose BWOC that the subset $\{\bar{v}_{i_1}, \dots, \bar{v}_{i_k}\}$ is lin. dep. Then, there exist scalars $c_{i_1}, \dots, c_{i_k}$, not all zero, s.t. $c_{i_1}\bar{v}_{i_1} + \dots + c_{i_k}\bar{v}_{i_k} = \bar{0}$ (1)

For all indices $i \notin \{i_1, \dots, i_k\}$, put $c_i = 0$.

Then, we have that $c_1\bar{v}_1 + \dots + c_n\bar{v}_n = \bar{0}$ (2), where for some index i_j, $c_{i_j} \neq 0$ from (1).

But then from (2), $\bar{v}_1, \dots, \bar{v}_n$ are l.d. $\Rightarrow\Leftarrow$. Result follows.

Q3. Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ in $\mathbb{R}^{2\times 2}$, are they linearly independent or lin. dependent?

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Suppose $\alpha A + \beta B + \gamma C = \bar{0}$ (1) where $\bar{0}$ indicates the zero matrix.

We then get the foll. system of 4 equations in 3 variables:

$$\alpha + \beta + \gamma = 0 \quad (1)$$
$$\alpha + \gamma = 0 \quad (2)$$
$$\alpha = 0 \quad (3)$$
$$\beta + \gamma = 0 \quad (4)$$

Solving, we get $\alpha = 0$ from (3), then $\gamma = 0$ from (2) and finally $\beta = 0$ from (1) or (4).

Thus the matrices are lin. indep. as vectors in $\mathbb{R}^{2\times 2}$.

Q4.

In the following, determine whether the vectors are linearly independent or linearly dependent:- $V = C[0, 2\pi]$,

$$f_1(x) = 1$$
$$f_2(x) = \sin(x)$$
$$f_3(x) = \sin(2x).$$

Ans: The vectors (functions) are <u>linearly</u> independent.

Suppose $\alpha \cdot f_1(x) + \beta \cdot f_2(x) + \gamma f_3(x) = \bar{0}(x)$, (*)

where $\bar{0}(x)$ is the zero function, i.e. zero identically for all $x \in [0, 2\pi]$.

We need to determine α, β, γ. Since these are 3 unknowns, we require three equations, which can be found by substituting 3 distinct values of x in (*). We take

$x = 0$, giving $\alpha = 0$ ①

$x = \frac{\pi}{2}$, giving $\alpha + \beta = 0$ ②

$x = \frac{\pi}{4}$, giving $\alpha + \frac{\sqrt{2}}{2}\beta + \gamma = 0$ ③

The only solution to the ~~system~~ linear homogeneous system ①, ②, ③ is:-

$\alpha = 0$, $\beta = 0$, $\gamma = 0$.

This proves linear independence.

Q5. Application of method of Steinitz Exchange Lemma.

Solution: $L = \{\bar{v}_1, \bar{v}_2\}$, $\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is a lin. indep. set, $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is a spanning set for $\mathbb{R}^3$.

Proceeding as in the proof of Prop. 12, we need to express $\bar{v}_1 = c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{e}_3$

or $$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so we easily see that $\bar{v}_1 = 0.\bar{e}_1 + 1.\bar{e}_2 + 1.\bar{e}_3$.

We have to take a vector on RHS with non-zero co-eff., i.e. <u>we cannot replace $\bar{e}_1$.</u>

We must take $\bar{e}_2$ or $\bar{e}_3$, and replace it by $\bar{v}_2$. Let us take $\bar{e}_2$ (other choice is equally correct).

So: $\bar{e}_2 = \bar{v}_1 - 0.\bar{e}_1 - 1.\bar{e}_3$, and we get a new spanning set, say $B_1 = \{\bar{v}_1, \bar{e}_1, \bar{e}_3\}$.

We now have to express $\bar{v}_2$ in terms of B_1, i.e. $\bar{v}_2 = d_1\bar{v}_1 + d_2\bar{e}_1 + d_3\bar{e}_3$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = d_1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + d_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We get $\bar{v}_2 = 1.\bar{v}_1 + 1.\bar{e}_1 + 0.\bar{e}_3$

Obviously, we cannot take either $\bar{v}_1$ or $\bar{e}_3$. So we have to exchange $\bar{v}_2$ and $\bar{e}_1$, since $\bar{e}_1 = \bar{v}_2 - \bar{v}_1 + 0.\bar{e}_3$

Our new basis $C = \{\bar{v}_1, \bar{v}_2, \bar{e}_3\}$

Q6. Prove Prop. 11 : $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$ is a basis of V iff every vector $\bar{v} \in V$ is uniquely expressible as a linear combination of the elements of B.

Proof: [$\Longrightarrow$] Given B is a basis.

Let $\bar{v} \in V$. Then, since B is a basis, i.e. a spanning set $\bar{v} = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$ (1)

Suppose BWOC that $\bar{v} = d_1\bar{v}_1 + \cdots + d_n\bar{v}_n$ (2)

Subtracting, we get:

$$\bar{0} = (c_1 - d_1)\bar{v}_1 + \cdots + (c_n - d_n)\bar{v}_n \quad (3)$$

But the $\bar{v}_i$ are l.i., i.e. all the coeffs. in (3) are zero, i.e. $c_1 - d_1 = 0 = c_2 - d_2 = \cdots = c_n - d_n$.

Hence $d_i = c_i \quad \forall\, i = 1, 2, \ldots, n \Rightarrow \Leftarrow$

Result follows.

[$\Longleftarrow$] Suppose every vector $\bar{v} \in V$ is uniquely expressible as a lin. comb. of the elements of B. So clearly B is a spanning set.

Furthermore $\bar{0} = 0\cdot\bar{v}_1 + \cdots + 0\cdot\bar{v}_n$ (4)

By uniqueness, this is the only way to express $\bar{0}$ as a lin. comb. of $\bar{v}_1, \ldots, \bar{v}_n$.

$\therefore\ \bar{v}_1, \ldots, \bar{v}_n$ are lin. indep.