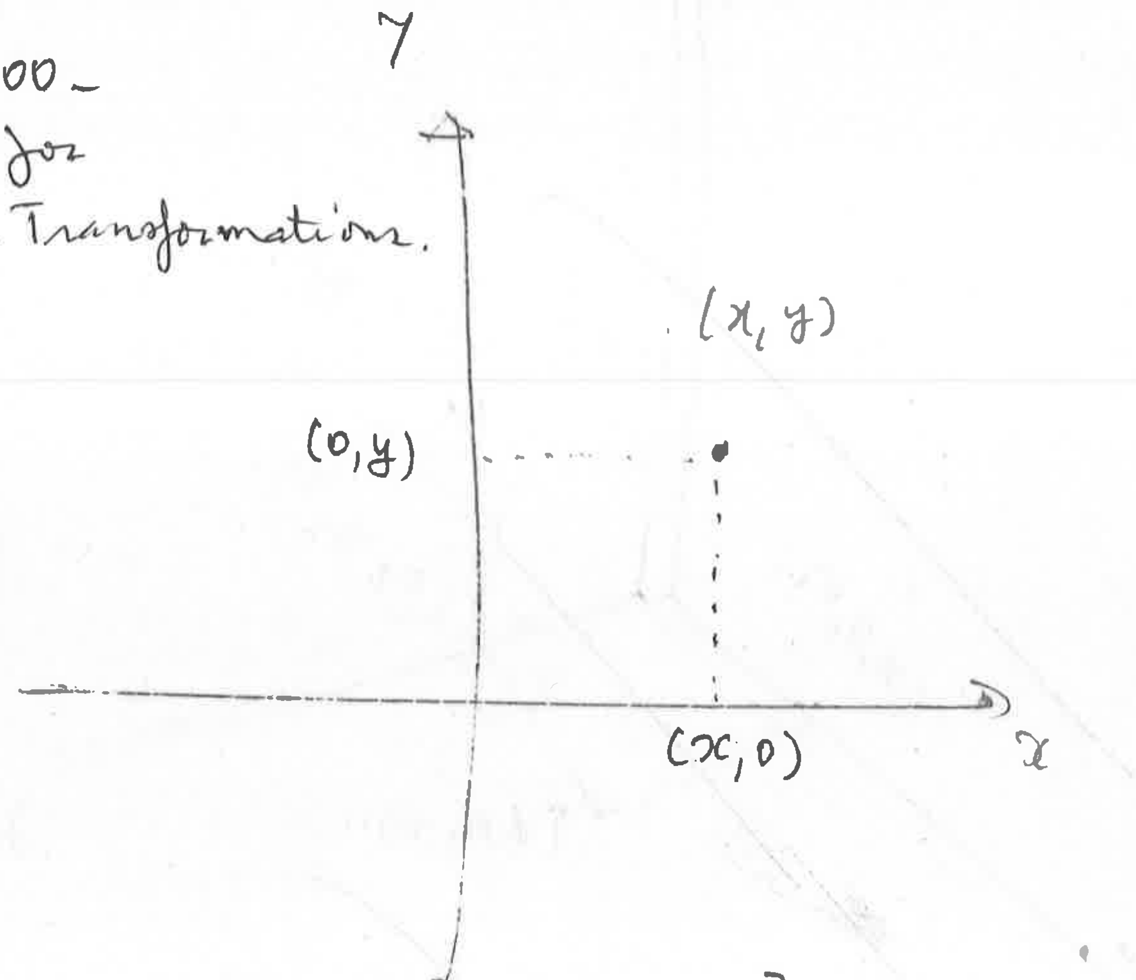
~~Explan~~ For projection:

MTH100 -
Notes for
linear Transformations.



$P_x : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

given by $(x, y) \xrightarrow{P_x} (x, 0)$

It is linear: Why?

(x_1, y_1), (x_2, y_2)

$P_x((x_1, y_1) + (x_2, y_2))$
$= P_x((x_1 + x_2, y_1 + y_2))$
$= (x_1 + x_2, 0)$

$P_x(x_1, y_1) + P_x(x_2, y_2)$
$= (x_1, 0) + (x_2, 0)$
$= (x_1 + x_2, 0)$

Similarly, for ~~scalar~~ the scalar ~~proper~~ mult. property

$\cancel{E}$

$$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 3x_3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) = T\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1 + y_1) + (x_2 + y_2) \\ (x_1 + y_1) + 3(x_3 + y_3) \end{bmatrix}$$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 3x_3 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_1 + 3y_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ x_1 + 3x_3 + y_1 + 3y_3 \end{bmatrix}$$

$$T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} cx_1 + cx_2 \\ cx_1 + 3cx_3 \end{bmatrix}$$

$$cT\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c\begin{bmatrix} x_1 + x_2 \\ x_1 + 3x_3 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 \\ cx_1 + 3cx_3 \end{bmatrix}$$

~~5~~ 3

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$T\bar{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T\bar{e}_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$T\bar{e}_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T(x\bar{e}_1 + y\bar{e}_2 + z\bar{e}_3)$$

$$= xT(\bar{e}_1) + yT\bar{e}_2 + zT\bar{e}_3$$

$$= x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 4 \\ 5 \end{bmatrix} + z\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} x + 4y + 4z \\ 2x + 5y + 5z \end{bmatrix}$$

[redacted]

Proof of Prop. 26 (b): Given a basis and list of vectors, to prove the existence of a unique linear transformation sending the basis to the list.

Proof: Let $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be a basis of V, and let $\bar{w}_1, \ldots, \bar{w}_n$ be a list of n vectors in W, not necessarily distinct.
We define a linear transformation $T: V \longrightarrow W$ as follows.

~~Firstly, define $T\bar{v}_1 = \bar{w}_1$, $T\bar{v}_2 = \bar{w}_2$, …, $T\bar{v}_n = \bar{w}_n$.~~

Now, let $\bar{v}$ be any vector in V.
Then, $\bar{v}$ can be uniquely expressed as

$\bar{v} = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$, where the c_i are scalars.

Define $T\bar{v} = c_1\bar{w}_1 + c_2\bar{w}_2 + \cdots + c_n\bar{w}_n$ (1)

Clearly, $T\bar{v}_i = \bar{w}_i$ (2) for all the vectors $\bar{v}_i \in B$.

Clearly, T is a well-defined function, $T: V \longrightarrow W$.

We need to show T is actually a linear transformation, i.e. additivity & homogeneity.

(i) Suppose $\bar{u} = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$ and $\bar{v} = d_1\bar{v}_1 + \cdots + d_n\bar{v}_n$ are any two

Prop. 26 (b) - cont'd :-

vectors in V.

Then $T(\bar{u}+\bar{v}) = T\left[(c_1\bar{v}_1 + \cdots c_n\bar{v}_n) + (d_1\bar{v}_1 + \cdots + d_n\bar{v}_n)\right] = T\left[(c_1+d_1)\bar{v}_1 + \cdots + (c_n+d_n)\bar{v}_n\right]$

$= (c_1+d_1)\bar{w}_1 + \cdots + (c_n+d_n)\bar{w}_n$, by the defn. (1)

$= (c_1\bar{w}_1 + \cdots + c_n\bar{w}_n) + (d_1\bar{w}_1 + \cdots + d_n\bar{w}_n)$

$= T\bar{u} + T\bar{v}$, again, from (1)

(ii) If now c is any scalar,

$T(c\bar{v}) = T\left[c(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n)\right]$

$= T\left[cc_1\bar{v}_1 + \cdots + cc_n\bar{v}_n\right]$

$= cc_1\bar{w}_1 + \cdots + cc_n\bar{w}_n$, from (1)

$= c(c_1\bar{w}_1 + \cdots + c_n\bar{w}_n) = c\,T\bar{v}_2$

Finally, we need ~~to~~ prove uniqueness. So, BWOC, suppose $\exists$ another lin. transformation $T_1 : V \to W$ such that $T_1\bar{v}_i = \bar{w}_i$ for all $\bar{v}_i \in B$.

Let $\bar{v} = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$ be any vector in V.

Then $T_1\bar{v} = c_1T_1\bar{v}_1 + \cdots + c_nT_1\bar{v}_n$ (by Remark 2)

$= c_1\bar{w}_1 + \cdots + c_n\bar{w}_n$

$= T\bar{v}$. (3)

Since $T\bar{v} = T_1\bar{v}$ for all $\bar{v} \in V$, it follows that $T = T_1$, proving uniqueness.