Numerical analysis: Homework 5

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Question 1:

Consider the penalized formulation for solving a quadratic problem with equality constraints

$$\min_x \frac{1}{2}x^T Hx + x^T c + \frac{1}{2\mu}\|Ax - b\|_2^2, \tag{1}$$

where $H \in \mathbb{R}^{n\times n}$ is symmetric, $c \in \mathbb{R}^n$, $\mu > 0$, and $A \in \mathbb{R}^{m\times n}$ with $m < n$ and full row-rank. Let $Z \in \mathbb{R}^{n\times(n-m)}$ be a matrix with orthonormal columns whose range is the null space of A. Show that if $Z^T HZ$ is positive definite, then for all sufficiently small μ (1) has a unique solution and that solution is a strict local minimizer.

Solution: To demonstrate that the quadratic problem

$$\min_x \frac{1}{2}x^T Hx + x^T c + \frac{1}{2\mu}\|Ax - b\|_2^2$$

has a unique solution and that solution is a strict local minimizer for sufficiently small μ, we will analyze the Hessian matrix.

The objective function is quadratic in x, and its Hessian matrix is given by:

$$\nabla^2 f(x) = H + \frac{1}{\mu}A^T A.$$

We aim to show that this matrix is positive definite for sufficiently small μ.

To establish that $H + \frac{1}{\mu}A^T A$ is positive definite, we need to show that for any non-zero vector $v \in \mathbb{R}^n$,

$$v^T \left(H + \frac{1}{\mu}A^T A\right) v > 0; \forall \; \|v\|_2 = 1$$

We decompose v into two orthogonal components:

$$v = v_{\mathcal{N}} + v_{\mathcal{R}},$$

where $v_{\mathcal{N}}$ is in the null space of A and $v_{\mathcal{R}}$ is in the range of A^T. Thus, $Av_{\mathcal{R}} \neq 0$ and $Av_{\mathcal{N}} = 0$. It follows that $\|v\|_2^2 = \|v_{\mathcal{N}}\|_2^2 + \|v_{\mathcal{R}}\|_2^2$.

Consider the quadratic form:

$$v^T \left(H + \frac{1}{\mu}A^T A\right) v = (v_{\mathcal{N}} + v_{\mathcal{R}})^T \left(H + \frac{1}{\mu}A^T A\right)(v_{\mathcal{N}} + v_{\mathcal{R}}).$$

Expanding this expression, we get:

$$v_{\mathcal{N}}^T Hv_{\mathcal{N}} + 2v_{\mathcal{N}}^T Hv_{\mathcal{R}} + v_{\mathcal{R}}^T Hv_{\mathcal{R}} + \frac{1}{\mu}v_{\mathcal{R}}^T A^T Av_{\mathcal{R}}.$$

To demonstrate positive definiteness, we need to establish a lower bound for the quadratic form. Using eigenvalues and singular values, we have:

$$\lambda_{\min}(Z^T HZ)\|v_{\mathcal{N}}\|_2^2 - 2\sigma_{\max}(H)\|v_{\mathcal{R}}\|_2\|v_{\mathcal{N}}\|_2 + \frac{1}{\mu}\sigma_{\min}(A)^2\|v_{\mathcal{R}}\|_2^2,$$

where $\lambda_{\min}$ is the minimum eigenvalue of $Z^T HZ$ and $\sigma_{\min}$, $\sigma_{\max}$ are the minimum and maximum singular values of A, respectively.
Since $\|v\|_2^2 = 1$, we write:

$$\|v_{\mathcal{N}}\|_2^2 = 1 - \|v_{\mathcal{R}}\|_2^2.$$

Thus, the quadratic form becomes:

$$\lambda_{\min}(Z^T HZ)(1 - \|v_{\mathcal{R}}\|_2^2) - 2\sigma_{\max}(H)\|v_{\mathcal{R}}\|_2\sqrt{1 - \|v_{\mathcal{R}}\|_2^2} + \frac{1}{\mu}\sigma_{\min}(A)^2\|v_{\mathcal{R}}\|_2^2.$$

To ensure the positive definiteness of the Hessian matrix, consider the quadratic term in $\|v_{\mathcal{R}}\|_2^2$:

$$\left(\frac{1}{\mu}\sigma_{\min}(A)^2 - \lambda_{\min}(Z^T HZ)\right)\|v_{\mathcal{R}}\|_2^2 - 2\sigma_{\max}(H)\|v_{\mathcal{R}}\|_2\sqrt{1 - \|v_{\mathcal{R}}\|_2^2} + \lambda_{\min}(Z^T HZ).$$

This quadratic form will be positive if:

$$\frac{1}{\mu}\sigma_{\min}(A)^2 - \lambda_{\min}(Z^T HZ) > 0 \quad \text{and} \quad \sigma_{\max}(H)^2 < \left(\frac{1}{\mu}\sigma_{\min}(A)^2 - \lambda_{\min}(Z^T HZ)\right)\lambda_{\min}(Z^T HZ).$$

For sufficiently small μ, these conditions are satisfied, ensuring that the Hessian $H + \frac{1}{\mu}A^T A$ is positive definite. This guarantees that the penalized problem has a unique solution, and this solution is a strict local minimizer.