Numerical analysis: Homework 6

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Question:

The question relates to the QR factorization. Let A be a matrix of dimension $m \times n$. Assume that A has linearly independent columns. Given two QR factorizations of the form $A = QR$ and $A = ST$, show that $Q = S$ and $R = T$. Essentially, we want to show that the QR factorization is unique.

Solution: We can express Q as $Q = [q_1\, q_2\, \ldots\, q_n]$. Similarly, $S = [s_1\, s_2\, \ldots\, s_n]$. After expressing the matrices in terms of their columns, we observe that $Q^T Q = I_n = S^T S$. This holds because both Q and S have orthonormal columns. Hence, we can simply show that $S = Q$, because this then gives $T = S^T A = Q^T A = R$.

Now, since $S^T S = I_n$, the equation $QR = S^T$ gives $S^T Q = TR^{-1}$, and this matrix is expressed as

$$S^T Q = TR^{-1} = [t_{ij}]$$

This matrix is clearly upper triangular with positive diagonal elements (since this is true for both R and T). As a result, $t_{ii} > 0$ for each i and $t_{ij} = 0$ if $i > j$. On the other hand, the (i, j)-entry of $S^T Q$ is $s_i^T q_j = s_i \cdot q_j$, so we have $s_i \cdot q_j = t_{ij}$ for all i and j. But each q_j is in $\text{span}\{s_1, s_2, \ldots, s_n\}$ because $Q = S(TR^{-1})$. Hence the expansion theorem gives

$$q_j = (s_1 \cdot q_j)s_1 + (s_2 \cdot q_j)s_2 + \ldots + (s_n \cdot q_j)s_n = t_{1j}s_1 + t_{2j}s_2 + \ldots + t_{jj}s_j$$

because $s_i \cdot q_j = t_{ij} = 0$ if $i > j$.
Expanding the terms, the first few equations we get are

$$\begin{aligned}
q_1 &= t_{11}s_1 \\
q_2 &= t_{12}s_1 + t_{22}s_2 \\
q_3 &= t_{13}s_1 + t_{23}s_2 + t_{33}s_3 \\
q_4 &= t_{14}s_1 + t_{24}s_2 + t_{34}s_3 + t_{44}s_4 \\
&\vdots
\end{aligned}$$

The first one gives $1 = \|q_1\| = \|t_{11}s_1\| = |t_{11}|\|s_1\| = t_{11}$. This means $q_1 = s_1$.
Now, we have $t_{12} = s_1 \cdot q_2 = q_1 \cdot q_2 = 0$, so the second equation is $q_2 = t_{22}s_2$. Following a similar logic, we get $q_2 = s_2$, resulting in $t_{13} = 0$ and $t_{23} = 0$. This then gives us $q_3 = t_{33}s_3$ and $q_3 = s_3$.

We can take this induction forward to get $q_i = s_i$ for all i. This means that $Q = S$, which is what we needed to show. This also implies that $T = S^T A = Q^T A = R$. Essentially, $QR = S^T$ and the QR factorization is unique. This concludes the proof.