Numerical analysis: midterm exam

Instructor: Anil Damle
Due: March 20, 2024

Policies

You may not discuss this exam with anyone aside from me, and must complete it on your own. If you believe something on the exam is unclear or requires clarification please ask me in person, via email, or via a private post on Ed Discussion. Your exam solutions must be typeset and submitted along with your code to Gradescope by **11:59pm on March 20, 2024**. Remember, slip days may not be used for the exam and late solutions will not be accepted.

The exam is open book, note, internet, etc. (any inanimate resource). However, if you use a result/statement that was not covered in class as part of your solution for a problem please cite your source for it. (This does not apply for "common knowledge" from class or prerequisite courses, e.g., you do not have to cite that matrix vector multiplication is generically $\mathcal{O}(n^2)$ or that computing a (reduced) QR factorization is $\mathcal{O}(mn\min(m,n))$.) Please err on the side of caution when decided whether or not to cite something, a few extra citations are always okay.

Notes

This exam has 5 questions (and a few page breaks to make it more readable). As this is an exam, the goal is to thoroughly test different levels/layers of understanding of the course material. Therefore, some of these questions may be challenging. Do not worry if there are parts of questions that you do not completely, cleanly work out. Do your best to answer the questions and good luck!

Question 1 (8 points):

For what values of x may naïve computation of the expression result in poor accuracy (you may assume you are given a way to accurately compute square roots and trigonometric functions for any value of x). In each case, devise a strategy to more accurately compute the expression in the problematic region.

(a) $\csc(x) - \cot(x)$ for $x \in (-\pi, \pi)$. Using your more accurate expression compute and report the value of this function for $x = 10^{-10}$.

(b) $\sqrt{x+1} - \sqrt{x}$ for $x > 0$

Question 2 (12 points)

Please clearly show your work for the following:

(a) Given a symmetric positive definite matrix A and its Cholesky factorization $A = LL^T$ (Where L is lower triangular) prove that

$$\|L\|_2 = \|A\|_2^{1/2}.$$

(b) If $A = QR$ is a reduced QR factorization for $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ prove that the singular values of A and R are the same.

(c) For any orthogonal projector P prove that $\|x\|_2^2 = \|Px\|_2^2 + \|(I - P)x\|_2^2$.

Question 3 (10 points)

Assume $A \in \mathbb{R}^{n \times n}$ is non-singular and we have already computed the partially pivoted LU factorization $PA = LU$. We now consider the linear system

$$\begin{bmatrix} A & B \\ C^T & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{1}$$

where $B, C \in \mathbb{R}^{n \times \ell}$, $D \in \mathbb{R}^{\ell \times \ell}$, $b_1 \in \mathbb{R}^n$, and $b_2 \in \mathbb{R}^\ell$. Throughout this problem we will assume that the block matrix in (1) is non-singular and that $\ell < n$ is independent of n. We are also assuming that you are given L, U, and P, so you do not need to take the time to compute them into account in your answers.

(a) Devise a scheme to solve for x in $\mathcal{O}(n^2\ell)$ time. Clearly articulate your scheme and prove that it: (1) always returns the unique solution to the desired linear system and (2) achieves the stated complexity.

(b) If A is sparse and we further assume that $PA = LU$ has factors L and U with at most a linear number of non-zeros in n can we solve (1) asymptotically faster than $\mathcal{O}(n^2\ell)$? If yes, provide an algorithm that achieves a faster complexity.

QUESTION 4 (8 POINTS)

Implement simultaneous iteration (also known as subspace iteration and orthogonal iteration). You may use built in routines to compute the requisite QR factorization at each iteration and compute all the desired eigenvalues of $(V^{(k)})^T AV^{(k)}$, where $V^{(k)}$ represent the iterates.

(a) Build a symmetric matrix A of size 100×100 with random eigenvectors and eigenvalues $\lambda_i = 1/i$ for $i = 1, \ldots, 100$. To accomplish this we can generate the individual pieces of the spectral decomposition $A = V\Lambda V^T$ and then take their product. To generate V, generate a random $Z \in \mathbb{R}^{n\times n}$, compute its QR factorization $Z = QR$, and let $V = Q$, i.e., the eigenvectors come from the orthogonal factor in a QR factorization of a random matrix. Λ can be constructed as a diagonal matrix with λ_i on the diagonal.

Let $V_1 \in \mathbb{R}^{n\times\ell}$ represent the eigenvectors associated with λ_1 through λ_ℓ. For $\ell = 5$ and $\ell = 20$ run simultaneous iteration to approximate V_1 and plot the convergence of $V^{(k)})$ to V_1 as a function of iteration. How do you expect the error to behave and do your results match your expectations? Why or why not?

(b) Given a square matrix A and some scalar function f the eigenvalues of $f(A)$ are $f(\lambda_i)$, where λ_i are the eigenvalues of A and the eigenvectors are unchanged (i.e., if (λ, v) is an eigenvalue/vector pair for A then $(f(\lambda), v)$ is an eigenvalue/vector pair for $f(A)$).[1] Let's assume you are able to run subspace iteration on $f(A)$ instead of A for any function f and someone suggests the following two possible functions:

$$
\begin{aligned}
&(1) \quad f(x) = \sin(\pi x/2)\\
&(2) \quad f(x) = 1/(1 + e^{-100(x-.175)}).
\end{aligned}
$$

For the problem in part (a) with $\ell = 5$, would running subspace iteration with these functions help us converge to V_1 in fewer iterations? Please answer the question for both functions and justify your response.

[1] There is a whole theory of functions of matrices that, while quite interesting, is beyond the scope of this midterm. All you need to know here is the fact about eigenvalues/vectors.

Question 5 (12 points)

Consider $A \in \mathbb{R}^{m \times n}$ with $m < n$.

(a) Prove that the least squares problem

$$\min_x \|Ax - b\|_2^2$$

has an infinite number of solutions for any b.

(b) Prove that for any non-singular $M \in \mathbb{R}^{n \times n}$ the regularized problem

$$\min_x \|Ax - b\|_2^2 + \|Mx\|_2^2 \tag{2}$$

has a unique solution.

(c) Provide an algorithm that solves (2) in $\mathcal{O}(n^3)$ time. Be sure to show why your algorithm returns the desired result.

(d) If M is upper triangular can we solve (2) asymptotically faster than $\mathcal{O}(n^3)$? If we can, provide an algorithm to do so, prove its complexity, and show why it returns the desired result.