CS 590: Algorithm

Lecture 3: Sorting Algorithm and Computation Complexity 1

Reading Assignment: Chapter 2, 3

Next Week: Chapter 4, 5

Outline of Lecture:

- 3.1. Analyzing Algorithms
- 3.2. Growth of Function
- 3.3. Insertion Sort
- 3.4. Merge Sort
- 3.5. Homework Discussion

3.1. Analyzing Algorithms

3.1. Analyzing Algorithms

Analyzing Algorithms – *predicting* the resources that the algorithm requires.

- Common resources are memory, communication bandwidth, or computer hardware
- Most commonly, we measure the running time.

We need a *computational model* to predict the running time.

- For technology resources and their costs.
- **Random Access Machine (RAM) – summing up** the number of steps needed to execute the algorithm on a set of data.
  - **No concurrent operations –** Instructions are executed one after another.

3.1. Analyzing Algorithms

RAM:

❑ **Issue:**

- **We want to be precise, but…**
- It is difficult and tedious to define every instruction and its associated time costs.
- For example, if the instruction says to sort, how can we define the cost?

3.1. Analyzing Algorithms

RAM:

❑ **Issue Solutions:**

1. We use instructions commonly found in real computers:
   - **Arithmetic**: add, subtract, multiply, divide, remainder, floor, ceiling, shift left, or shift right.
   - **Data movement**: load, store, copy.
   - **Control**: conditional/unconditional branch, subroutine call, and return.
2. We use integer and floating-point types.
3. We do not model the memory hierarchy (no caches or virtual memory).
   - They are significant in real programs on real machines.
   - They are relatively more complex than RAM.

3.1. Analyzing Algorithms

❑ **Goal:**

- To identify the most efficient algorithm among algorithms that can solve the problem.
  - Algorithms may differ by methodology.
  - They may have different orders of operations.
  - They may require some specific conditions on input and output.

3.1. Analyzing Algorithms

❑ ***Input size:***

- It is the most commonly realistic independent variable in analysis than any other resource in algorithms.
  - Operations depend on the size of the input.
  - But the size of the input is not dependent on the operations.

❑ ***Running time:***

- It is the number of primitive machine-independent primitive operations (steps) executed.
- We assign some constant cost c_i for individual operations at the i-th line in pseudocode.
- The running time for each operation is the product of cost and the number of executions.

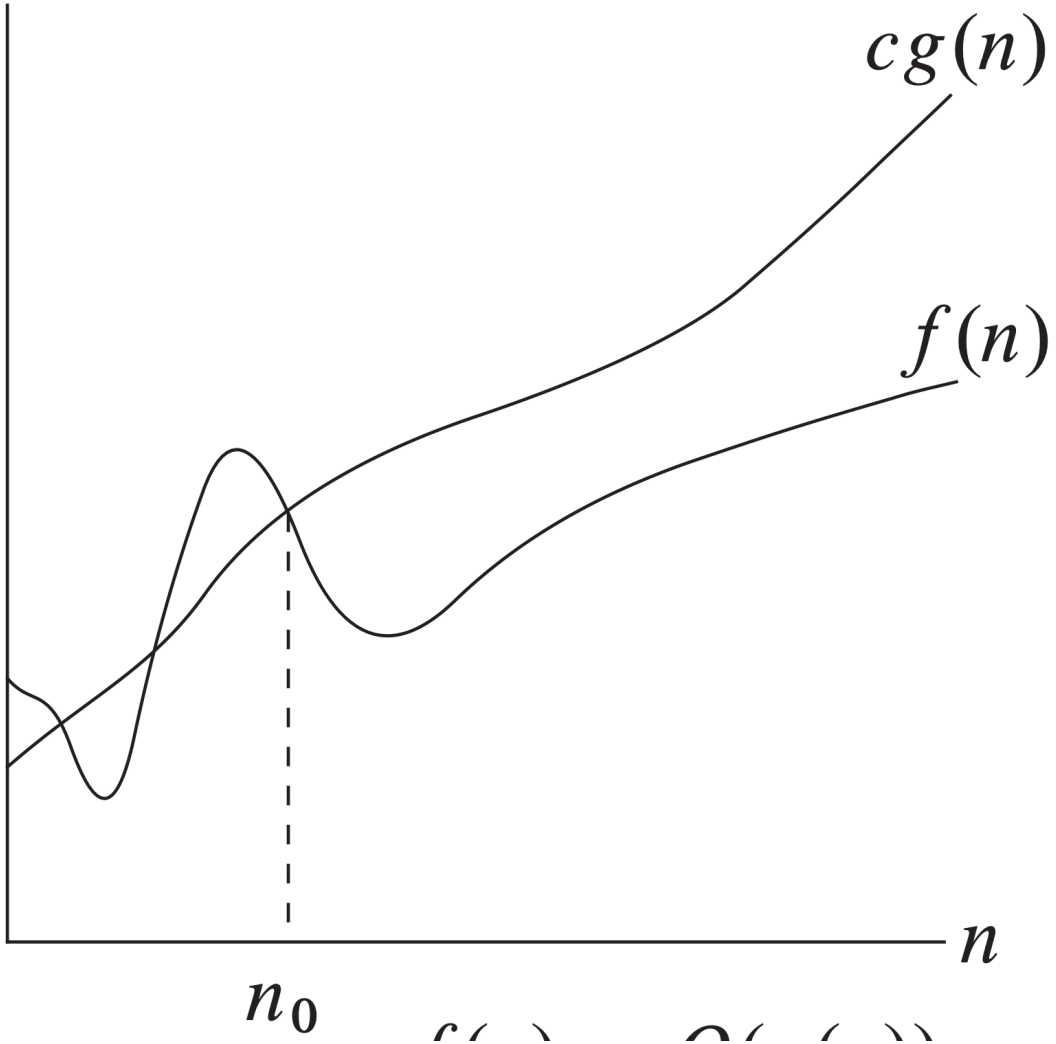
3.2. Growth of Functions

3.2. Growth of Functions

- It is not easy to measure the numerical running time without running an actual program.
- It is not easy to describe or forecast the actual real-time behavior of algorithms.
- At the same time, we do not want to analyze the algorithms too roughly, so we cannot compare them to other relative performances of alternative algorithms.
- We need mathematical tools to express the running time behavior with some boundary conditions asymptotically.
  - Asymptotic notations

3.2. Growth of Functions

Asymptotic Notation: O −notation



- If $f(n) \in cg(n)$ for a constant $c > 0$, then $f(n) = O(g(n))$
  - $0 \leq f(n) \leq cg(n) \; \forall \; n \geq n_0$
- $g(n)$ is an asymptotic **upper bound** for $f(n)$ when $n \geq n_0$.

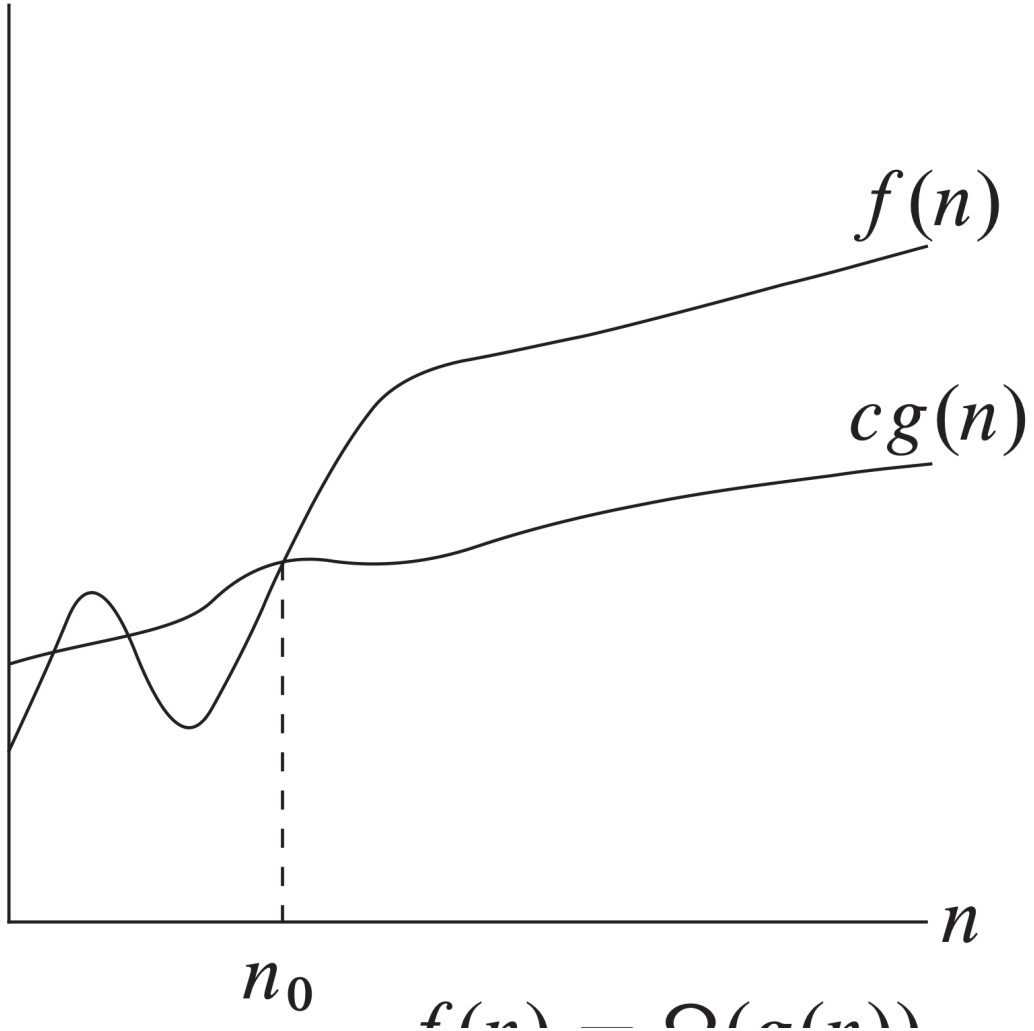
3.2. Growth of Functions

Asymptotic Notation: O −notation

- $2n^2 = O(n^3)$ with $c = 1$ and $n_0 = 2$
  - Let $f(n) = 2n^2$ and $g(n) = n^3$.
  - $g(n) \geq f(n)$ when $n \geq 2$. Otherwise, $f(n) > g(n)$.
  - For example,
    - ✓ $2(1)^2 > 1^3$.
    - ✓ $2(2)^2 = 2^3$.
    - ✓ $2(3)^2 < 3^3$.
  - The $c = 1$ condition is easy to identify that the boundary condition will satisfy when $c \geq 1$. (e.g., $2(3^2) > 0.5(3^3)$.)
- Any polynomial function, $f(n)$, can be described as $O\big(g(n)\big)$ where $g(n)$ is a function of n with the highest power.
  - $n^2, n^2 + n, n^2 + 1000n$, etc… are all $\in O(n^2)$.

3.2. Growth of Functions

Asymptotic Notation: Ω −notation



- If $f(n) \in cg(n)$ with a constant $c > 0$ and $n \geq n_0$, then $f(n) = \Omega(g(n))$.
  - $0 \leq cg(n) \leq f(n)\ \forall n \geq n_0$
- $g(n)$ is an asymptotic **lower bound** for $f(n)$.

3.2. Growth of Functions

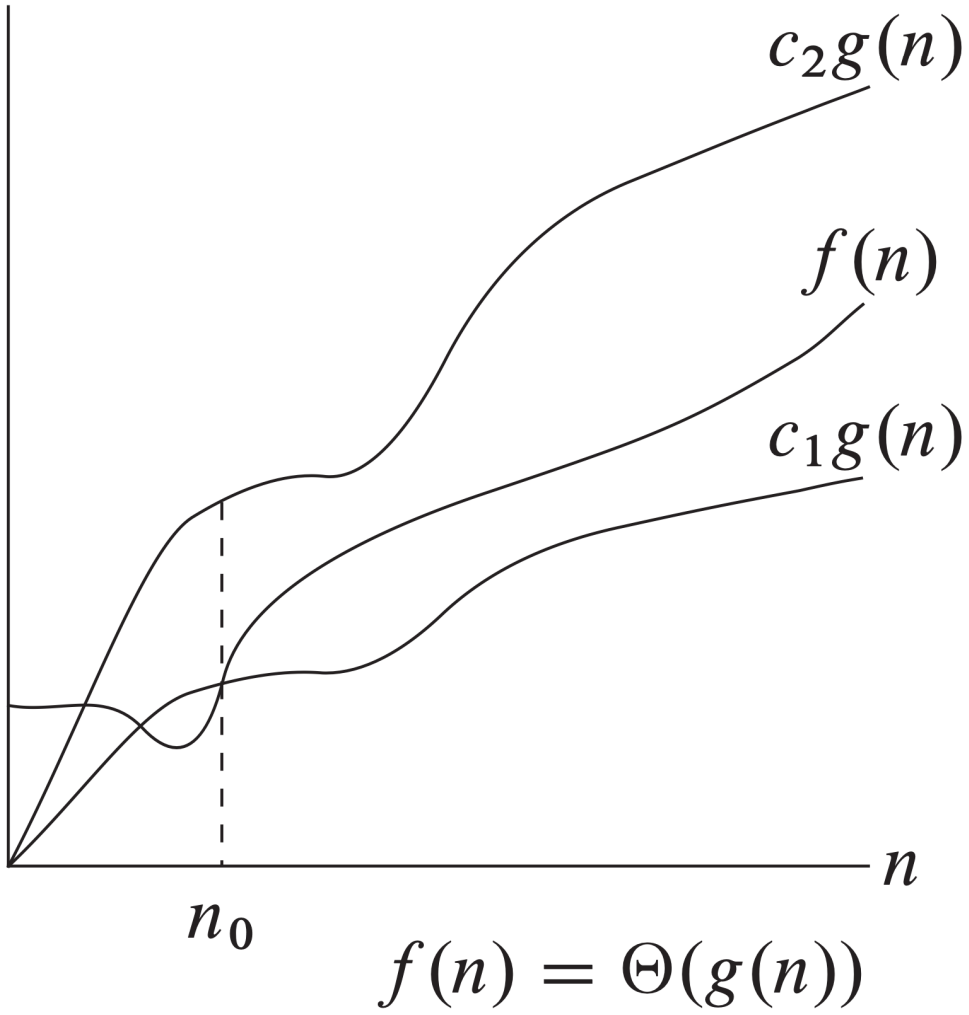
Asymptotic Notation: Ω −notation

- $\sqrt{n} = \Omega(\lg n)$ with $c = 1$ & $n_0 = 16$.
  Let $f(n) = \sqrt{n}$ and $g(n) = \lg n$.
  $g(n) \leq f(n)$ for $n \geq 16$. Otherwise, $g(n) > f(n)$.
  For example,
  - ✓ If $n = 64$, $\sqrt{64} > \lg(64)$.
  - ✓ If $n = 16$, $\sqrt{16} = \lg(16)$.
  - ✓ If $n = 4$, $\sqrt{4} = \lg(4)$.
  - ✓ If $n = 2$, $\sqrt{2} > \lg(2)$.

- For any polynomials, $f(n) \geq cg(n)\ \forall n \geq n_0$ and $c > 0$.
  - $n^2, n^2 + n, n^2 - n, n^2 + 1000n, n^2 - 1000n$, etc… are all $\in \Omega(n^2)$.

3.2. Growth of Functions

Asymptotic Notation: Θ −notation



$f(n) = \Theta(g(n))$

- $g(n)$ is an asymptotic **tight bound** for $f(n)$.
- If and only if $f = O(g(n))$ & $f = \Omega(g(n))$, $f(n) = \Theta\big(g(n)\big)$
  - For some constants, $c_1, c_2 > 0$,
  - $0 < c_1 g(n) \leq f(n) \leq c_2 g(n)\ \forall\ n \geq n_0$.

3.2. Growth of Functions

Asymptotic Notation: Θ −notation

- $\frac{n^2}{2} - 2n = \Theta(n^2)$ with $c_1 = \frac{1}{4}, c_2 = \frac{1}{2}, \& \, n_0 = 8.$
  - $f(n) = \frac{n^2}{2} - 2n$ and $g(n) = n^2.$
  - For $n = 10, \frac{1}{4}(10^2) < \frac{10^2}{2} - 2(10) < \frac{1}{2}(10^2).$
  - For $n = 8, \frac{1}{4}(8^2) = \frac{8^2}{2} - 2(8) < \frac{8^2}{2}.$
  - For $n = 4, \frac{4^2}{2} - 2(4) < \frac{4^2}{4} < \frac{4^2}{2}.$

- For all polynomials $f(n) = O\big(g(n)\big)\ and = \Omega\big(g(n)\big), f(n) = \Theta\big(g(n)\big).$
- $n^2, n^2 + n, n^2 - n, n^2 + 1000n, n^2 - 1000n,$ etc… are all $\in O(n^2), \Omega(n^2)$
  - $\therefore f(n) = \Theta(n^2).$

3.2. Growth of Functions

Chain rule: Asymptotic notations in equations:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n) = \Theta(n^2)$$

Interpretation:

1. There exists $f(n) \in \Theta(n)$ s.t. $2n^2 + 3n + 1 = 2n^2 + f(n)$.
2. For all $g(n) \in \Theta(n)$, there exists $h(n) \in \Theta(n^2)$ s.t. $2n^2 + g(n) = h(n)$.

3.2. Growth of Functions

Asymptotic Notation: o −notation

- If $f(n) < cg(n)$ for $c, n_0 > 0$ and $\forall n \geq n_0$, then $f(n) = o\big(g(n)\big)$.
- $g(n)$ is an **upper bound that is not asymptotically tight** for $f(n)$.
- $f(n)$ becomes insignificant relative to $g(n)$ as *n* approaches infinity:
  - $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
- Consider $f(n) = 2n.$
  - $f(n)$ is tightly and asymptotically bounded to $O(n)$ or
  - But not to $o(n^2)$.
  - We still can say $2n = o(n^2) = O(n)$.
  - $\lim_{n\to\infty} \frac{2n}{n^2} = 0$
  - Note: $2n^2 \neq o(n^2)$ but $= o(n^3) = O(n^2)$.

3.2. Growth of Functions

Asymptotic Notation: ω −notation

- If $f(n) > cg(n) \geq 0$ for $c, n_0 > 0$ and $\forall n \geq n_0$, then $f(n) = \omega\big(g(n)\big)$.
- $g(n)$ is a **lower bound that is not asymptotically tight** for $f(n)$.
- $f(n) \in \omega\big(g(n)\big)$ if and only if $g(n) \in o\big(f(n)\big)$.
- $f(n)$ becomes arbitrarily large relative to $g(n)$ as n approaches infinity:
  - $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$
- Consider $f(n) = \frac{n^2}{2}$,
  - $\frac{n^2}{2} > cn \Rightarrow f(n) = \omega(n) = \Omega(n^2)$.
  - But $f(n) \neq \omega(n^2)$.

3.2. Growth of Functions

The analogy between asymptotic comparison of two functions f and g and the comparison of two real numbers a and b:

$f(n) = O(g(n))$	$a \leq b$
$f(n) = \Omega(g(n))$	$a \geq b$
$f(n) = \Theta(g(n))$	$a = b$
$f(n) = o(g(n))$	$a < b$
$f(n) = \omega(g(n))$	$a > b$

- $f(n)$ is asymptotically **smaller** than $g(n)$ if $f(n) = o\big(g(n)\big)$.
- $f(n)$ is asymptotically **larger** than $g(n)$ if $f(n) = \omega\big(g(n)\big)$.

3.2. Growth of Functions

Transitivity	$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$ (same for $O, \Omega, o,$ & ω)
Reflexivity	$f(n) = \Theta(f(n)), f(n) = O(f(n)), f(n) = \Omega(f(n))$. (not for o & ω)
Symmetry	$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
Transpose symmetry	$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$. $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

3.2. Growth of Functions

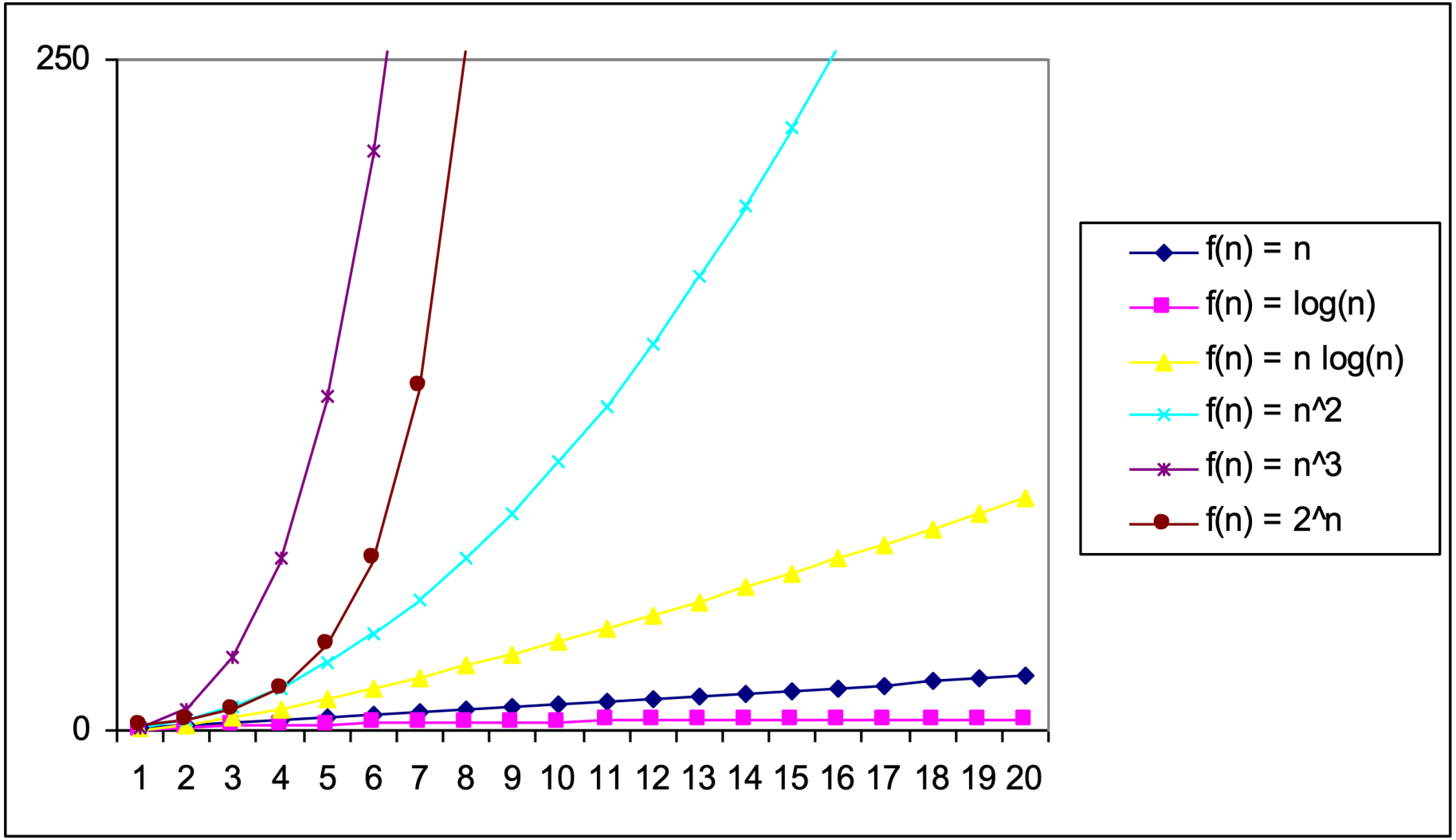
Monotonicity:

- $f(n)$ is **monotonically** increasing if $m \leq n$ implies that $f(m) \leq f(n)$.
- $f(n)$ is **monotonically** decreasing if $m \leq n$ implies that $f(m) \geq f(n)$.
- $f(n)$ is **strictly** increasing if $m < n$ implies that $f(m) < f(n)$.
- $f(n)$ is **strictly** decreasing if $m < n$ implies that $f(m) > f(n)$.

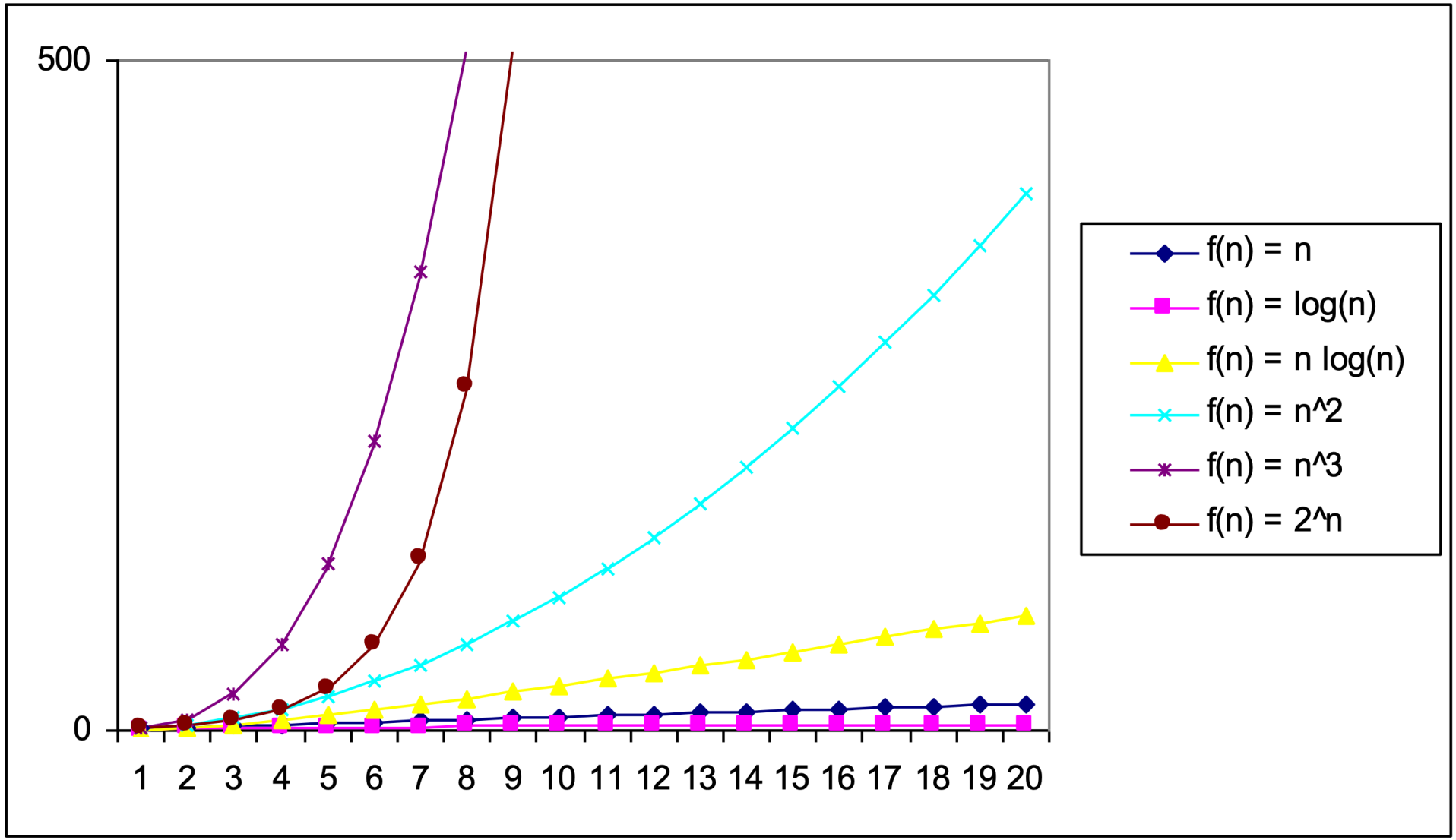
Floors and Ceilings:

- For any real number x, we denote **the greatest integer less than or equal** to by $\lfloor x \rfloor$.
  - We read "**the floor of x**". E.g., $\lfloor 2.3 \rfloor = 2$.
- For any real number x, we denote **the least integer greater than or equal** to by $\lceil x \rceil$.
  - We read "**the ceiling of x**". E.g., $\lceil 2.3 \rceil = 3$.
- $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
- The floor function $f(x) = \lfloor x \rfloor$ is monotonically increasing, as is the ceiling function $f(x) = \lceil x \rceil$.

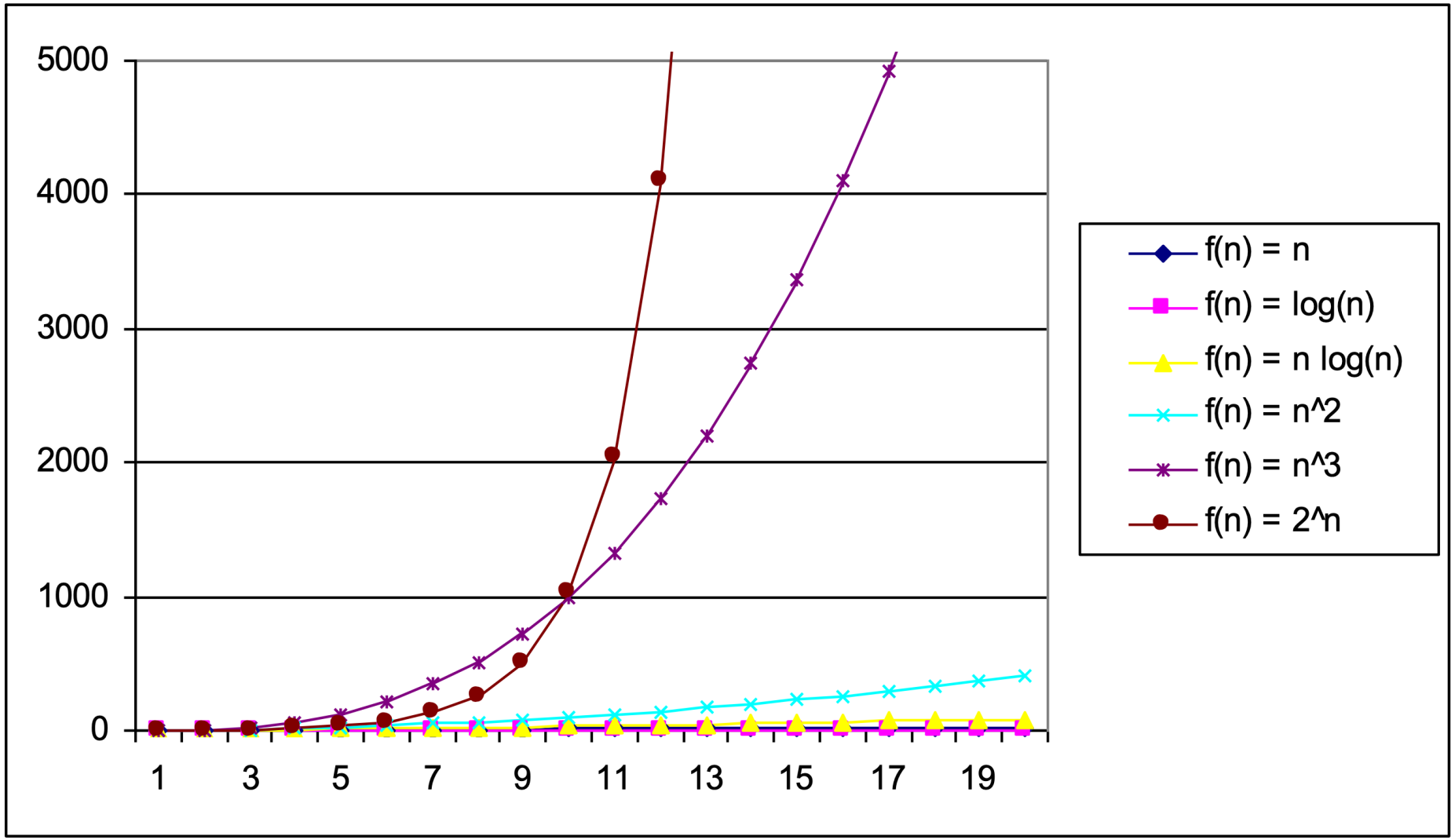
3.2. Growth of Functions



3.2. Growth of Functions



3.2. Growth of Functions



3.2. Growth of Functions

- Order of growth gives a simple characterization of the algorithm's efficiency.
- Allows us to compare the relative performance of alternative algorithms.
- For large enough n, the running time $\Theta(n \lg n)$ beats $\Theta(n^2)$: merge sort beats insertion sort for large enough n. (We will see this later)
- Asymptotic efficiency of algorithms ⇒ input size large enough s.t. order of growth is running time relevant.
- Asymptotically more efficient algorithm usually the best choice for all n (except for small n).
- Used in analysis of algorithms.

3.3. Insertion Sort

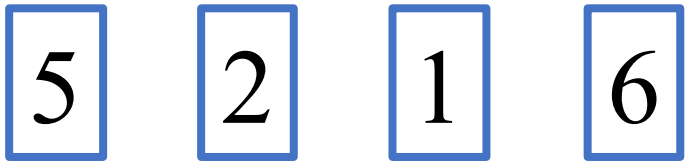
3.3. Insertion Sort

- Input: A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$
- Output: A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that
  - $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Definition:

- **Keys** – the numbers that we wish to sort

In this problem, even though we are sorting a sequence, the input comes to us in the form of an array with n elements.



3.3. Insertion Sort

Algorithm (Insertion sort)

INPUT: $A[1 \ldots n]$
OUTPUT: $A[1 \ldots n]$ *sorted*

```
(1)  for (2 ≤ j ≤ n) do
(2)      key = A[j]
(3)      /* Insert A[j] into the sorted sequence A[1..j − 1]. */
(4)      i = j − 1
(5)      while (i > 0 and A[i] > key) do
(6)          A[i + 1] = A[i]
(7)          i = i − 1
(8)      A[i + 1] = key
(9)  od
```

3.3. Insertion Sort

5	2	1	6

Line 1: j = 2

Line 2: j =2, key = A[2]=2

Line 4: i = 1

Line 5: j = 2, key = 2, i = 1, and A[1] > 2

Line 6: j = 2, key = 2, A[2] = A[1] ,

A[1]=? & A[2]=?

Line 7: j =2, key = 2, i = 0

Line 8: j =2, key = 2, A[2] = 5, i = 0, A[1] = 2 (why?)

Algorithm (Insertion sort)

INPUT: $A[1 \ldots n]$
OUTPUT: $A[1 \ldots n]$ *sorted*

```
(1)  for (2 ≤ j ≤ n) do
(2)     key = A[j]
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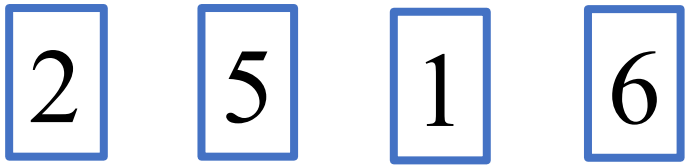
(4)     i = j − 1
(5)     while (i > 0 and A[i] > key) do
(6)        A[i + 1] = A[i]
(7)        i = i − 1
(8)     A[i + 1] = key
        od
```

2	5	1	6

3.3. Insertion Sort



Algorithm (Insertion sort)

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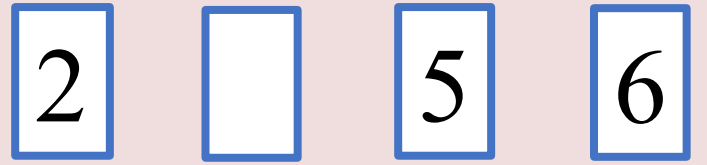
(9)  od
```



Line 1: j =3, key = 2, i = 0, A[1] = 2

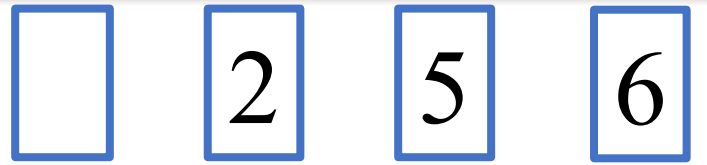
Line 2: j =3, key = A[3]=1

Line 4: i = 2

Line 5: j = 3, key = 1, i = 2, A[2]=5 > 1

Line 6: j = 3, key = 1, i = 2, A[3] = A[2]=5

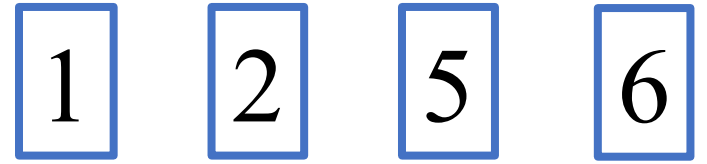
Line 7: j =3, key = 1, A[2] = ?, A[3]=5, i = 1, A[1]=2



Line 5: j = 3, key = 1, i = 1, A[1]=2 > 1

Line 6: j = 3, key = 1, i = 1, A[2] = A[1]=2

Line 7: j =3, key = 1, A[2] = 2, A[1]=?, i = 0



Line 5: j = 3, key = 1, i = 0, Fail the while-loop condition

Line 8: j =3, i = 1, key = 1, A[1]=1

3.3. Insertion Sort

1	2	5	6

Line 1: j =4, key = 1, i = 1, A[1] = 1

Line 2 & 4: j =4, key = 6, i = 3

Line 5: j =4, key = 6, i = 3, A[3] < 6

Step 6 &7: SKIP

Line 8: j =4, i = 3, A[4] = 6, key = 6

1	2	5	6

Algorithm (Insertion sort)

INPUT: $A[1 \ldots n]$
OUTPUT: $A[1 \ldots n]$ *sorted*

```
(1)  for (2 ≤ j ≤ n) do
(2)     key = A[j]
(3)     /* Insert A[j] into the sorted sequence A[1..j − 1]. */
(4)     i = j − 1
(5)     while (i > 0 and A[i] > key) do
(6)        A[i + 1] = A[i]
(7)        i = i − 1
(8)     A[i + 1] = key
(9)  od
```

3.3. Insertion Sort

We often use a **loop invariant** to help us understand why an algorithm gives the correct answer.

Loop Invariant:

- At the start of each iteration of the "outer" for loop – the loop indexed by j.
- The subarray $A[1, \cdots, j-1]$ consists of the elements originally in $A[1, \cdots, j-1]$ but in sorted order.

To prove correctness, we must show three things about loop invariant:

- **Initialization** – It is true prior to the first iteration of the loop.
- **Maintenance** – If it is true before an iteration of the loop, it remains true before the next iteration.
- **Termination** – When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

3.3. Insertion Sort

Initialization:

- Show the loop invariant holds before the 1st loop: $j = 2$
- The subarray $A[1, \cdots, j-1]$ consists the single element $A[1]$ which is the original element in $A[1]$.
- This subarray is sorted showing the loop invariant holds prior to the first iteration of the loop.

Algorithm (Insertion sort)

INPUT: $A[1 \ldots n]$
OUTPUT: $A[1 \ldots n]$ *sorted*

```
(1)  for (2 ≤ j ≤ n) do
(2)     key = A[j]
(3)     /* Insert A[j] into the sorted sequence A[1..j − 1]. */
(4)     i = j − 1
(5)     while (i > 0 and A[i] > key) do
(6)        A[i + 1] = A[i]
(7)        i = i − 1
(8)     A[i + 1] = key
(9)  od
```

3.3. Insertion Sort

Maintenance:

- Each iteration maintains the loop invariant – for loop works by moving $A[j-1], A[j-2], A[j-3]$, ... by one position to the right until it finds the proper position for $A[j]$ at point it inserts the value of $A[j]$
- The subarray consists in sorted order.
- Increment j for the next iteration of the for loop then preserves the loop invariant.

Algorithm (Insertion sort)

INPUT: $A[1 \ldots n]$
OUTPUT: $A[1 \ldots n]$ *sorted*

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(6)        A[i + 1] = A[i]
(7)        i = i − 1
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(9)  od
```

3.3. Insertion Sort

Termination:

- The for loop terminates when $j > A.length = n$ and we have $j = n + 1$ at that time.
- We have subarray $A[1, \cdots, n]$ in sorted order.

Algorithm (Insertion sort)

INPUT: $A[1 \ldots n]$
OUTPUT: $A[1 \ldots n]$ *sorted*

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3.3. Insertion Sort

Algorithm (Insertion sort)

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(9)  od
```

Why n?

Line	Cost	Times
(1)	c_1	n
(2)	c_2	$n-1$
(4)	c_4	$n-1$
(5)	c_5	$\sum_{j=2}^{n} t_j$
(6)	c_6	$\sum_{j=2}^{n}(t_j-1)$
(7)	c_7	$\sum_{j=2}^{n}(t_j-1)$
(8)	c_8	$n-1$

- Assume the j^{th} line takes time **constant** c_j.
- Let t_j be the number of times that the while loop test executed for that value of j.
- Note that when a *for* or *while* loop exists in the usual way, **the test is executed one time more than the loop body.**

3.3. Insertion Sort

The running time $T(n)$ of the algorithm is $\sum(\textbf{cost}) \cdot (\#\ \textbf{of times is executed})$:

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- Running time depends on **size of the input,** n**,** and t_j.
- Value of t_j depends on the size and the input itself.

3.3. Insertion Sort

If the array is already sorted (**the best case**), $t_j = 1$,

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

$$\sum_{j=2}^{n} t_j = (n-1) \qquad \sum_{j=2}^{n} (t_j - 1) = 0$$

$$= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$$

$$= (c_1 + c_2 + c_4 + c_5 + c_8)n + (c_2 + c_4 + c_5 + c_8)$$

- $T(n)$ is a **linear function** of n.
- $T(n) = \Theta(n)$

3.3. Insertion Sort

If an array is **reverse sorted order** (**worst-case**):

- Always find that $\boldsymbol{A[i] > key}$ in while loop test.
- Has to compare key with all elements to the left of the jth position.
- Since the while loop exists because i reaches 0, there is one additional test after $j-1$ tests $\Rightarrow$ $t_j = j$.
- $\sum_{j=2}^{n} t_i = \sum_{j=2}^{n} j$ and $\sum_{j=2}^{n}(t_i - 1) = \sum_{j=2}^{n}(j-1)$.
- Using $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$,

$$\sum_{j=2}^{n} j = \left(\sum_{j=1}^{n} j\right) - 1 = \frac{n(n+1)}{2} - 1.$$

$$\sum_{j=2}^{n}(j-1) = \left(\sum_{j=1}^{n}(j-1)\right) - 1 = \frac{n(n-1)}{2}.$$

3.3. Insertion Sort

The running time $T(n)$ is is a quadratic function of n:

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

$$\sum_{j=2}^{n} t_j = \frac{n(n+1)}{2} - 1 \qquad \sum_{j=2}^{n} (t_j - 1) = \frac{n(n-1)}{2}$$

$$= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) + (c_6 + c_7)\left(\frac{n(n-1)}{2}\right) + c_8(n-1)$$

$$= \frac{n^2}{2}(c_5 + c_6 + c_7) + n\left(c_1 + c_2 + c_4 + c_8 + \frac{1}{2}(c_5 - c_6 - c_7)\right) - (c_2 + c_4 + c_5 + c_8)$$

- $T(n)$ is a **quadratic function** of n.
- $T(n) = \mathrm{O}(n^2)$

3.3. Insertion Sort

We usually concentrate of finding the worst-case running time:

- Worst-case gives us a guaranteed upper bound on the running time for any input.
- For some algorithms, the worst case occurs fairly often, e.g., searching for absent information might happen quite often.
- Average case is often as bad as the worst case:
  - On average half of the elements in $A[1, \cdots, j-1]$ are less than $A[j]$ and half are greater.
    - $\Rightarrow t_j$ is on average $\frac{j}{2}$ and the while loop has to look halfway through the already sorted subarray $A[1, \cdots, j-1]$.
    - $\Rightarrow$ The average-case running time is a quadratic function of n.

3.3. Insertion Sort

We used simplified abstractions:

- Ignored the actual cost of each statement.
- But still observed that constants gave more than needed.
- We still can make this analysis even simpler – order of growth of the running time
  - Consider only the *leading* term
  - Ignore the coefficient of leading term
  - Example: $T(n) = an^2 + bn + c \Rightarrow T(n) \approx an^2 \Rightarrow T(n) \rightarrow \Theta(n^2)$

3.4. Merge Sort

3.4. Merge Sort

For insertion sort, we used an incremental approach.
We can also use “divide-and-conquer” method to design a sorting algorithm.

- The worst-case running time is much less than the incremental approach.
- They break the problem into several subproblems that are similar to the original problem but
  - Divide smaller in size
  - Conquer the subproblems recursively
  - Combine there solutions to create a solution to the original problem.

3.4. Merge Sort

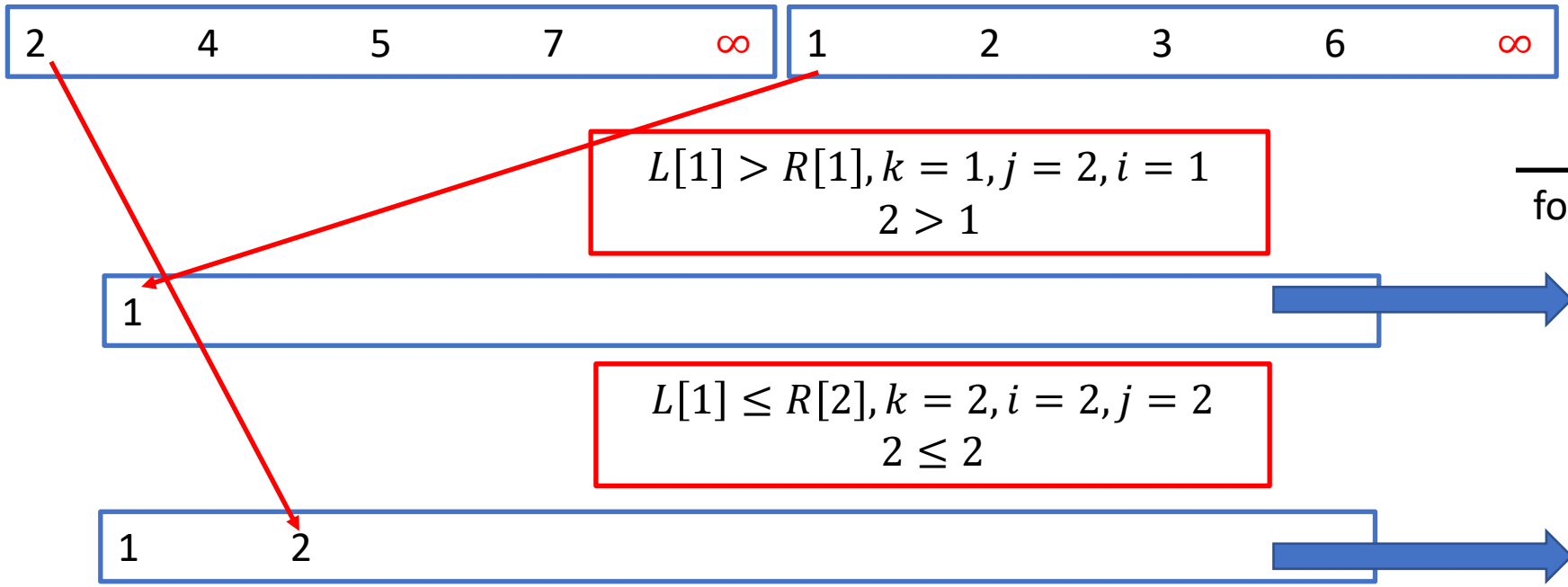
Merge sort is based in divide-and-conquer – worst-case running time is better than insertion sort.

- Divide by splitting into two subarrays $A[p, \cdots, q]$ and $A[q+1, \cdots, r]$ where q is the middle point of $A[p, \cdots, r]$.
- Conquer by recursively sorting these two subarrays.
- Combine by merging two sorted subarrays $A[p, \cdots, q]$ and $A[q+1, \cdots, r]$ to generate a single sorted subarray. We define procedure $\text{Merge}(A, p, q, r)$ to accomplish that step.

The recursion stops when the subarray consists of only one element. One element arrays are trivially sorted.

3.4. Merge Sort

2	4	5	7	1	2	3	6	A
2	4	5	7	1	2	3	6	L & R
1	2	2	3	4	5	6	7	A (sorted)



for $(1 \leq k \leq 8)$ do

  if $(L[i] \leq R[j])$ then

    $A[k] = L[i]$

    $i = i + 1$

  else

    $A[k] = R[j]$

    $j = j + 1$

Done

3.4. Merge Sort

2	4	5	7	1	2	3	6	A
2	4	5	7	1	2	3	6	L & R
1	2	2	3	4	5	6	7	A (sorted)

2	4	5	7	∞

1	2	3	6	∞

$k = 3, L[2] > R[2] \rightarrow 4 > 2, i = 2, j = 3$
$k = 4, L[2] > R[3] \rightarrow 4 > 3, i = 2, j = 4$
$k = 5, L[2] \leq R[4] \rightarrow 4 \leq 6, i = 3, j = 4$
$k = 6, L[3] \leq R[4] \rightarrow 5 \leq 6, i = 4, j = 4$
$k = 7, L[4] > R[4] \rightarrow 7 > 6, i = 4, j = 5$
$k = 8, L[4] \leq R[5] \rightarrow 7 \leq \infty, i = 4, j = 6$

1	2	2	3	4	5	6	7

$i = 1, j = 1$
for $(p \leq k \leq r)$ do
    if $(L[i] \leq R[j])$ then
        $A[k] = L[i]$
        $i = i + 1$
    else
        $A[k] = R[j]$
        $j = j + 1$

3.4. Merge Sort – Merge(A,p,q,r)

(1)	$n_1 = q - p + 1, n_2 = r - q$	Computes lengths
(2)	Let $L[1 \dots n_1 + 1]$ and $R[1 \dots n_2 + 1]$ be new arrays	Create arrays L and R
(3)	for $(1 \leq i \leq n_1)$ do	Copies the subarray into L
(4)	$L[i] = A[p + i - 1]$	
(5)	for $(1 \leq j \leq n_2)$ do	Copies the subarray into R
(6)	$R[j] = A[q + j]$	
(7)	$L[n_1 + 1] = \infty, R[n_2 + 1] = \infty$	Put sentinels at the ends of the arrays L & R
(8)	$i = 1, j = 1$	
(9)	for $(p \leq k \leq r)$ do	The subarray $A[p \dots k - 1]$ contains the $k - p$ smallest
(10)	if $(L[i] \leq R[j])$ then	Elements of $L[i \dots n_1 + 1]$ & $R[1 \dots n_2 + 1]$, in sorted orde
(11)	$A[k] = L[i]$	$L[i]$ and $R[j]$ are the smallest elements of their
(12)	$i = i + 1$	Arrays that have not been copied back into A.
(13)	else	
(14)	$A[k] = R[j]$	
(15)	$j = j + 1$	
(16)	Done	

*MERGE takes time $\Theta(n)$ where $n = r - p + 1$ is the total number of elements being merged.

- Line 1-2 & 7-8 takes constant time.
- Line 3-6 takes $\Theta(n_1 + n_2) = \Theta(n)$
- Line 12-17 takes constant time in for loop n iterations

3.4. Merge Sort

Each iteration of the loop maintains the invariant that provides a useful property to show correctness when the loop terminates.

Initialization:

- Prior to the first iteration of the loop, we have k=p and $A[p \ldots k-1]$ is empty.
- This contains the smallest elements of L and R.
- Since $i = j = 1$, both L[i] and R[j] are the smallest elements of their arrays that have not been copied back into A.

3.4. Merge Sort

Maintenance:

- Suppose $L[i] \leq R[j]$, then $L[i]$ is the smallest element not yet copied back into A because $A[p \ldots k-1]$ contains the k-p smallest elements.
- After line 11 copies L[i] into A[k], the subarray $A[p \ldots k]$ will contain the k-p+1 smallest elements. The for loop updates and reestablishes the loop invariant for the next iteration.
- Same idea for a case of $L[i] > R[j]$.

Termination:

- Terminates when $k = r + 1$.
- By loop invariant, the subarray contains the smallest element of $L[1 \ldots n_1 + 1]$ and $R[1 \ldots n_2 + 1]$ in sorted order.
- Together contains $n_1 + n_2 + 2 = r - p + 3$ elements.
- All but the two largest (the sentinels) have been copied back into A.

3.4. Merge Sort

Using the Merge procedure, we can make MERGE-SORT(A, p, r):

- If $p \geq r$, the subarray has at most one element and is therefore already sorted.
- Otherwise, divide step simply computes an index q that partitions $A[p \ldots r]$ into two subarrays.
- $A[p \ldots q]$ contains n/2 elements and $A[q + 1 \ldots r]$ contains n/2 elements.

```
MERGE-SORT(A,p,r)
(1)   if (p < r) then
(2)       q = ⌊(p + r)/2⌋
(3)       MERGE-SORT(A,p,q)
(4)       MERGE-SORT(A,q+1,r)
(5)       MERGE(A,p,q,r)
```

The initial call is MERGE-SORT(A,1,n)

3.4. Merge Sort

```
(1)  if (p < r) then
(2)      q = ⌊(p + r)/2⌋
(3)      MERGE-SORT(A,p,q)
(4)      MERGE-SORT(A,q+1,r)
(5)      MERGE(A,p,q,r)
```

$$MS(A, 1,8): p = 1, r = 8, q = 4$$
$$MS(A, p = 1, q = 4) \mid MS(A, q + 1 = 5, r = 8):$$
$$MS(A, p = 1, q = 2) \mid MS(A, q + 1 = 3, r = 4) \mid MS(A, p = 5, q = 6) \mid MS(A, q + 1 = 7, r = 8)$$
$$MS(A, p = 1, q = 1) \mid MS(A, q + 1 = 2, r = 2) \mid MS(A, p = 3, q = 3) \mid MS(A, q + 1 = 4, r = 4)$$
$$M(A, p = 1, q = 1, r = 2) \mid M(A, p = 3, q = 4, r = 4)$$
$$M(A, p = 1, q = 2, r = 4)$$
$$MS(A, p = 5, q = 5) \mid MS(A, q + 1 = 6, r = 6) \mid MS(A, p = 7, q = 7) \mid MS(A, q + 1 = 8, r = 8)$$
$$M(A, p = 5, q = 6, r = 6) \mid M(A, p = 7, q = 7, r = 8)$$
$$M(A, p = 5, q = 6, r = 8)$$
$$M(A, p = 1, q = 4, r = 8)$$

3.4. Merge Sort

```
(1)  if (p < r) then
(2)      q = ⌊(p + r)/2⌋
(3)      MERGE-SORT(A,p,q)
(4)      MERGE-SORT(A,q+1,r)
(5)      MERGE(A,p,q,r)
```

5	2	4	7	1	3	2	6	(original array)
5	2	4	7	1	3	2	6	(recursion)
5	2	4	7	1	3	2	6	(recursion)
5	2	4	7	1	3	2	6	(sub problems sorted, merge)
2	5	4	7	1	3	2	6	(merge)
2	4	5	7	1	2	3	6	(merge)
1	2	2	3	4	5	6	7	(elements sorted)

3.4. Merge Sort

When an algorithm contains a recursive call to itself, we use a recurrence equation (more commonly, recurrence) to describe the running time of a divide-and-conquer algorithm.

- The recurrence equation is

$$T(n) = \begin{cases} \Theta(1) & if\ n \leq c, \\ aT\left(\frac{n}{b}\right) + D(n) + C(n) & otherwise. \end{cases}$$

- Base case: if the problem size is small enough, $n \leq c$, the base takes constant time $\Theta(1)$.
- Otherwise, divide into a number of a subproblems of size $1/b$
  - There are a subproblems to solve, each of size n/b.
  - Each subproblem takes $T(n/b)$ times to solve and we spend $aT(n/b)$ time solving subproblems.
- The time to divide a size-n problem is $D(n)$.
- The time to combine solutions is $C(n)$.

3.4. Merge Sort

For simplicity, we assume that $n = 2^m$.

- Two subproblems of size $\frac{n}{2}$ in each step.
- Base case for $n = 1$, $T(n) = \Theta(1)$.
- For $n \geq 2$:
  - Divide: $D(n) = \Theta(1)$ (Compute q as the average of p and r is constant time).
  - Conquer: Solve 2 subproblems of size $n/2$ recursively: $a = b = 2 \Rightarrow 2T\left(\frac{n}{2}\right)$.
  - Combine: $C(n) = \Theta(n)$. MERGE on an n-element subarray takes linear time.
  - The total running Time is

$$T(n) = \begin{cases} \Theta(1) & if\ (n = 1), \\ 2T\left(\frac{n}{2}\right) + \Theta(n) + \Theta(1) & if\ (n > 1). \end{cases}$$

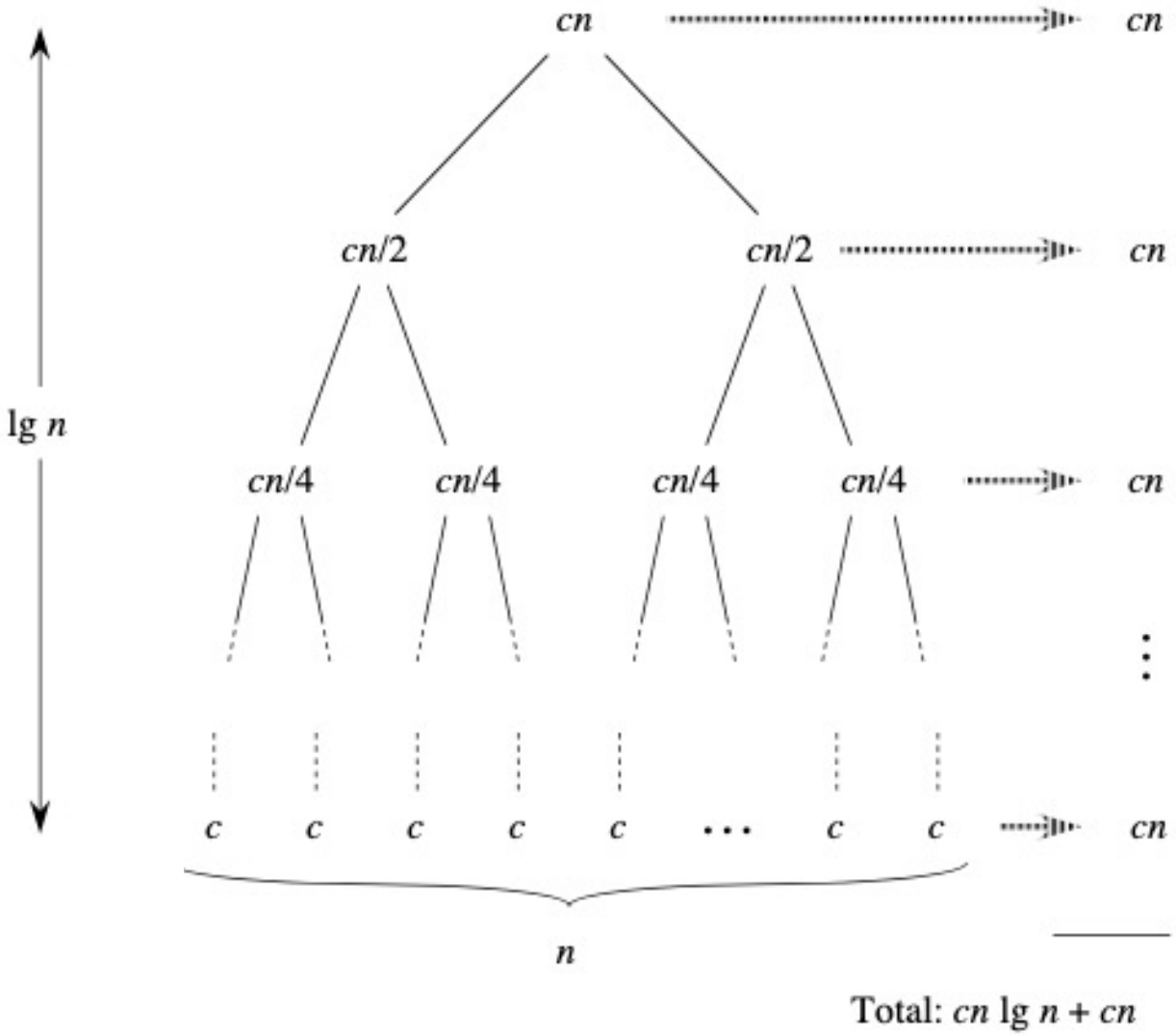
3.4. Merge Sort

- Running time for merge-sort is $T(n) = \Theta(n \lg n)$ where $\lg n = \log_2 n$.
- Rewrite recurrence as

$$T(n) = \begin{cases} c & if\ (n = 1), \\ 2T\left(\frac{n}{2}\right) + c_1 n & if\ (n > 1). \end{cases}$$

- Draw the recursion tree to solve.

3.4. Merge Sort



Each level has cost cn.

- Each time we go down one level, the number of subproblems doubles but the cost per subproblem halves ⇒ cost/level stays the same.

There are $\lg n + 1$ levels (height is $\lg n$).
Total cost is sum of costs at each level.

- Total cost is $cn \lg n + cn \Rightarrow \Theta(n \lg n)$.

3.4. Merge Sort

Recall the divide-and-conquer technique:

- Divide the problem into a number of subproblems that are small instances of same problem.
- Conquer the subproblem by use of recursion. Small enough or trivial subproblems (base case) are solved in a straightforward manner.
- Combine the subproblems solutions into the solution for the original problem.

We use recurrence to characterize the running time of a divide-and-conquer algorithm and its solution gives us the asymptotic running time.

Homework Discussion

3.5. Homework Discussion

The first homework will be open this Friday @ 12 AM.
Information

- Due: 2/24th Friday 11:59 PM
- Headers and other necessary files will be provided in a zip file together with the description in docx file.
- Students need to work on the source file "sort.cpp".
- This assignment requires a report.
  - Include the result tables and plots
  - Explain the major finding - the behavior of running time.
  - Evaluate the result – agreement or disagreement with theoretical running time.
- When you submit, make sure to compress only code files.
  - **Do not just submit the sort.cpp file.**
  - Submit the compressed file and the report in a docx file (not in pdf).
- Keep in mind that
  - The description is written for C++ users. If you are going to use other than C++, you have to measure the running time on your own.
  - There will be a penalty (-10%) for not using C++ from HW#2.

3.5. Homework Discussion

Homework Description

You are given an integer vector which is represented by *int** an array of integers and its dimension *n* as a separate parameter. We are interested in sorting arrays of integer vectors according to a pre-defined notion of vector length. You therefore are given the function *ivector_length(v, n)* that computes and returns the length of vector *v* with dimension *n* as $\sum_{i=1}^{n} |v_i|$.

You are given a naive (and very inefficient) implementation of insertion sort for arrays of integer vectors.

3.5. Homework Discussion

Questions (100 points)

1. Develop an improved implementation of **insertion sort** for integer vector (*insertion_sort_im*) that precomputes the length of each vector before the sorting. Keep in mind that the vectors are sorted according to their length (see *ivector_length* function). You can test the correctness of your sorting algorithm using the provided *check_sorted* function.

2. Implement a **merge sort** for an array of integer vectors. For this implementation of the merge sort, as is the case for the improved insertion sort algorithm, you should precompute the length of the vectors before the sorting, and the sorting is done according to the vector lengths. Test the correctness of your merge sort implementation using the provided *check_sorted* function.

3.5. Homework Discussion

3. Measure the runtime performance of insertion sort (naive and improved) and merge sort for random, sorted, and inverse sorted inputs of size m = 10000; 25000; 50000; 100000; 250000; 500000; 1000000; 2500000 and vector dimension n = 10; 25; 50. You can use the provided functions *create_random_ivector*, *create_sorted_ivector, create reverse_sorted_ivector.*

Repeat each test a number of times (usually at least 10 times) and compute the average running time for each combination of algorithm, input, size m, and vector dimension n. Report and comment on your results.

3.5. Homework Discussion

Sample test case:
Consider input [[4,2,1], [9,2,1], [1,2,1]],
Output should be [[1,2,1], [4,2,1], [9,2,1]]

Explanation:

- 1+2+1 < 4+2+1 < 9+2+1
- Sorting occurs according to sum of numbers at that index as shown above.
- Vector just means an array. Eg. [4,2,1] is a vector
- Length means sum of numbers in that vector. Eg. Length of vector [4,2,1] is 7
  - $[[4,2,1],[9,2,1],[1,2,1]] \rightarrow [7,12,4] \rightarrow [[1,2,1],[4,2,1],[9,2,1]]$

3.5. Homework Discussion

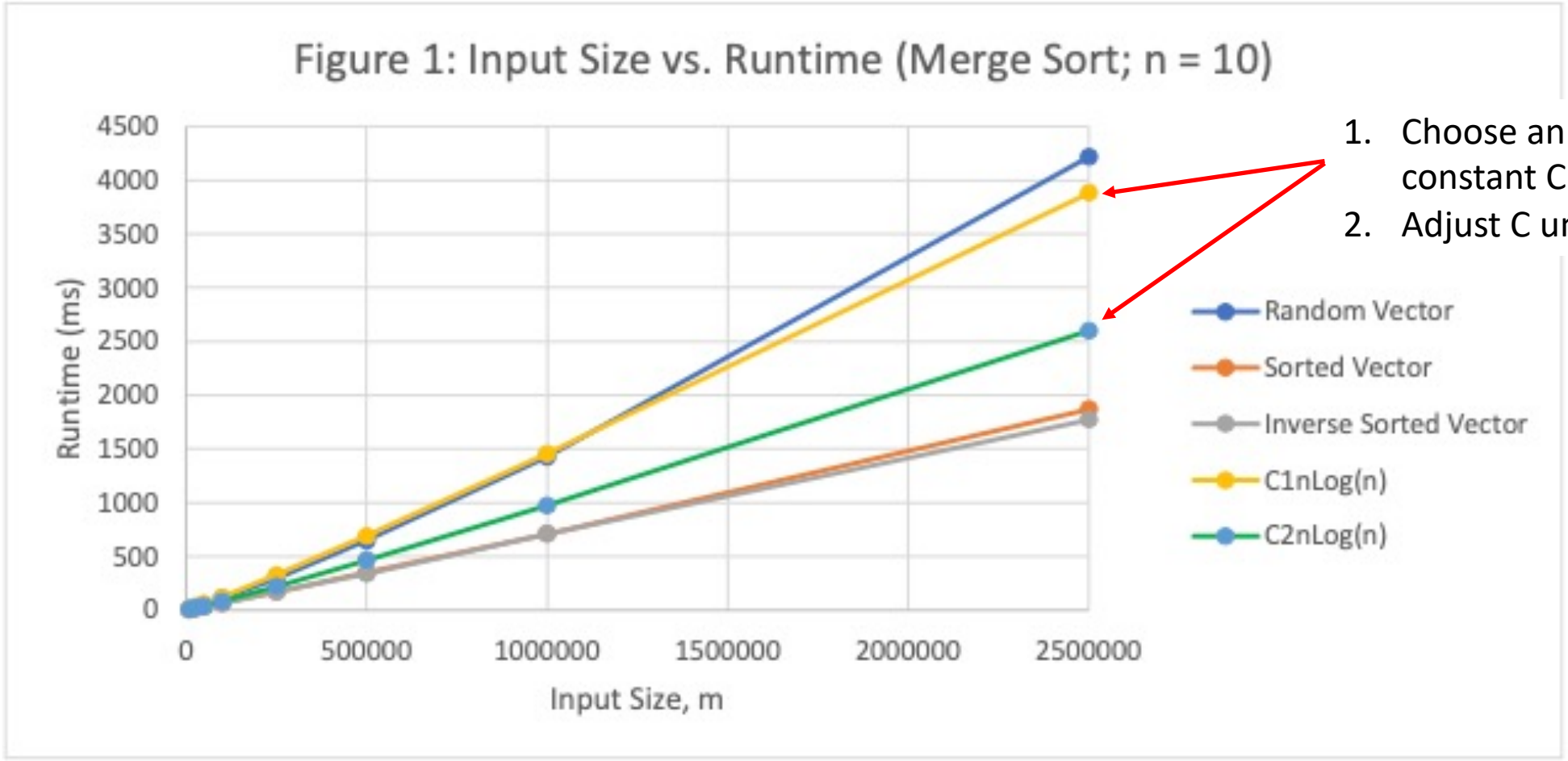
Table 1: Merge Sort Runtime in ms (Average Summary)

	n = 10			n = 25			n = 50		
m	Random Vector	Sorted Vector	Inverse Sorted Vector	Random Vector	Sorted Vector	Inverse Sorted Vector	Random Vector	Sorted Vector	Inverse Sorted Vector
10000	9.7	6.5	6.1	18.5	11.5	11.7	33.1	20.3	18.4
25000	25.7	17.3	17.6	48.1	30.2	28.6	89.2	51.6	48.6
50000	51.7	32.4	31.2	102	58	56.1	187.3	106.2	102.3
100000	109.7	68.4	65.6	212.2	120.5	118.1	402.1	217.7	208.7
250000	292.8	167.3	172.2	595.7	305.1	310.4	1110	550.2	536.5
500000	646.1	348.3	338.4	1251	615.5	618.5	2288.4	1170	1133.8
1000000	1425.6	709.6	710.7	2760.5	1271	1279.7	4936.3	2507	2319.3
2500000	4216.3	1868	1770.3	7730.1	3679	3532.8	14040.5	6820	6628.8

Table 2: Insertion Sort Runtime in ms (Average Summary)

	n = 10			n = 25			n = 50		
m	Random Vector	Sorted Vector	Inverse Sorted Vector	Random Vector	Sorted Vector	Inverse Sorted Vector	Random Vector	Sorted Vector	Inverse Sorted Vector
10000	1135.6	0.5	2172.9	2582.8	1.2	5288.7	5575.9	2.5	10652.4
25000	7760.7	1.2	14010.2	17816.3	2.8	33459.6	38181.2	5.5	69056.1
50000	31252.2	2.1	53042.3	67483.8	5	122766.5	165510	11.4	262562.3
100000	122095.5	4.6	217219.2	352616	12.7	-	-	27.5	-
250000	1202565	11.8	-	-	33	-	-	58.1	-
500000	-	26.7	-	-	57.5	-	-	113.7	-
1000000	-	53.4	-	-	109.6	-	-	225.8	-
2500000	-	120.9	-	-	261.6	-	-	563.1	-

3.5. Homework Discussion



1. Choose an arbitrary constant C.
2. Adjust C until fits.