CS 590: Algorithms

Lecture 10 & 11: Graphs & Graph's algorithm

Graphs

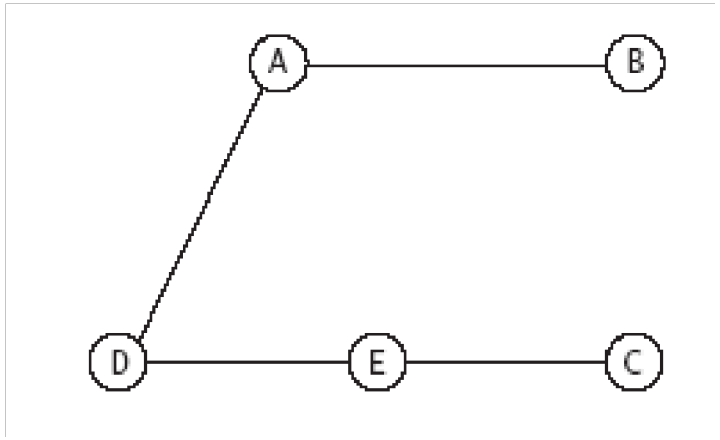
- Trees are limited in that a data structure can only have one parent
- Graphs overcome this limitation
- Graphs were being studied long before computers were invented
- Graphs algorithms run
  - large communication networks
  - the software that makes the Internet function
  - programs to determine the optimal placement of components on a silicon chip
- Graphs describe
  - roads maps
  - airline routes
  - course prerequisites

Graph Terminology

- A graph is a data structure that consists of a set of vertices (or nodes) and a set of edges (relations) between pairs of vertices
- Edges represent paths or connections between vertices
- Both the set of vertices and the set of edges must be finite
- Either set may be empty (if the set of vertices is empty, the set of edges also must be empty)
- We restrict our discussion to simple graphs in which there is at least one edge from a given vertex to another vertex

Visual Representation of Graphs

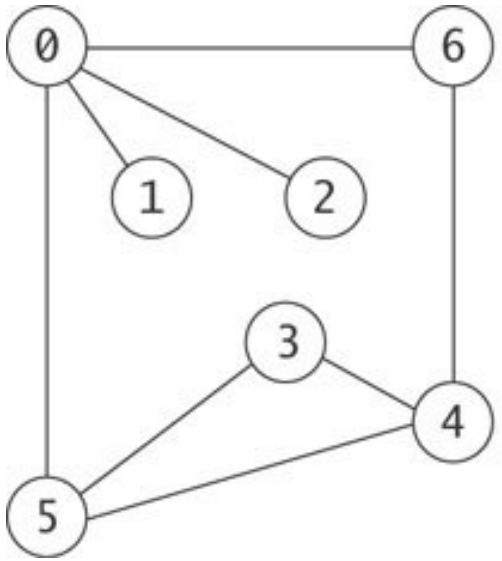
- Vertices are represented as points or labeled circles, and edges are represented as line segments joining the vertices
- Each edge is represented by the two vertices it connects
- If there is an edge between vertices x and y, there is a path from x to y and vice versa

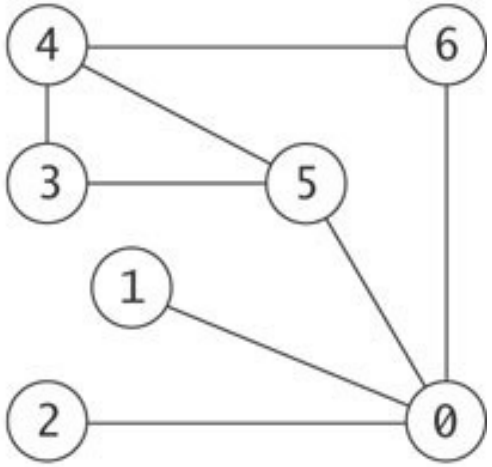


- V = {A, B, C, D, E}
- E = {{A, B}, {A, D}, {C, E}, {D, E}}

Visual Representation of Graphs

- The physical layout of the vertices and their labeling is not relevant

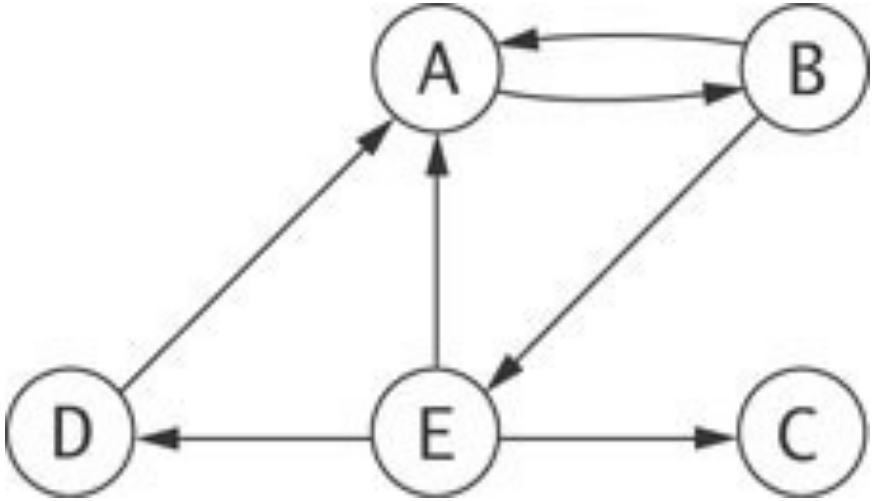




- V = {0, 1, 2, 3, 4, 5, 6}
- E = {{0, 1}, {0, 2}, {0, 5}, {0, 6}, {3, 5}, {3, 4}, {4, 5}, {4, 6}}

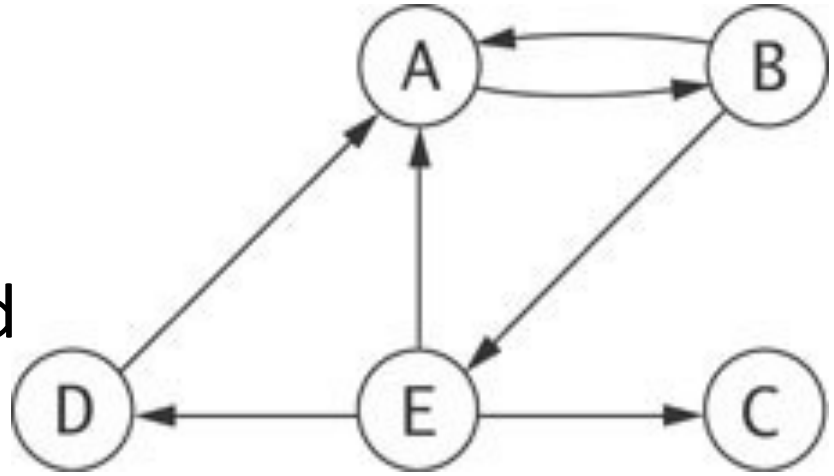
Directed and Undirected Graphs

- The edges of a graph are directed if the existence of an edge from A to B does not necessarily guarantee that there is a path in both directions
- A graph with directed edges is called a directed graph or digraph
- A graph with undirected edges is an undirected graph, or simply a graph



Directed and Undirected Graphs (cont.)

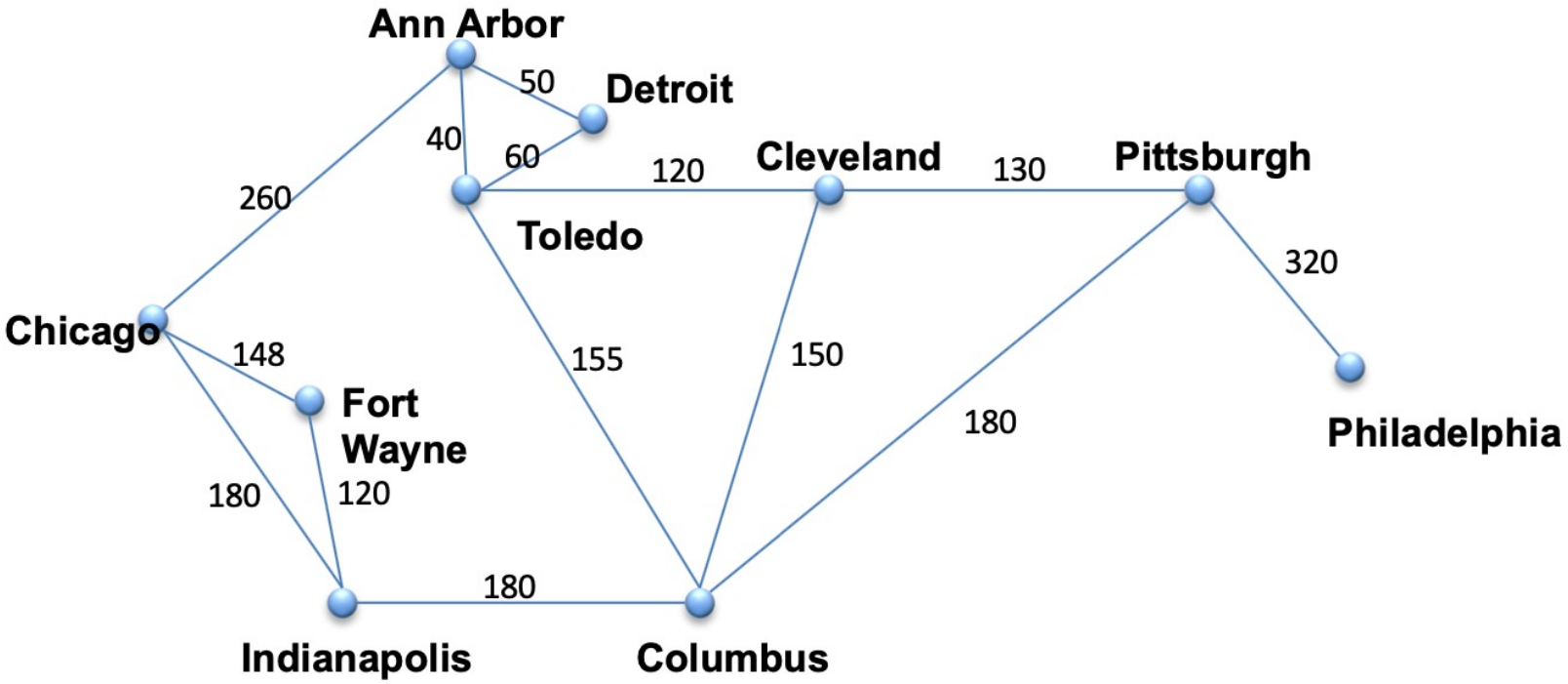
- A directed edge is like a one way street; you can travel in only one direction
- Directed edges are represented as lines with an arrowhead on one end (undirected edges do not have an arrowhead at either end)



- Directed edges are represented by ordered pairs of vertices {source, destination}; the edges for the digraph on this slide are:
  - {{A, B}, {B, A}, {B, E}, {D, A}, {E, A}, {E, C}, {E, D}}

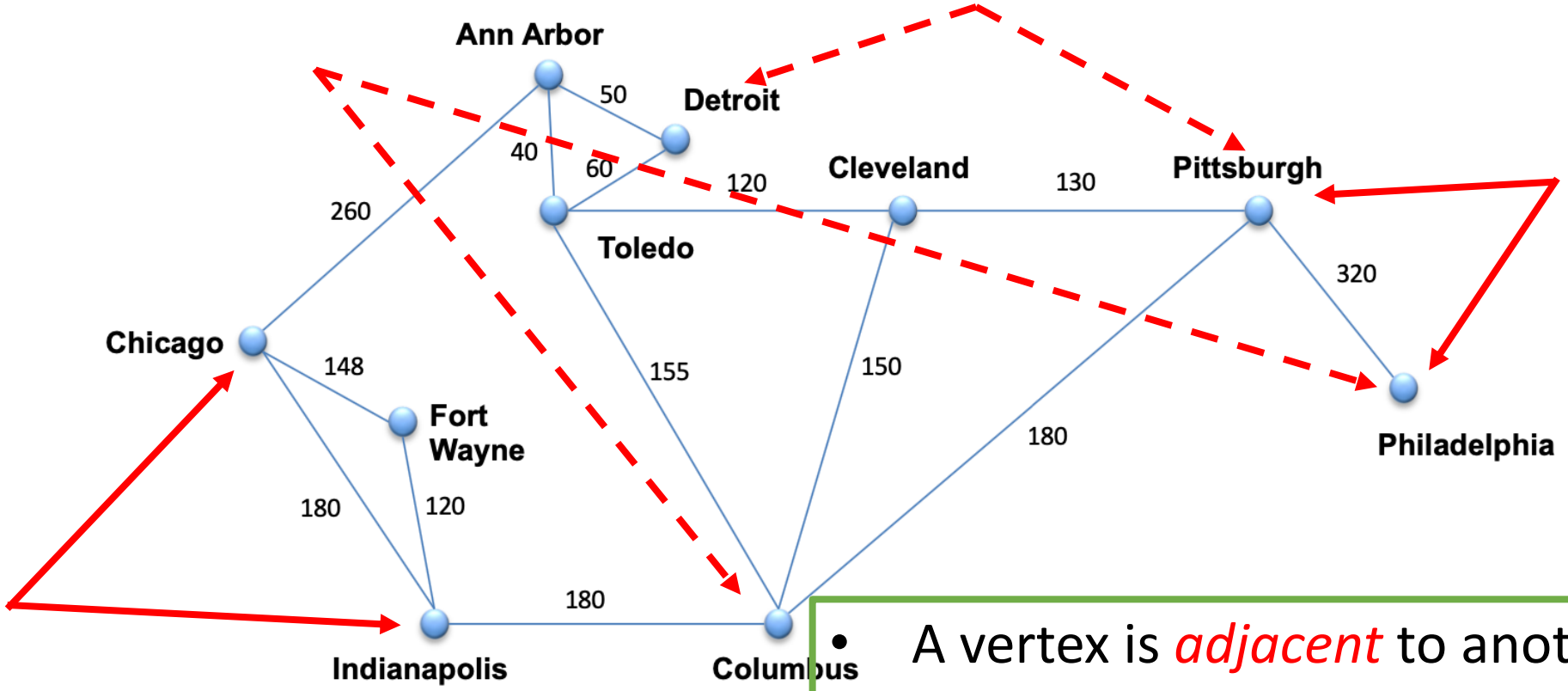
Directed and Undirected Graphs (cont.)

- The edges in a graph may have associated values known as their weights
- A graph with weighted edges is known as a weighted graph



Paths and Cycles

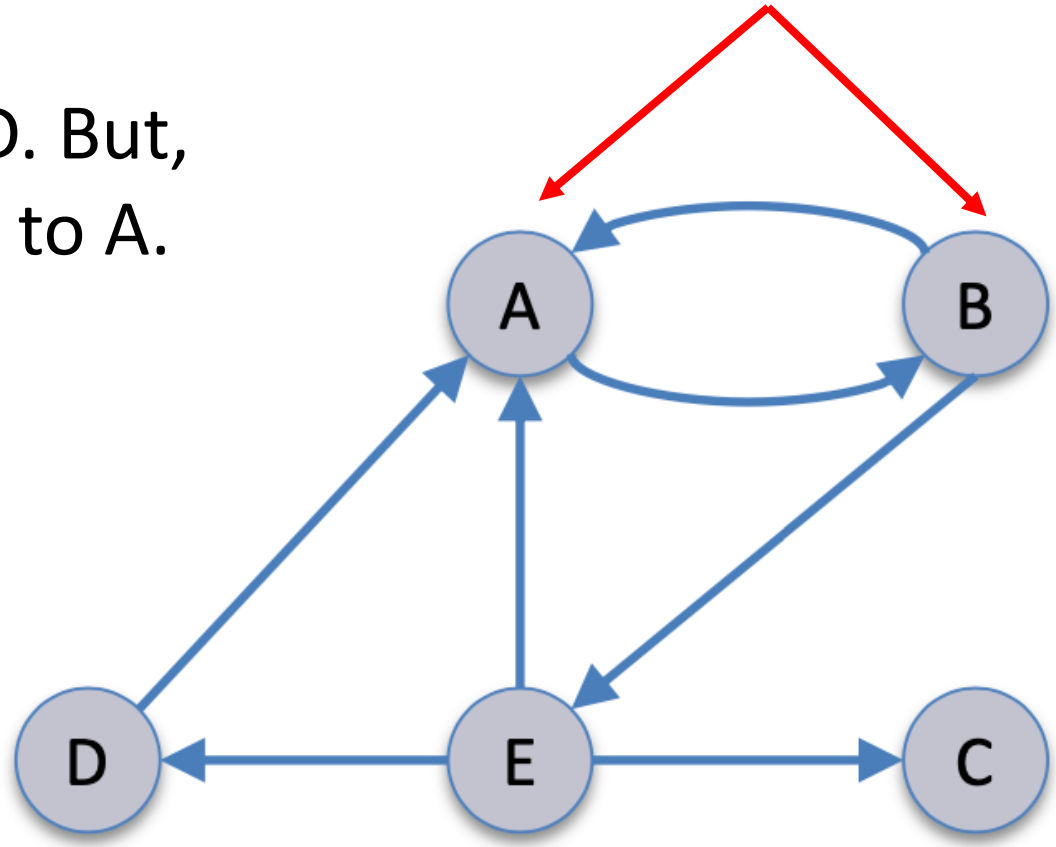
❑The following definitions describe pathways between vertices:



- A vertex is *adjacent* to another vertex if there is an edge to it from that other vertex.
- "Destination" is adjacent to "Source"

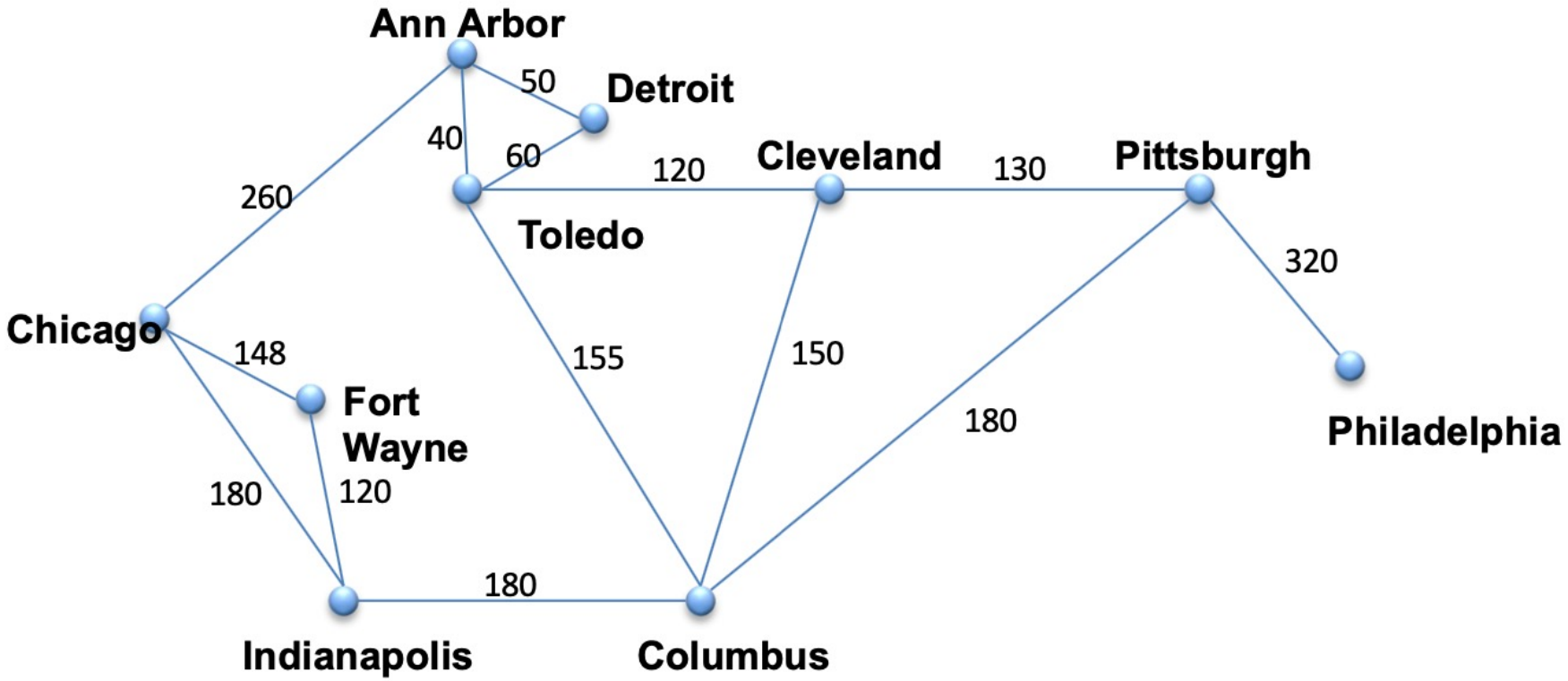
Paths and Cycles (cont.)

- A is adjacent to D. But,
- D is not adjacent to A.

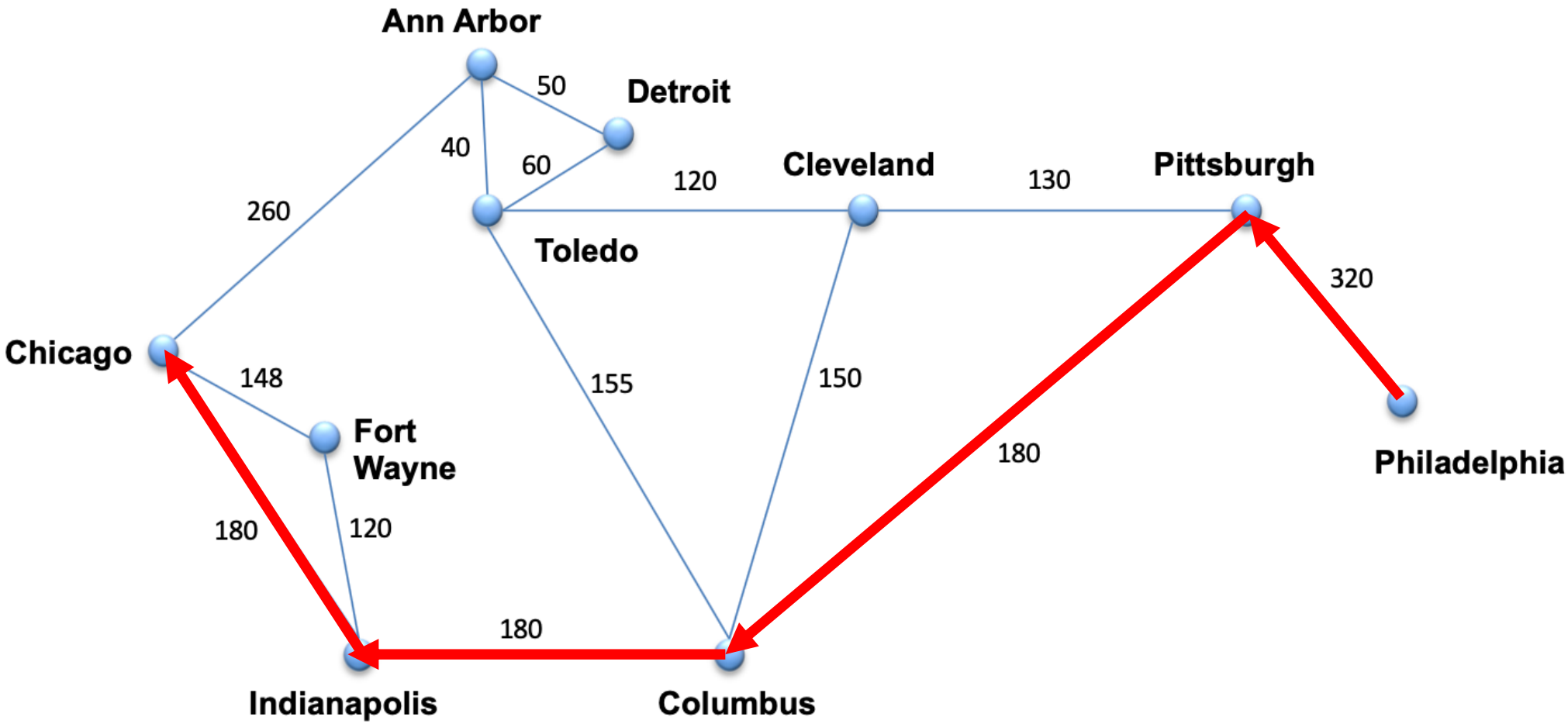


Paths and Cycles (cont.)

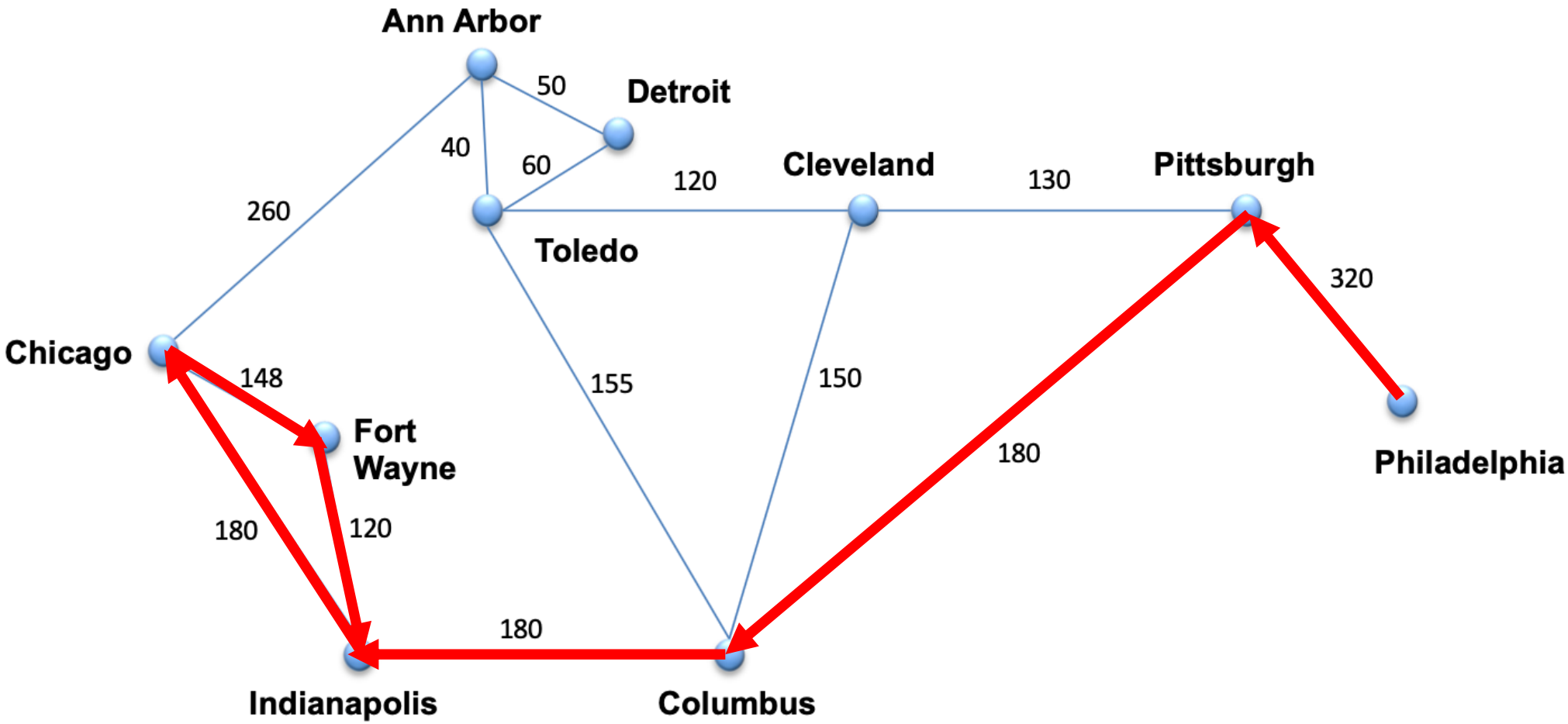
- A ***path*** is a sequence of vertices in which each successive vertex is **adjacent** to its predecessor.
- In a ***simple path***, the vertices and edges are **distinct**.
  - The first and last vertex may be the same – **cycle.**



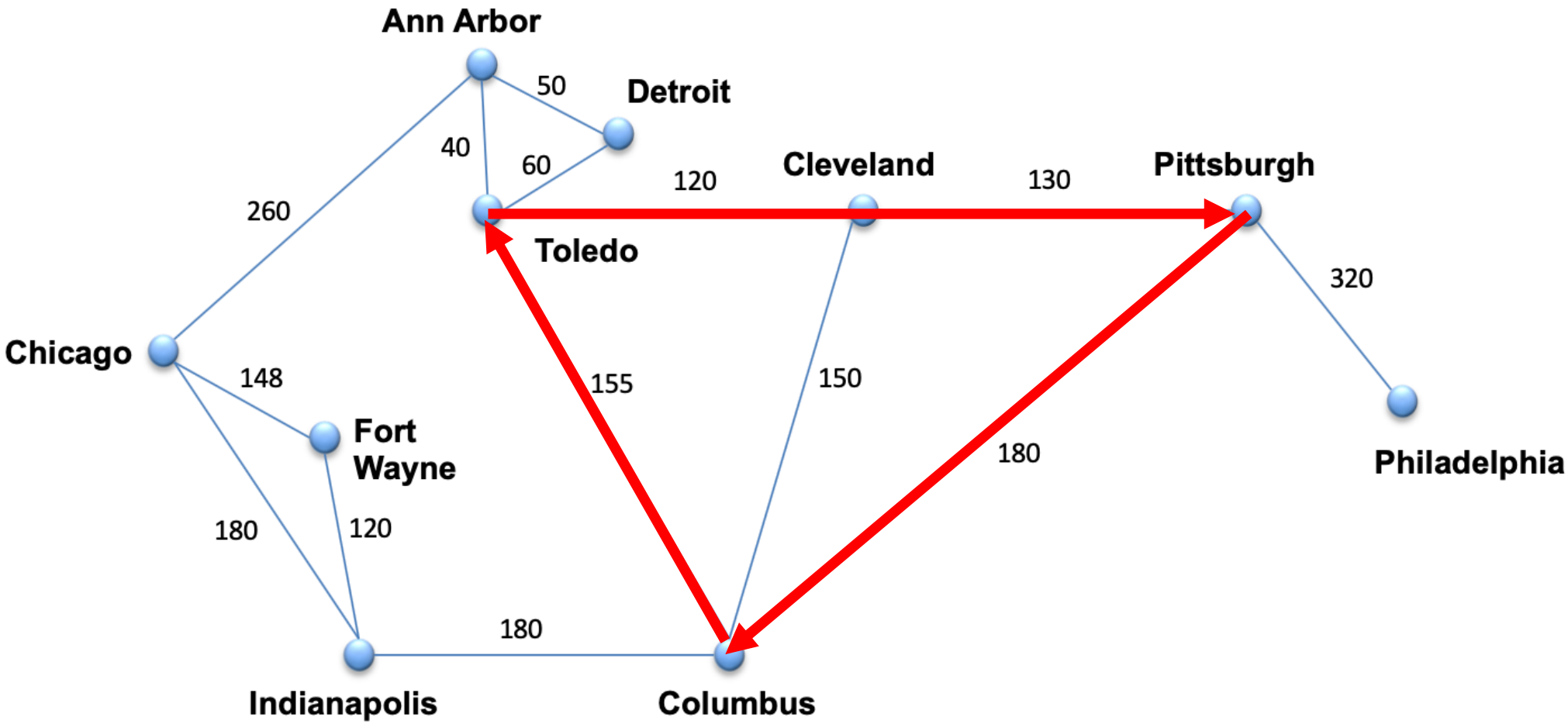
Paths and Cycles (cont.)



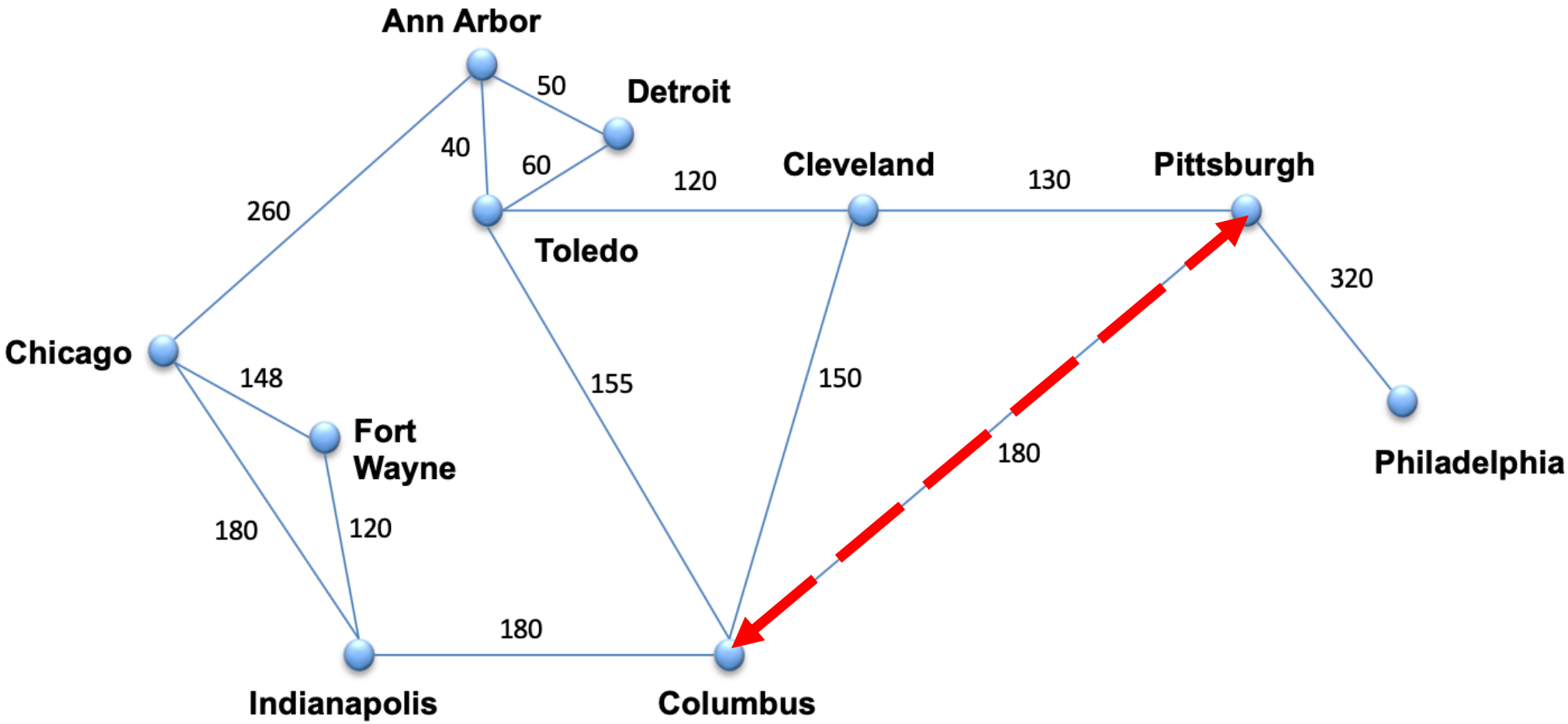
Paths and Cycles (cont.)



Paths and Cycles (cont.)

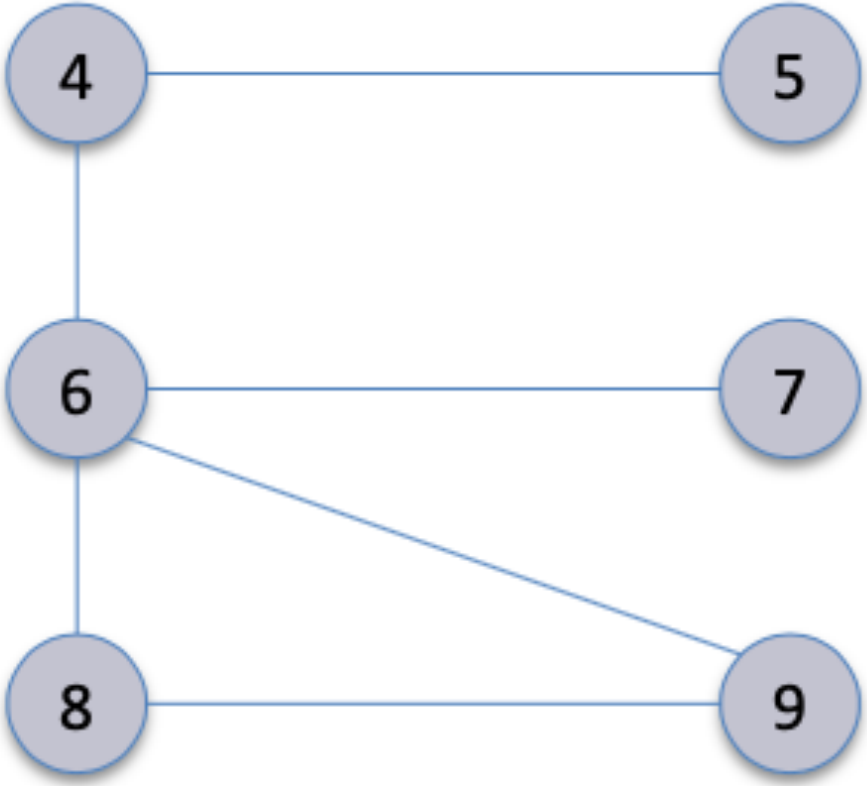


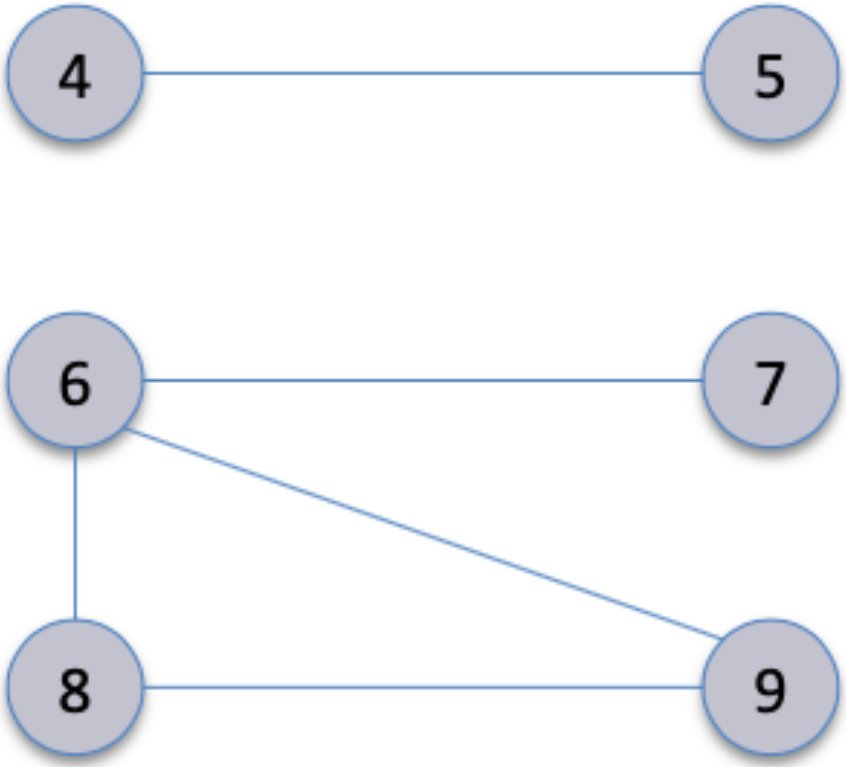
Paths and Cycles (cont.)



Paths and Cycles (cont.)

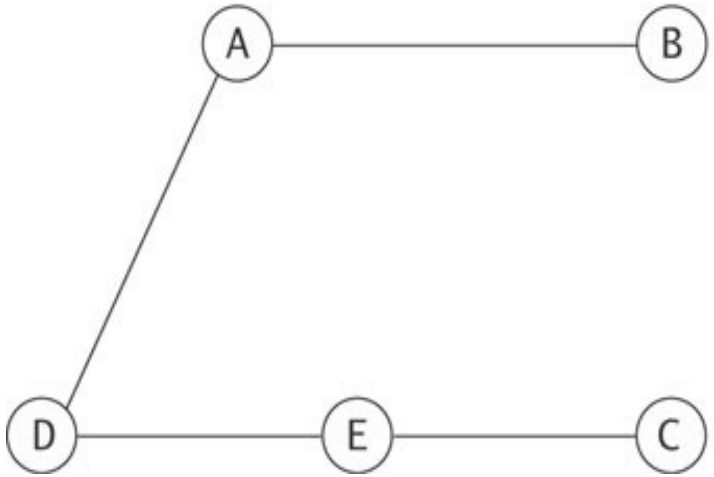
- If a graph is not connected, it is considered ***unconnected,*** but will still consist of connected components.

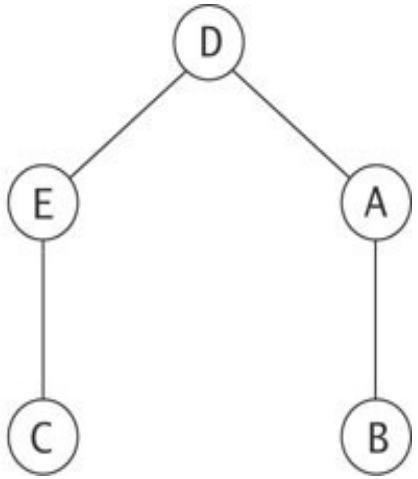




Relationship between Graphs and Trees

- A tree is a special case of a graph
- Any graph that is
  - Connected
  - contains no cycles can be viewed as a tree by making one of the vertices the root



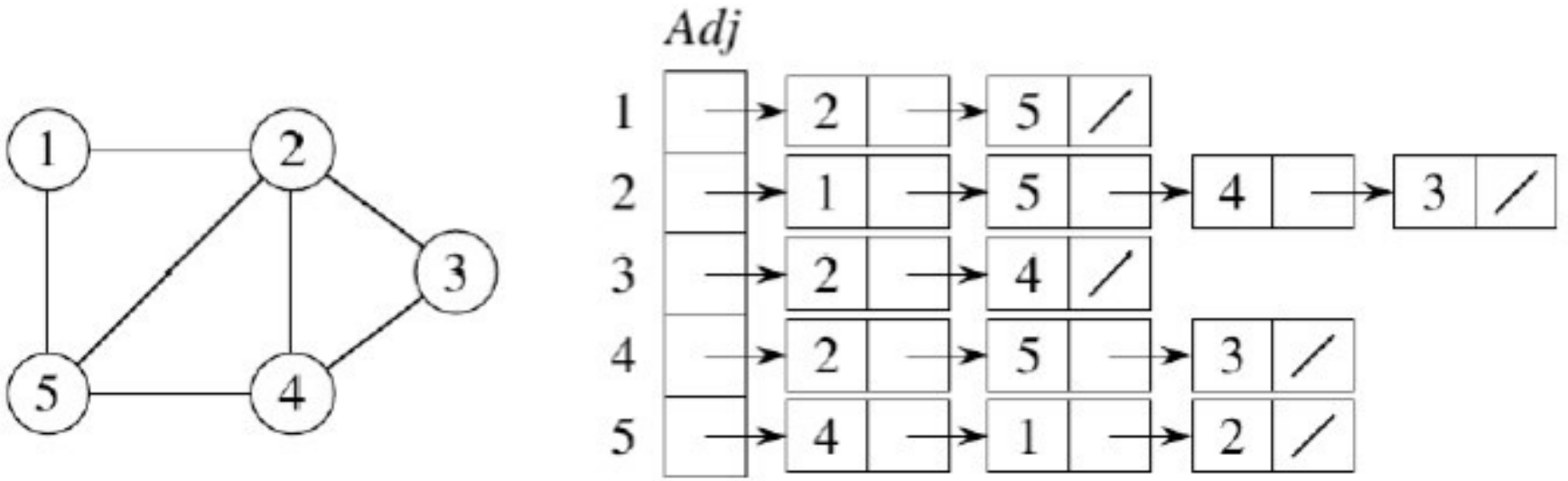


Graph Applications

- Graphs can be used to:
  - determine if one node in a network is connected to all the others
  - map out multiple course prerequisites (a solution exists if the graph is a directed graph with no cycles)
  - find the shortest route from one city to another (least cost or shortest path in a weighted graph)

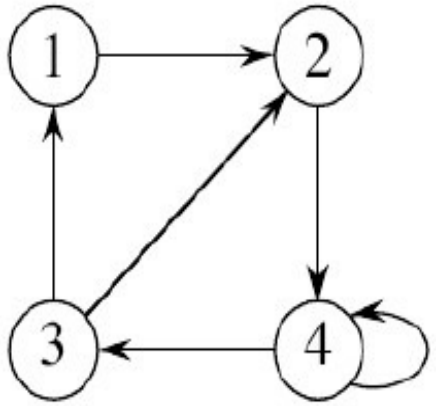
Adjacency Lists

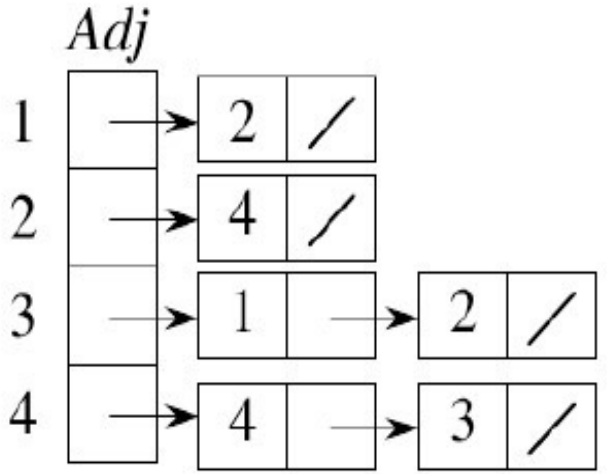
- Array Adj of |V| lists per vertex.
- List of vertex u has all vertices v such that (u,v) ∈ E ⇒ works both for directed and undirected graphs.
- We denote the array as attribute G.adj ⇒ we use the notation G.Adj[u].



Adjacency Lists

- We can put weights in the list in case the edges have weights. Weight $w: E \rightarrow \mathbb{R}$.
- Weights are later used for spanning trees and shortest path.
- Space: $\Theta(V + E)$
- Time: to list all vertices adjacent to u: $\Theta\big(\text{degree}(u)\big)$.
- Time: to determine whether $(u, v) \in E$: $\Theta\big(\text{degree}(u)\big)$.

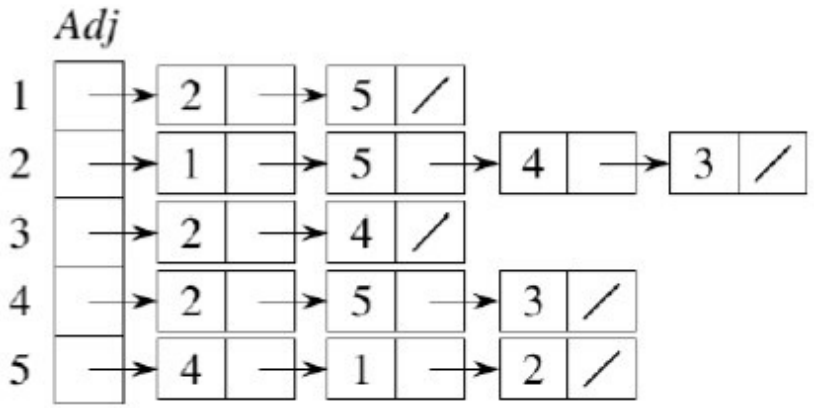


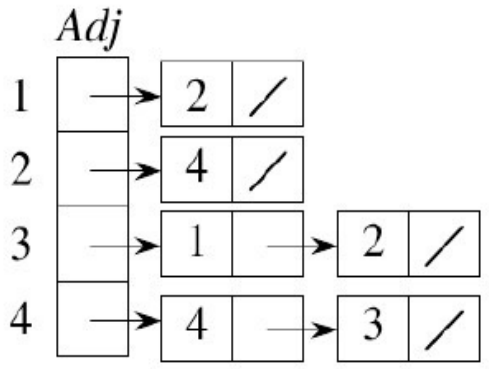


Adjacency Matrix

|V|×|V| matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$





	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

	1	2	3	4
1	0	1	0	0
2	0	0	0	1
3	1	1	0	0
4	0	0	1	1

- **Space:** $\Theta(V^2)$
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine whether $(u, v) \in E$: $\Theta(1)$.
- We can store weights instead of bits for weighted graphs.

$\Rightarrow$ We will use both representations.

Graph attributes

Representing graph attributes:

- Graph algorithms usually maintain attributes for vertices and/or edges.
- We use v.d to denote attribute d of vertex v.
- We use (u, v).f to denote attribute f of edge (u, v).

Implementing graph attributes:

- Depends on the programming language, the algorithm, and on the interaction of the rest of the program with the graph ⇒ no best way to implement.
- Example: Could use array d[1…|V|] parallel to Adj in order to store vertex attributes. Vertices adjacent to u are in Adj[u], store u.d in array d[u].
- Example: Represent vertex attributes as instance variables within a subclass of a Vertex class.

Elementary Graph Algorithms

Breadth-first search (BSF):

- Input: Graph G = (V, E), either directed or undirected and a source vertex $s \in V$.
- Output: v.d = distance (smallest # of edges) from s to v, for all $v \in V$.
- The textbook use $v.\pi$ such that (u, v) is the last edge on the shortest path $s \to v$.
  - u is v's predecessor.
  - Set of edges $\{(v.\pi, v): v \neq s\}$ forms a tree.
- We do a generalization of the BFS, with edge weights, later. Keep it simple for now.
  - We compute only the v.d, not the $v.\pi$.
  - Omit colors of vertices.

Elementary Graph Algorithms

Breadth-first search (BSF):

- We send a wave out from our source s.
  - The wave first hits all vertices 1 edge from s.
  - From there, the wave hits all vertices 2 edges from s,…
- We will use a FIFO queue Q to maintain the wavefront.
  - $v \in Q$ if and only if wave hit v but has not come out of v set.

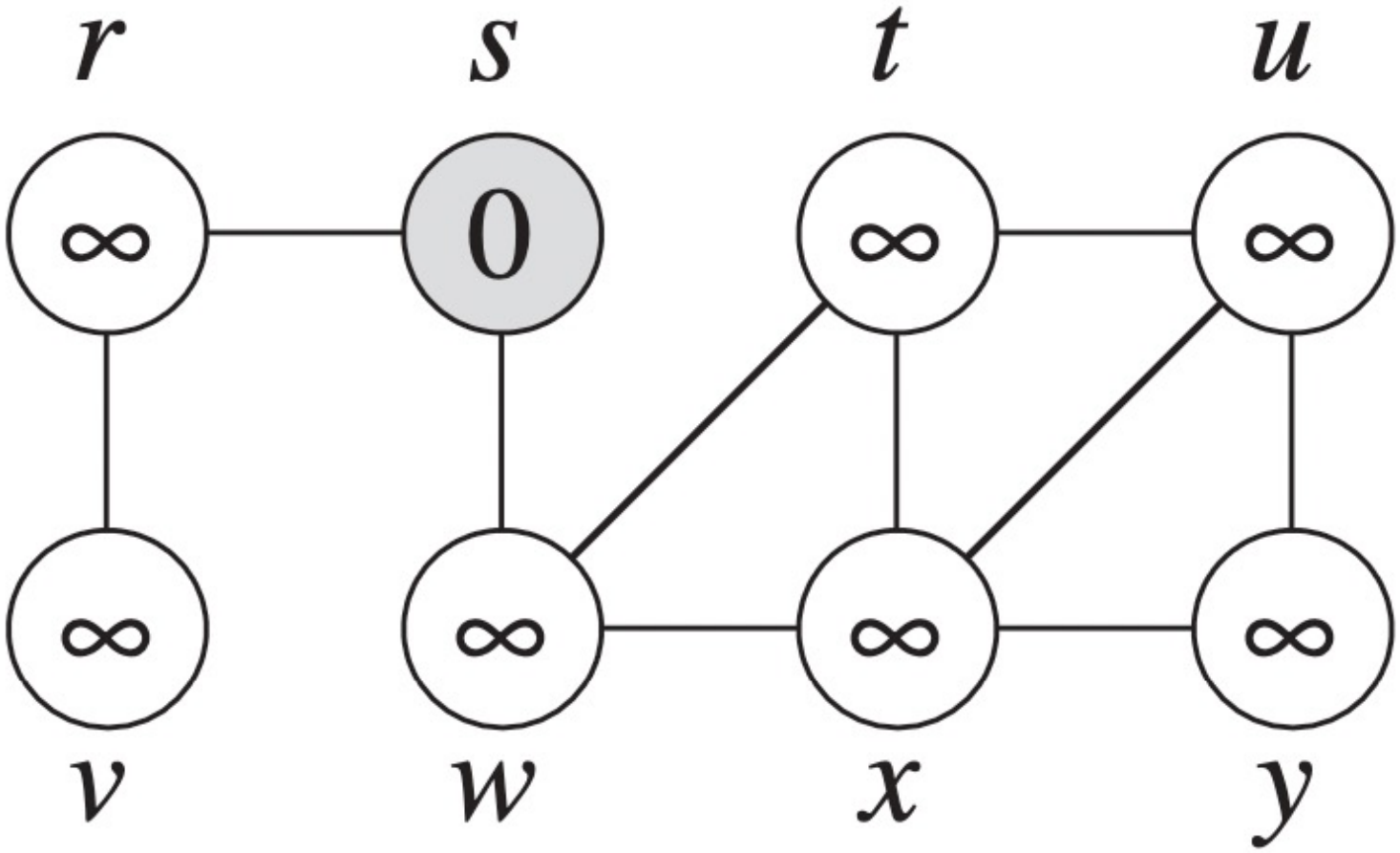
Elementary Graph Algorithms

Breadth-first search (BSF):

Algorithm (BFS(V,E,s))

```
1  foreach (u ∈ (V − {s})) do
2     u.d = ∞
3  s.d = 0
4  Q = ∅
5  ENQUEUE(Q, s)
6  while (Q ≠ ∅) do
7     u = DEQUEUE(Q)
8     foreach (v ∈ G.Adj[u]) do
9        if (v.d = ∞) then
10          v.d = u.d + 1
11          ENQUEUE(Q, v)
```

Elementary Graph Algorithms



Elementary Graph Algorithms

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Algorithm (BFS(V,E,s))
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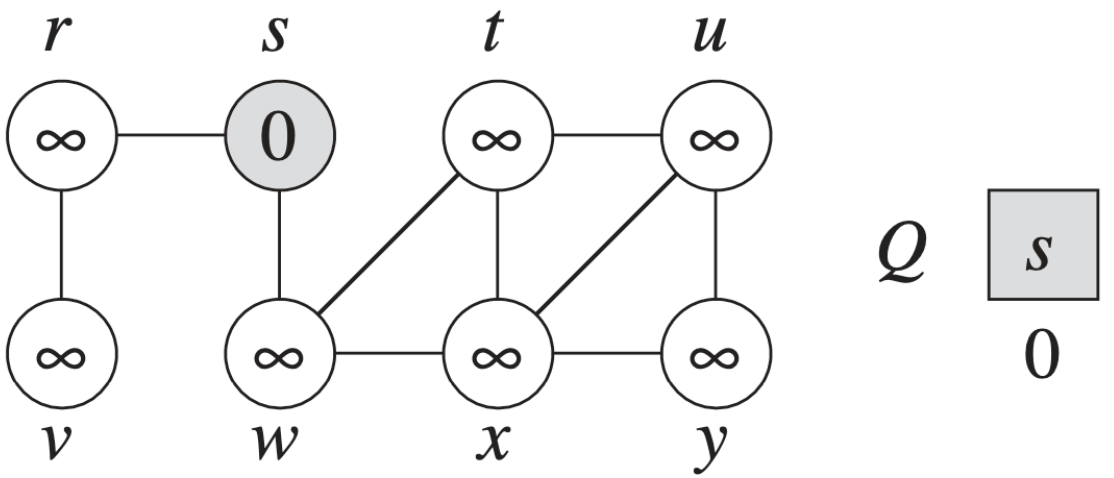
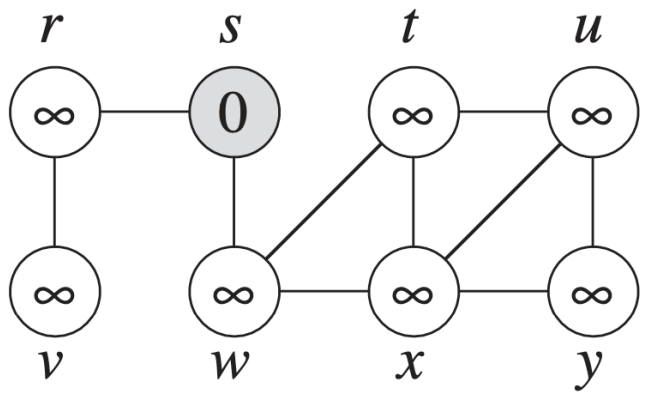


Elementary Graph Algorithms



Q | s

0

1	1
w	r

1	2	2
r	t	x

2	2	2
t	x	v

2	2	3
x	v	u

2	3	3
v	u	y

s	w	r
0	1	1

s	w	r	r	x
0	1	1	2	2

s	w	r	t	x	v
0	1	1	2	2	2

s	w	r	t	x	v	u
0	1	1	2	2	2	3

s	w	r	t	x	v	u	y
0	1	1	2	2	2	3	3

Algorithm (BFS(V,E,s))

```
1  foreach (u ∈ (V − {s})) do
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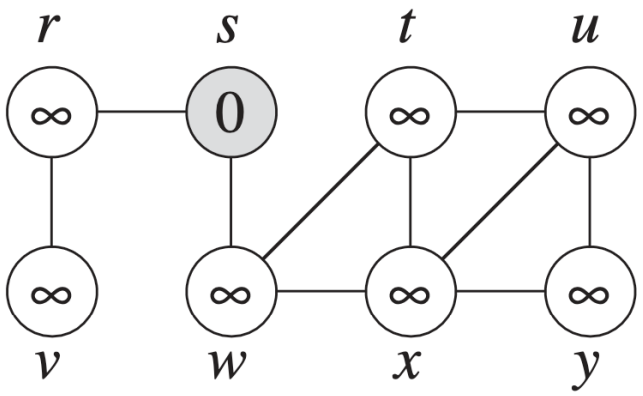
```

Elementary Graph Algorithms



Q | s | 0

Algorithm (BFS(V,E,s))

```
1  foreach (u ∈ (V − {s})) do
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5  ENQUEUE(Q, s)
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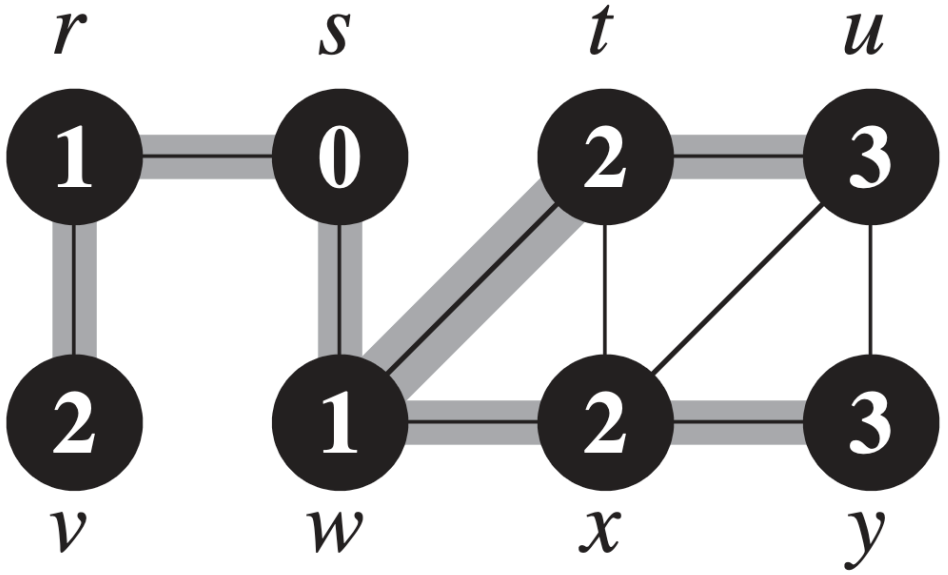
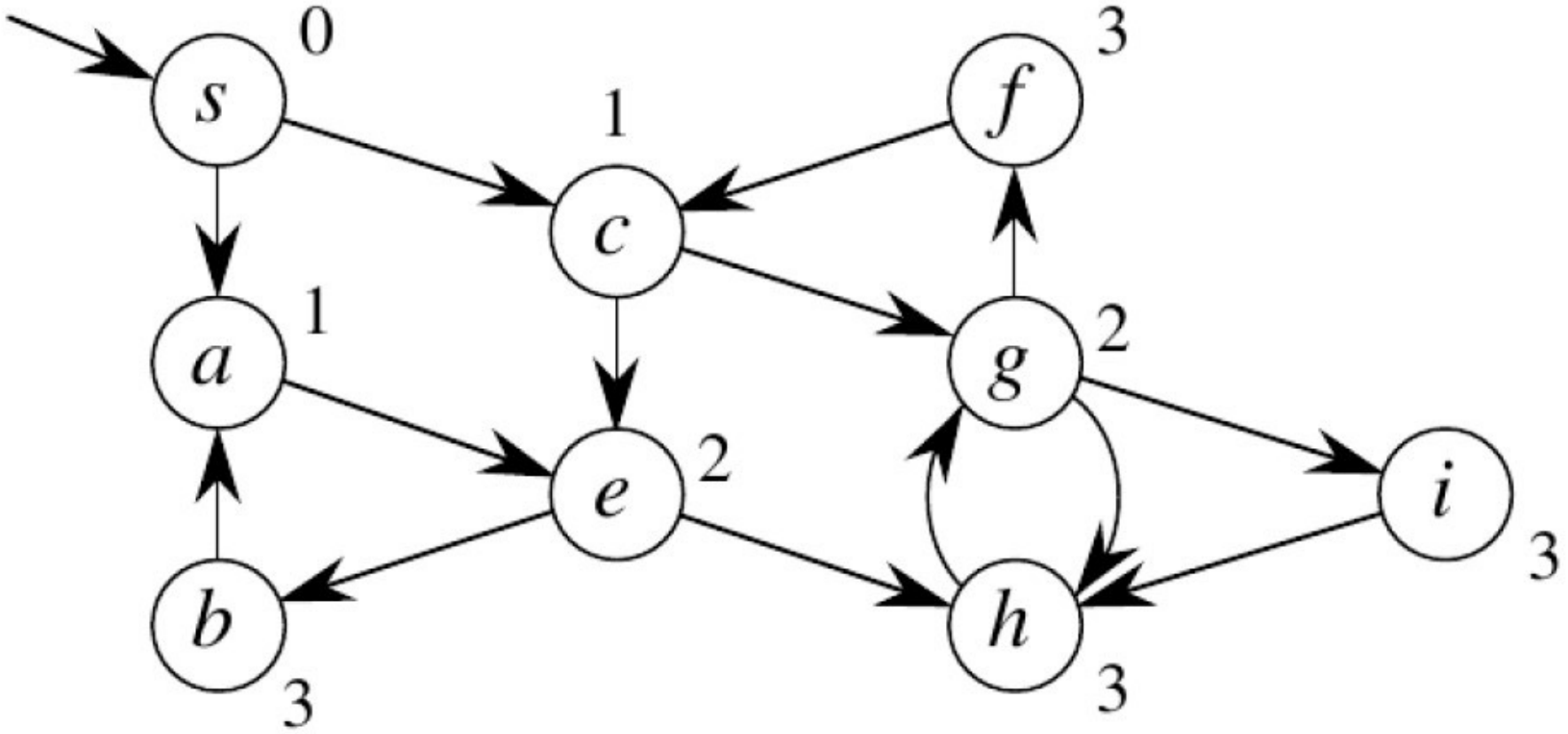


Q $\emptyset$

Elementary Graph Algorithms

Example: directed graph



Elementary Graph Algorithms

Breadth-first search (continued):

- We can show that Q consists of vertices with d values: $i \;\; i \;\; i \;\; \ldots \;\; i \;\; i+1 \;\; i+1 \;\; \ldots \;\; i+1$.
- The values assigned to vertices are monotonically increasing over time, since each vertex gets a finite d value at most once.
- See the book for the actual (a bit trickier) proof.
- BFS may not reach all vertices.
- **Time:** $O(V+E)$
  - $O(V)$ because every vertex is enqueued at most once.
  - $O(E)$ because every vertex is dequeued at most once and we then examine (u,v) only when u is dequeued $\Rightarrow$ every edge examined at most once if directed, at most twice if undirected.

Elementary Graph Algorithms

Depth-first search:

Input: $G = (V, E)$, directed or undirected, No source vertex is given.

Output: 2 **timestamps** on each vertex:

- $v.d =$ **discovery time**.
- $v.f =$ **finishing time**.

These timestamps will be useful for other algorithms later on (see textbook for computation of $v.\pi$).

- We will methodically explore *every* edge.
  - We will start over from different vertices as necessary.
- We will explore a vertex, as soon as we discover it.
  - Unlike BFS, which puts a vertex on a queue so that we explore from it later.

Elementary Graph Algorithms

Depth-first search (continued):

- Every vertex has a **color** as DFS progresses.
  - *WHITE* = undiscovered
  - *GRAY* = discovered, not finished (not done exploring from it).
  - *BLACK* = finished (have found everything reachable from it).
- Discovery and finishing times:
  - Unique integers from 1 to $2|V|$.
  - For all v, we have $v.d < v.f$.

  $\Rightarrow 1 \leq v.d < v.f \leq 2|V|$.

Algorithm (DFS):

- We use a global time stamp *time*.
- We use two routines DFS(G) and DFS-VISIT(G,u).
- DFS(G) ensures that every vertex is visited.
- DFS-VISIT(G,u) does the recursive depth-first exploration.

Elementary Graph Algorithms

Algorithm (DFS(G))

```
1 foreach (u ∈ G.V) do
2    u.color = WHITE
3 time = 0
4 foreach (u ∈ G.V) do
5    if(u.color = WHITE) then
6    DFS-VISIT(G,u)
```

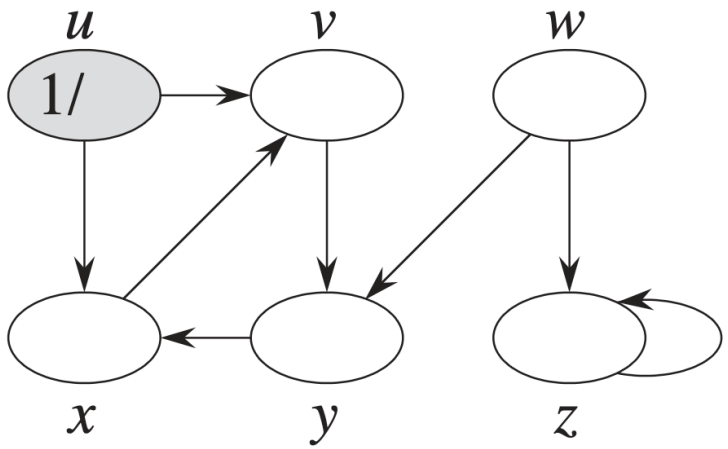
Algorithm (DFS-VISIT(G,u))

```
1 time = time + 1
2 u.d = time
3 u.color = GRAY
4 foreach (v ∈ G.Adj[u]) do
5    if(v.color = WHITE) then
6       DFS-VISIT(v)
7 u.color = BLACK
8 time = time + 1
9 u.f = time
```

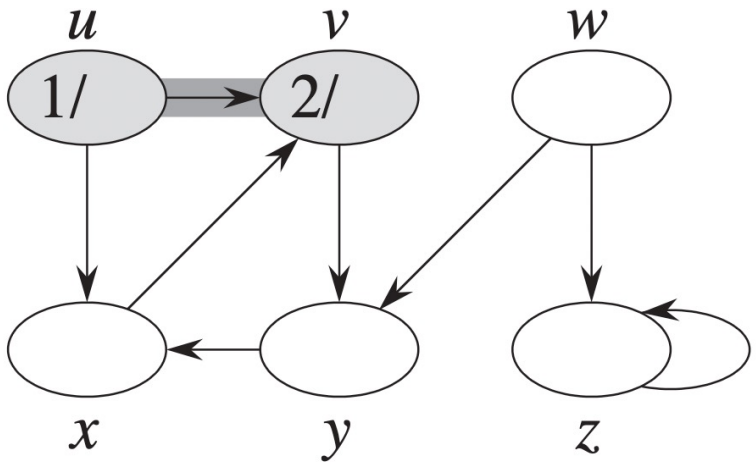
Elementary Graph Algorithms

Algorithm (DFS-VISIT(G,u))

```
1 time = time + 1
2 u.d = time
3 u.color = GRAY
4 foreach (v ∈ G.Adj[u]) do
5   if(v.color = WHITE) then
6      DFS-VISIT(v)
7 u.color = BLACK
8 time = time + 1
9 u.f = time
```

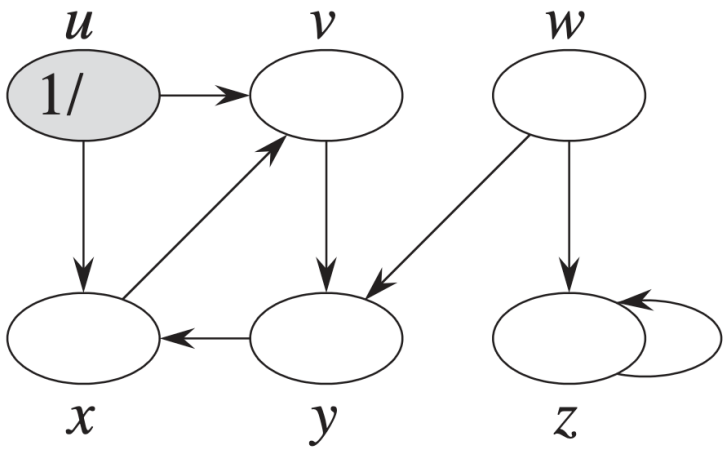


(a)

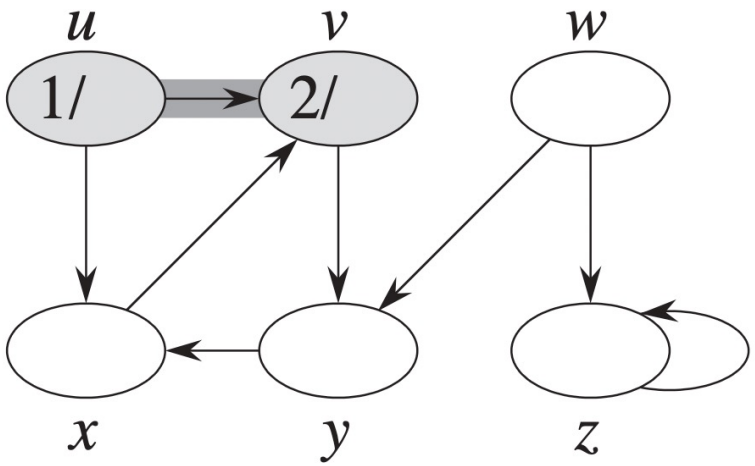


(b)

Elementary Graph Algorithms



(a)



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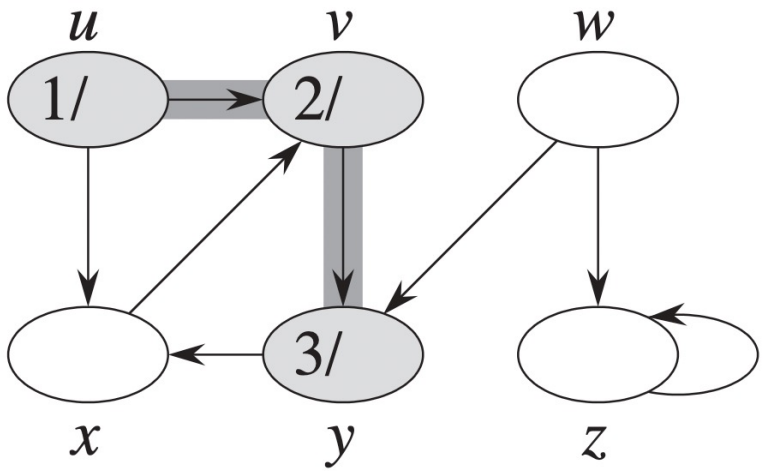
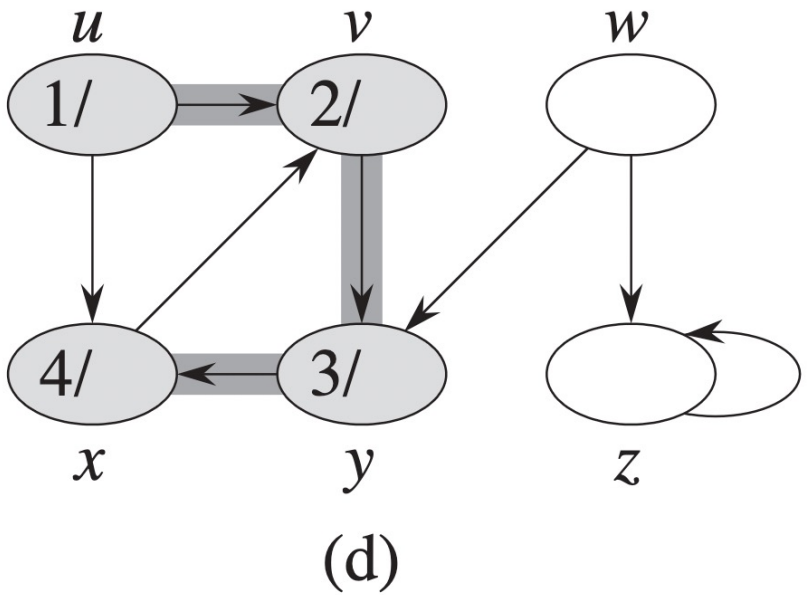
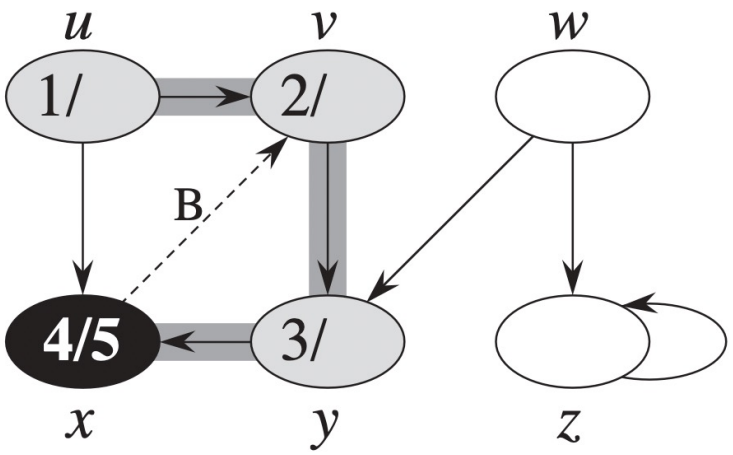


(c)

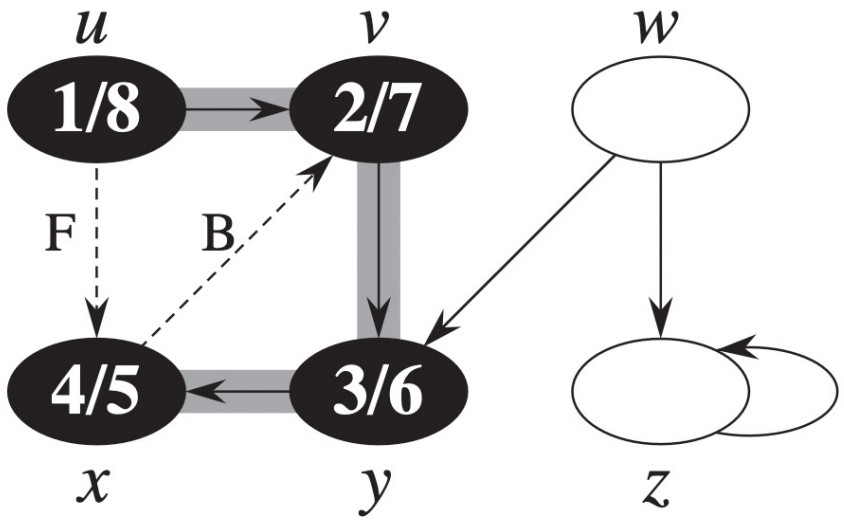


(d)

Elementary Graph Algorithms



(f)



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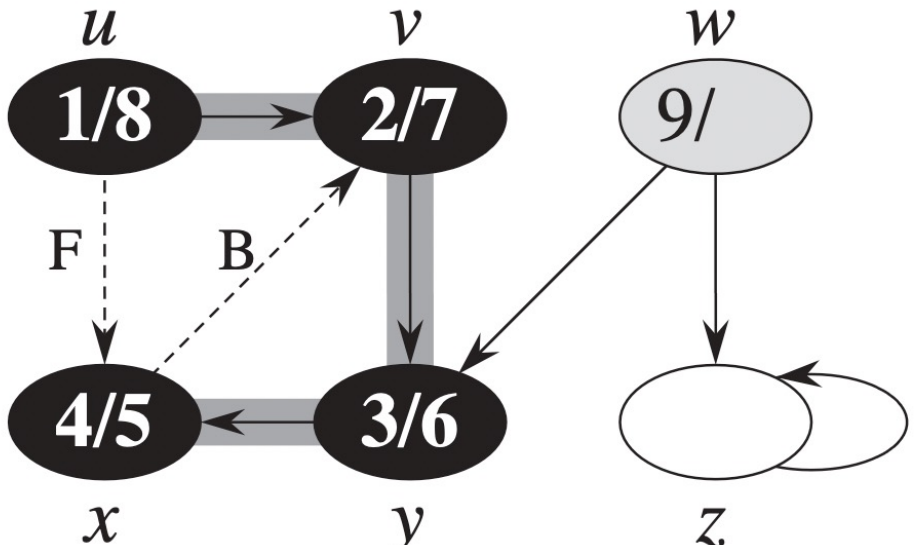
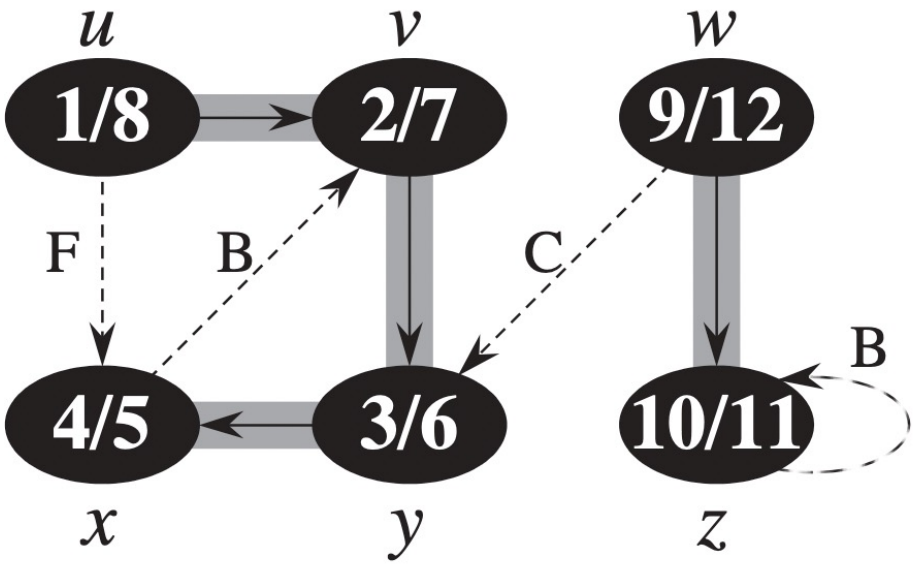
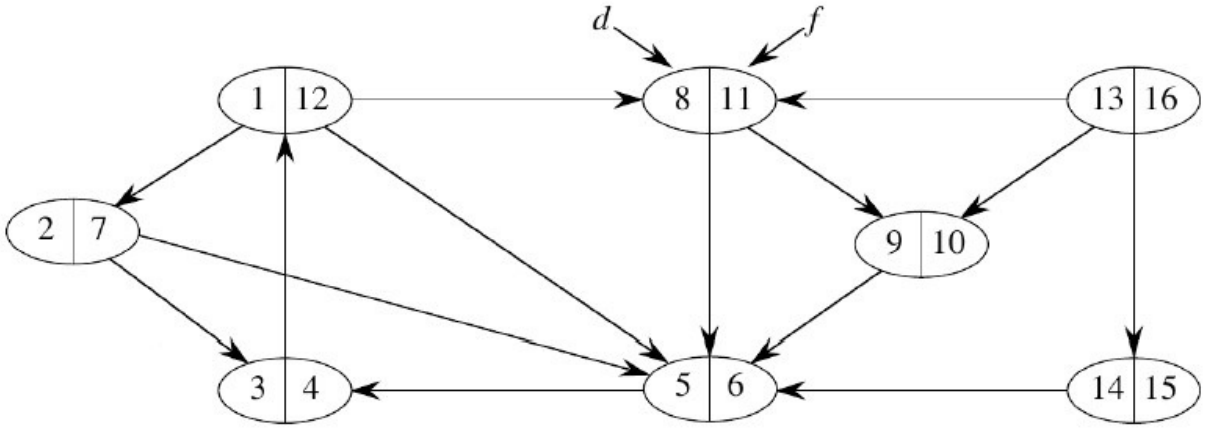




(p)

Elementary Graph Algorithms

Example: directed graph (2)



- **Time:** $\Theta(V + E)$.
  - Similar to the BFS analysis.
  - Θ, not just O, since it is guaranteed that every vertex and edge is examined.
- DFS does form a **depth-first forest** which is comprised of > 1 **depth-first trees**.
- Each tree is made of edges (u, v) such that u is gray and v is white when (u, v) is explored.

Elementary Graph Algorithms

Theorem (Parenthesis theorem)

For all u, v, exactly one of the following holds:

1. $u.d < u.f < v.d < v.f$ *or* $v.d < v.f < u.d < u.f$ *(i.e., the intervals* $[u.d, u.f]$ *and* $[v.d, v.f]$ *are disjoint) and neither of* u *and* v *is a descendant of the other.*
2. $u.d < v.d < v.f < u.f$ *and* v *is a descendant of* u.
3. $v.d < u.d < u.f < v.f$ *and* u *is a descendant of* v.

So $u.d < v.d < u.f < v.f$ *cannot happen.*

Elementary Graph Algorithms

Corollary

v is a proper descendant of u if and only if $u.d < v.d < v.f < u.f$.

Theorem

v is a descendant of u if and only if at time $u.d$, there is a path $u \rightsquigarrow v$ consisting of only white vertices.
(Except for u, which was just colored gray.)

Classification of edges:

Tree edge: in depth-first forest. Found by exploring (u, v).

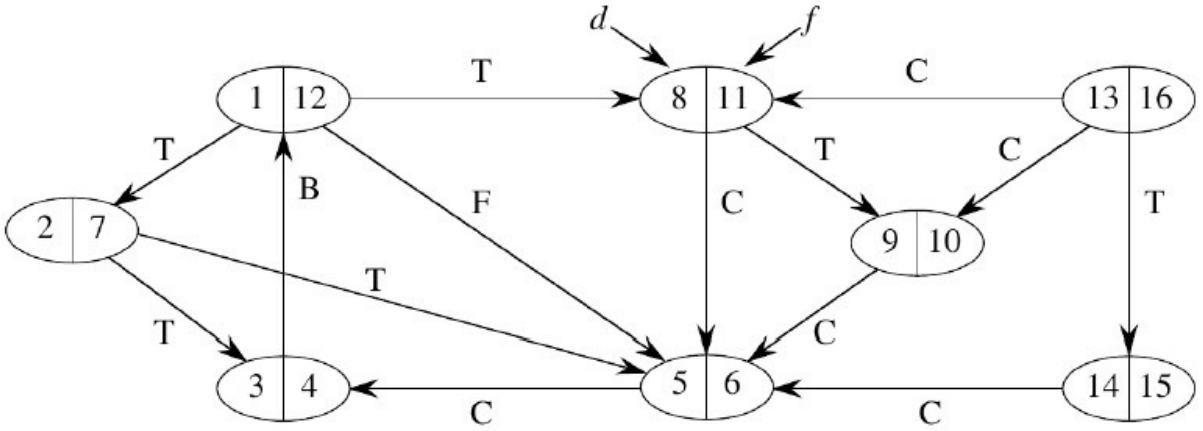
Back edge: (u, v), where u is a descendant of v.

Forward edge: (u, v), where v is a descendant of u, but not a tree edge.

Cross edge: any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Elementary Graph Algorithms

Example: directed graph (2)

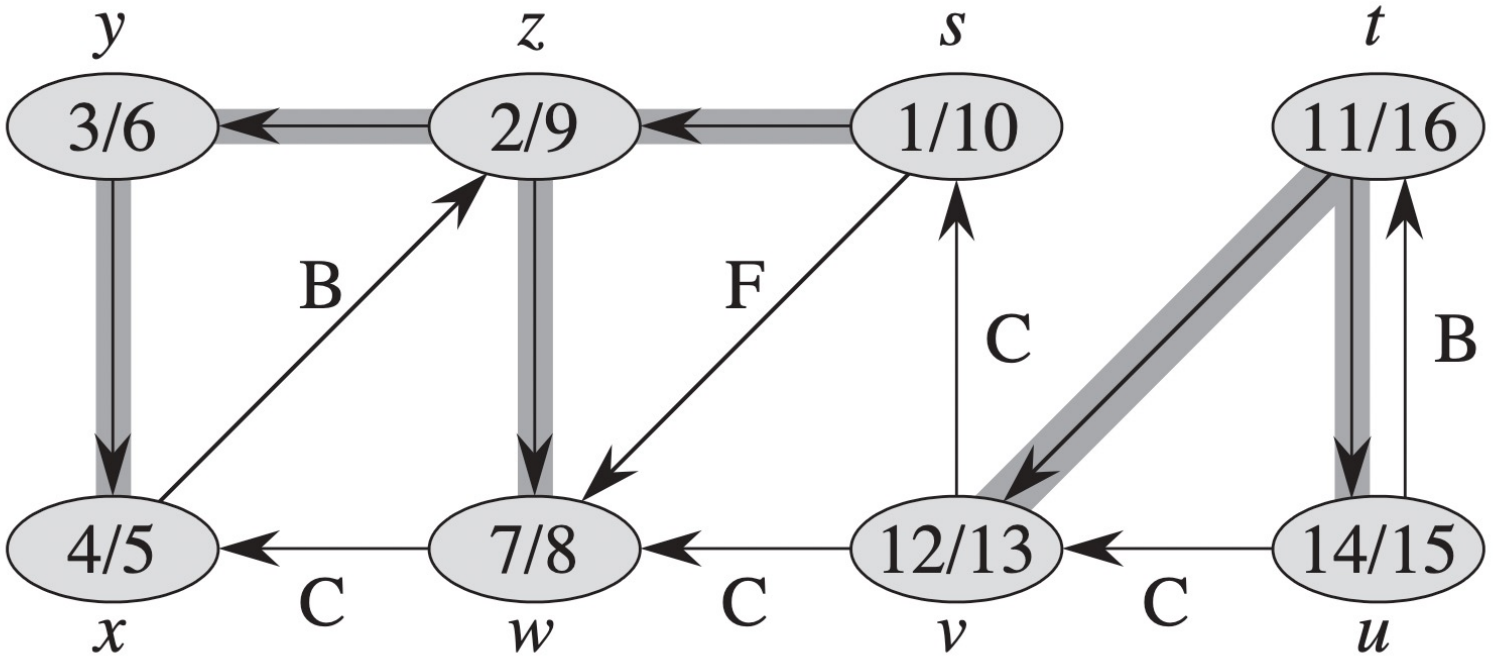


- In an undirected graph, there may be some ambiguity since (u, v) and (v, u) are the same edge.
- We classify by the first type above that matches.

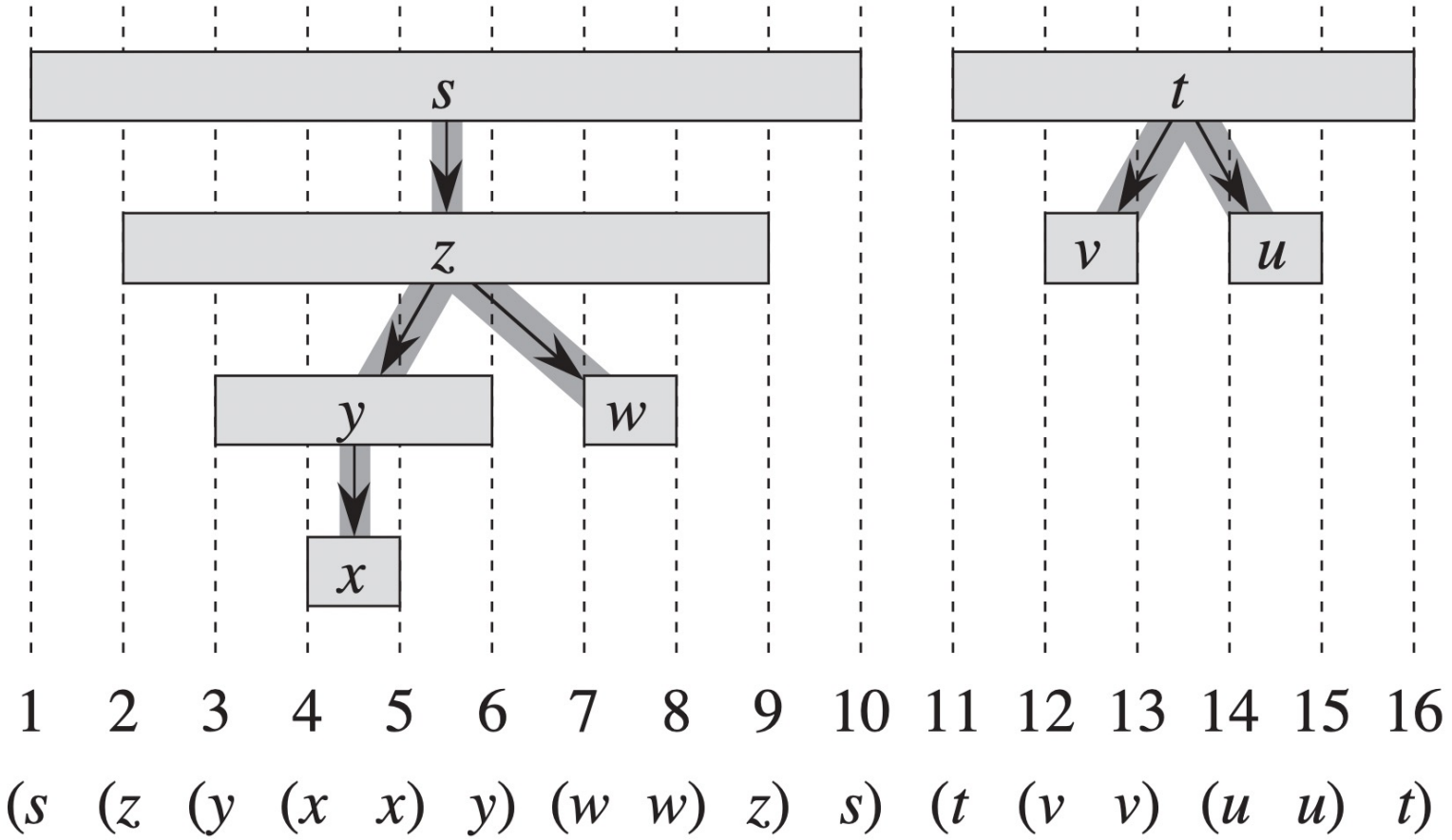
Theorem

In DFS of an undirected *graph, we get only tree and back edges. No forward or cross edges. (see textbook for proof)*

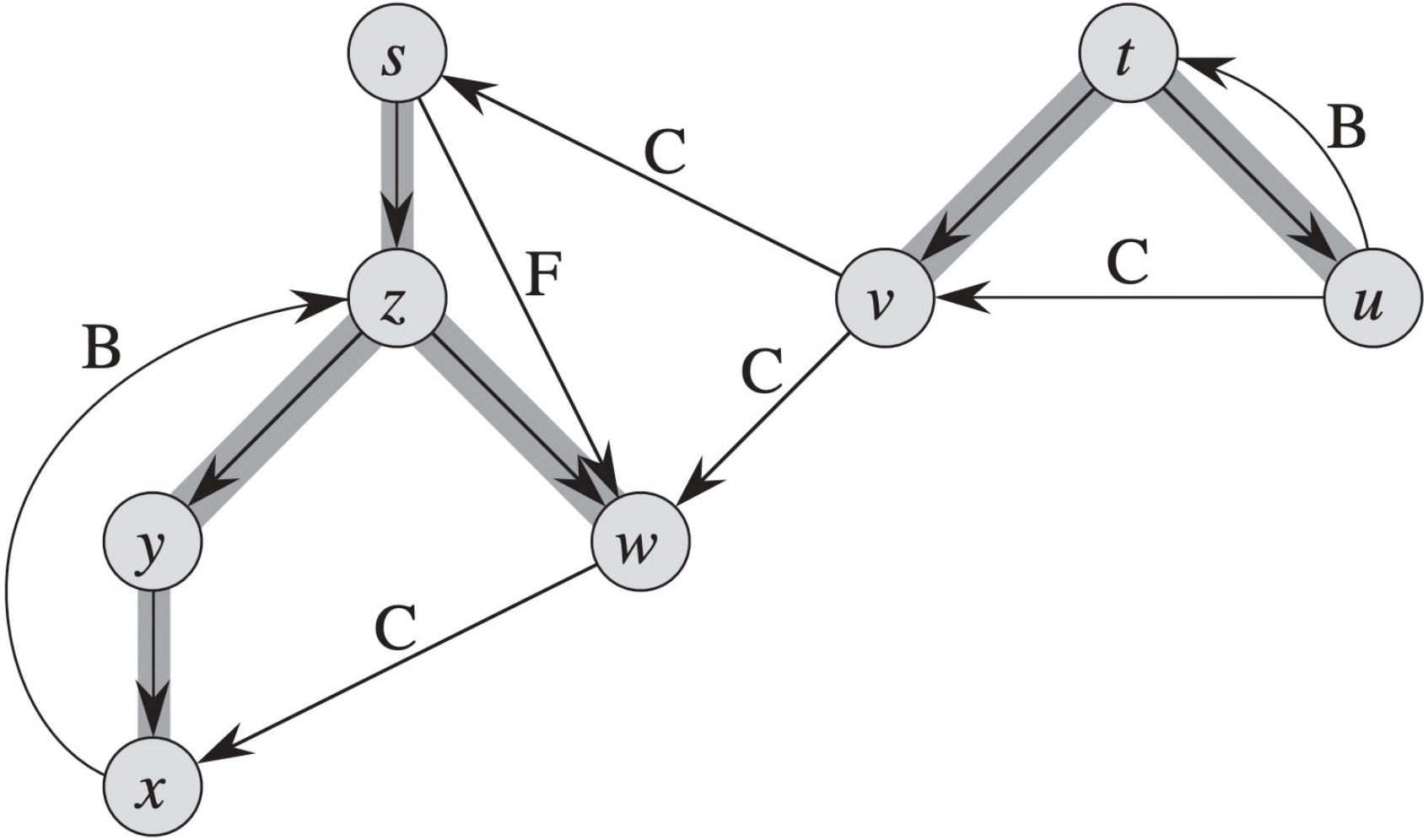
Elementary Graph Algorithms



Elementary Graph Algorithms



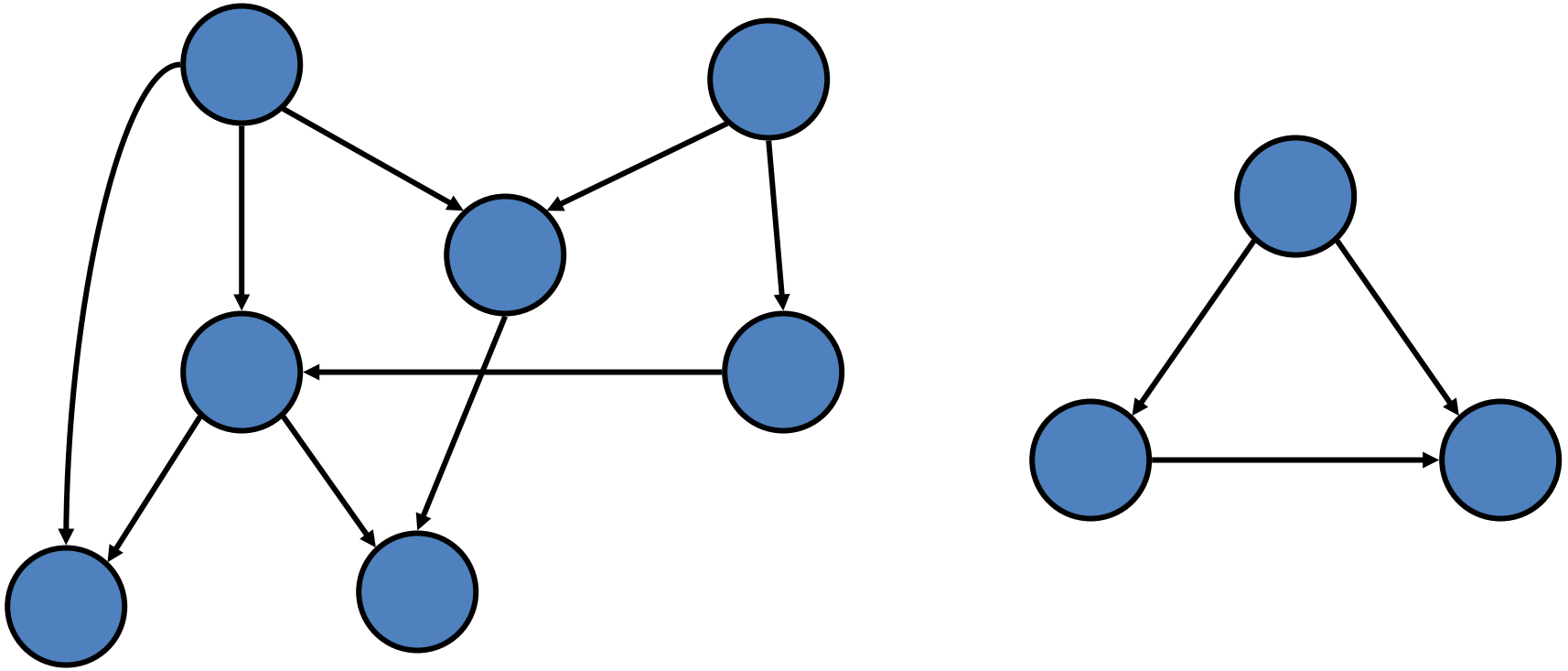
Elementary Graph Algorithms



Graph’s algorithm

Directed Acyclic Graphs

- A ***directed acyclic graph*** or ***DAG*** is a directed graph with no directed cycles:

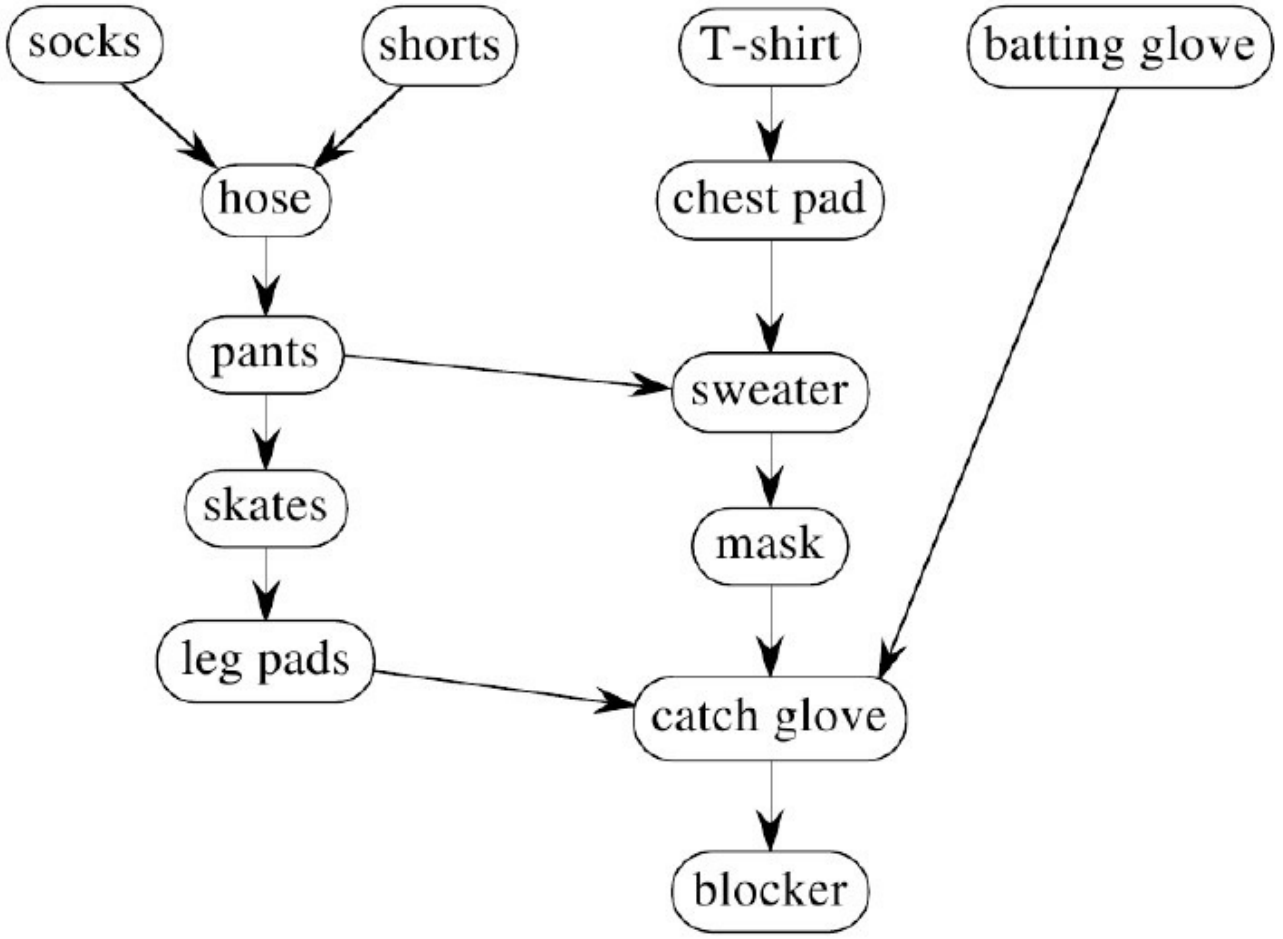


Topological Sort

- ***Topological sort*** of a DAG:
  - Linear ordering of all vertices in graph G such that vertex *u* comes before vertex *v* if edge $(u, v) \in G$.
- Real-world example: getting dressed

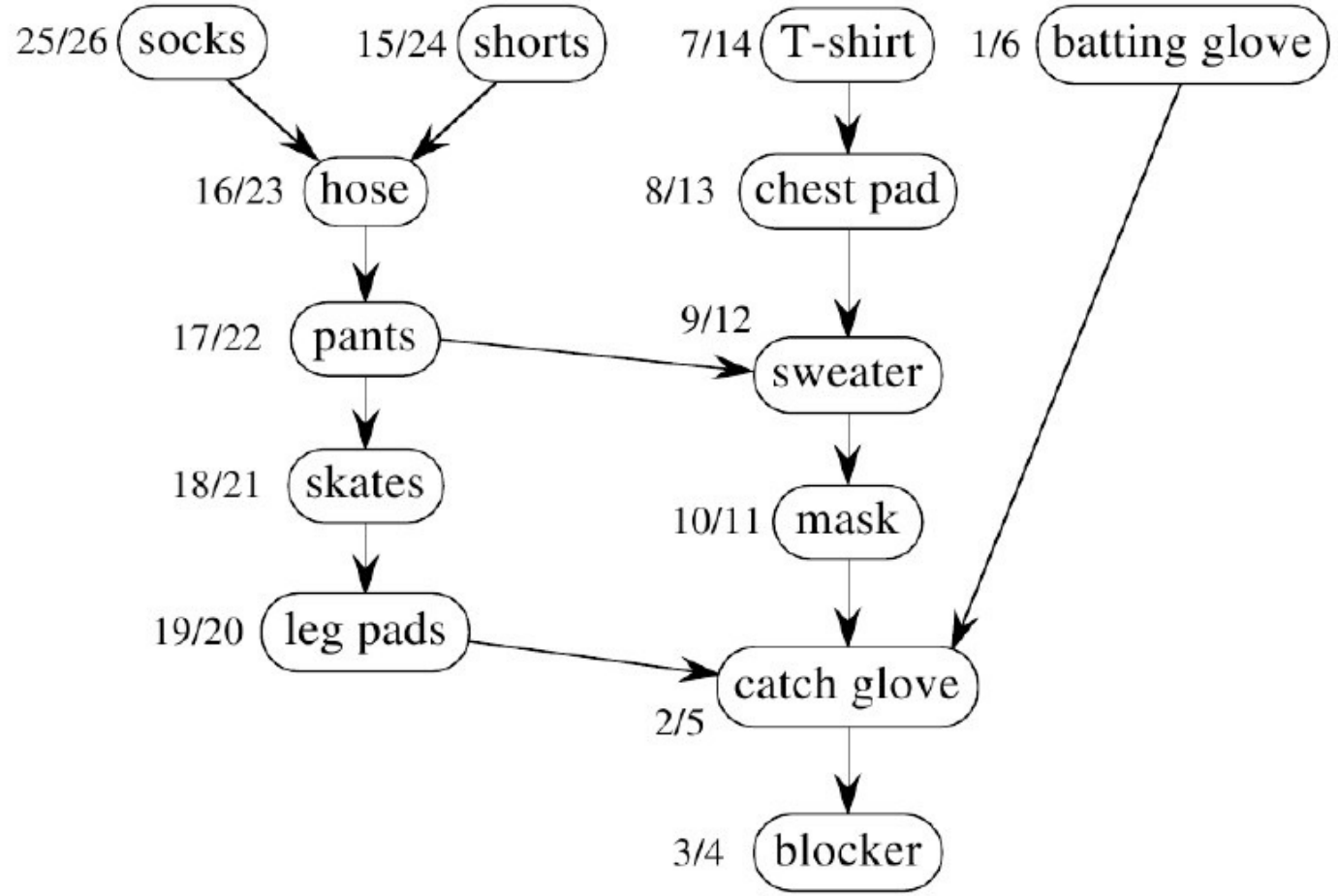
Graphs

Example:



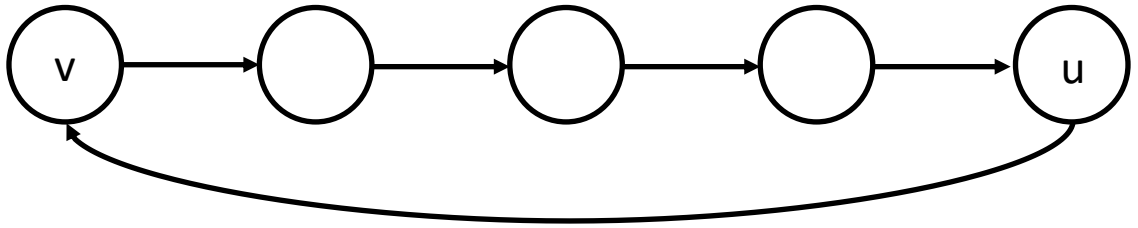
Graphs

Example:



Graphs

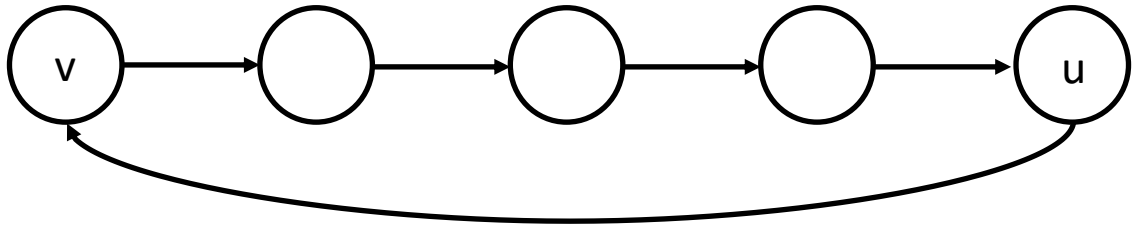
- A directed graph G is acyclic if and only if a DFS of G yields no back edges.
- Proof:
  - We need to show that back edge is a cycle.
  - Suppose there is a back edge (u, v). Then v is an ancestor of u in the DFS.



  - Therefore, there is a path $v \rightarrow u$, so $v \rightarrow v$ is a cycle.
  - ✓ Suppose G contains a cycle c. Let v be the first vertex discovered in c, and let (u, v) be the preceding edge in c.

Graphs

- A directed graph G is acyclic if and only if a DFS of G yields no back edges.



- Therefore, there is a path $v \rightarrow u$, so $v \rightarrow v$ is a cycle.
- ✓ Suppose G contains a cycle c. Let v be the first vertex discovered in c, and let (u, v) be the preceding edge in c.
- ✓ At time v.d vertices of c form a white path $v \rightarrow u$ (since v is the first vertex discovered in c).
- ✓ By the white-path theorem, u is a descendant of v in the depth-first forest.
- ✓ Therefore, (u, v) is a back edge.

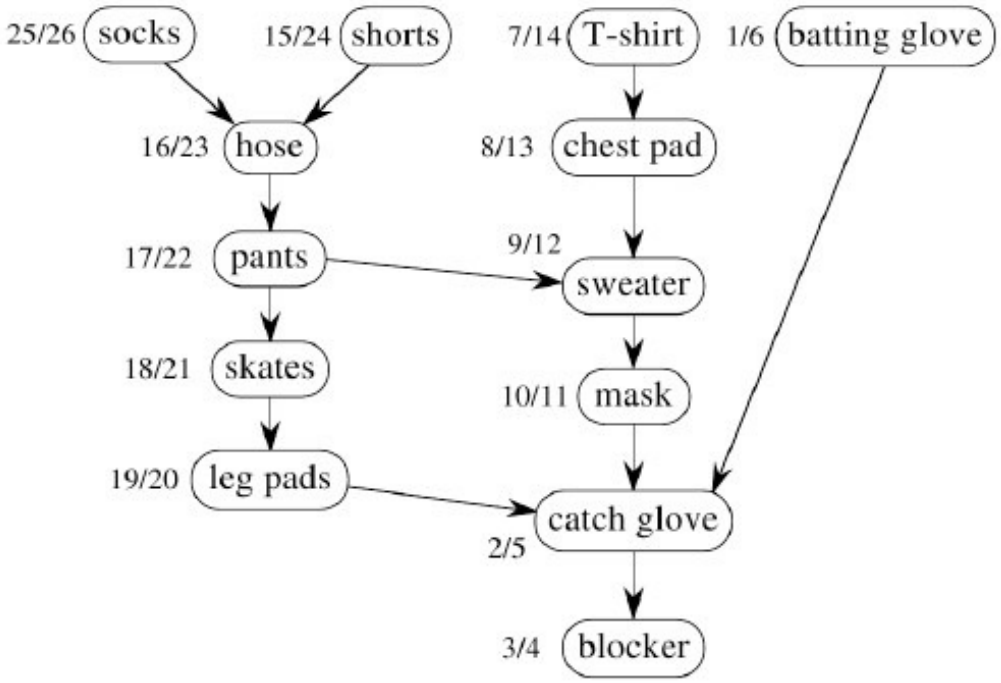
Topological-sort

- Topological sort of a DAG: a linear ordering of vertices such that if $(u, v) \in E$, then u appear somewhere before v.
- We call DFS(G) to compute the finishing time v.f for all $v \in G.V$.
- We output all vertices in order of decreasing finishing times.
- We do not need to sort by finishing time.
  - We can just output the vertices as they are finished and understand that we want the reverse of this list.
  - Another option is to put them in front of a liked list as they are finished. After the DFS, the list contains the vertices in topological sorted order.
  - Finished referrers to the DFS-VISIT(G, u) algorithm and the assignment of the finishing time u.f.
- Time: $\Theta(V + E)$

Graphs

Example (continued):

- Order:
  - 26 socks
  - 24 shorts
  - 23 hose
  - 22 pants
  - 21 skates
  - 20 leg pads
  - 14 t-shirt
  - 13 chest pad
  - 12 sweater
  - 11 mask
  - 6 batting glove
  - 5 catch glove
  - 4 blocker



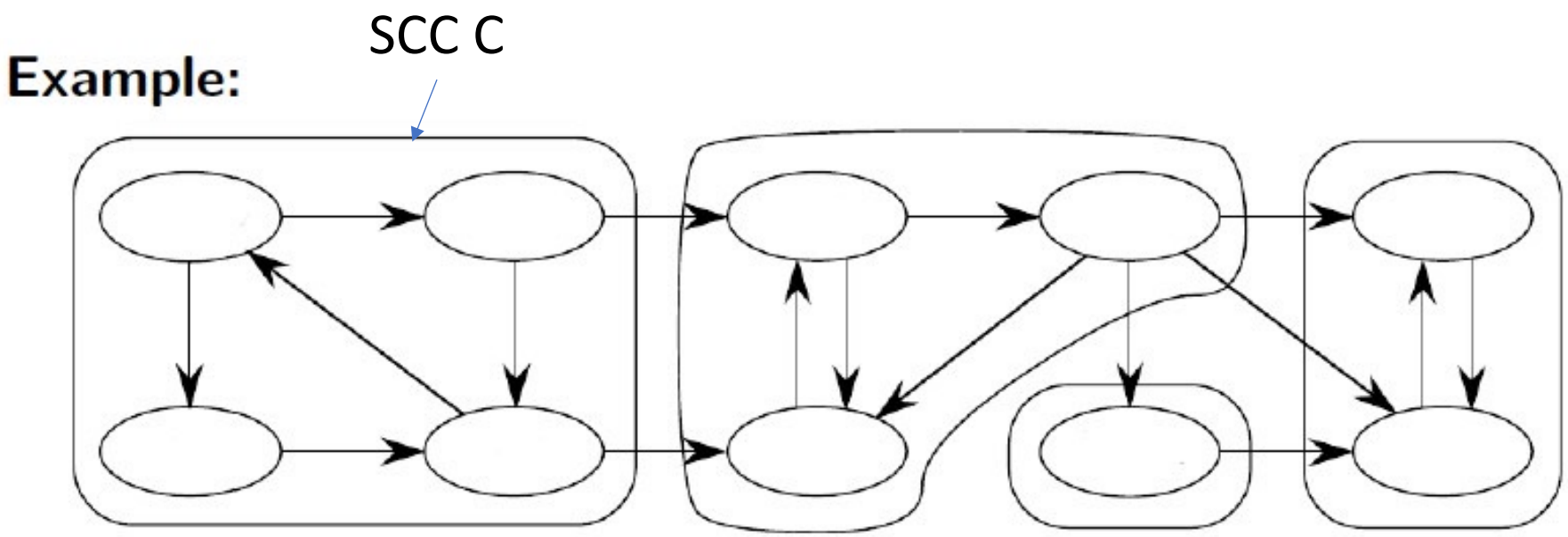
Graphs

Correctness:

- We just need to show: if $(u, v) \in E$, then $v.f < u.f$.
- What are the colors of u and v when we explore (u, v).
  - u is gray.
  - Is v also gray?
    - No, because then v would be an ancestor of u.
    - $\Rightarrow (u, v)$ is a back edge.
    - $\Rightarrow$ contradiction of lemma (DAG has no back edges).
  - Is v white?
    - Then v becomes a descendent of u. By parenthesis theorem, $u.d < v.d < v.f < u.f$.
  - Is v black?
    - Then v is already finished. Since we exploring (u, v), we have not yet finished u. Therefore, $v.f < u.f$.

Graphs – Strongly Connected Components (SCC)

- Given a directed graph $G = (V, E)$.
- A **strong connected component (SCC)** of G is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightarrow v$ and $v \rightarrow u$.



Graphs – Strongly Connected Components (SCC)

Algorithm for SCC:

- Algorithm uses G^T = transpose of G.
  - $G^T = (V, E^T), E^T = \{(u, v): (v, u) \in E\}$.
  - G^T is G with all edges reversed.
- Can create G^T in $\Theta(V + E)$ time if using adjacency lists.

Observation:

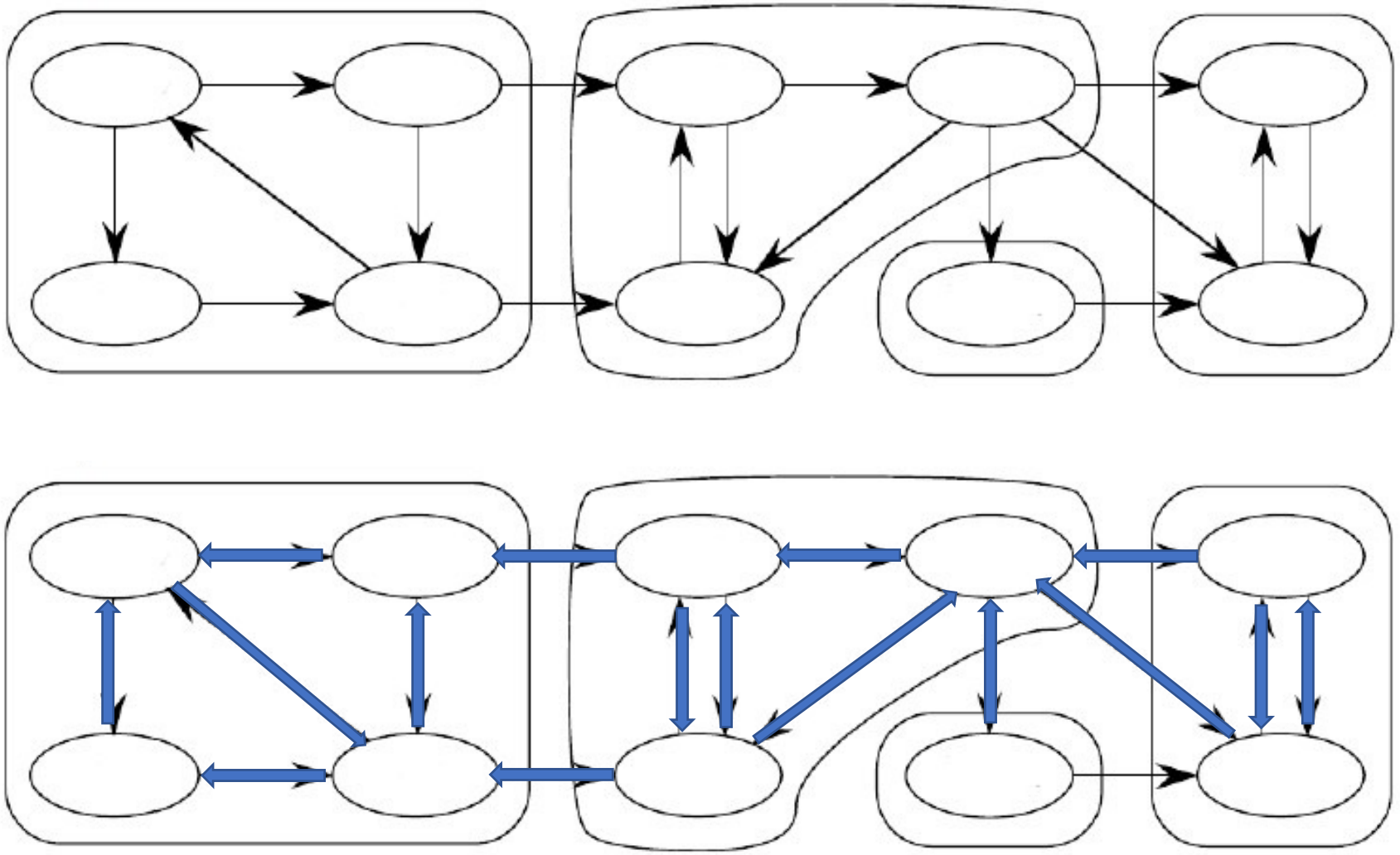
- G and G^T have the same SCC's.
- u and v are reachable from each other in G if and only if they are reachable from each other in G^T.

Component Graph:

- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- V^{SCC} has one vertex from each SCC in G.
- E^{SCC} has an edge if there is an edge between the corresponding SCC's in G.

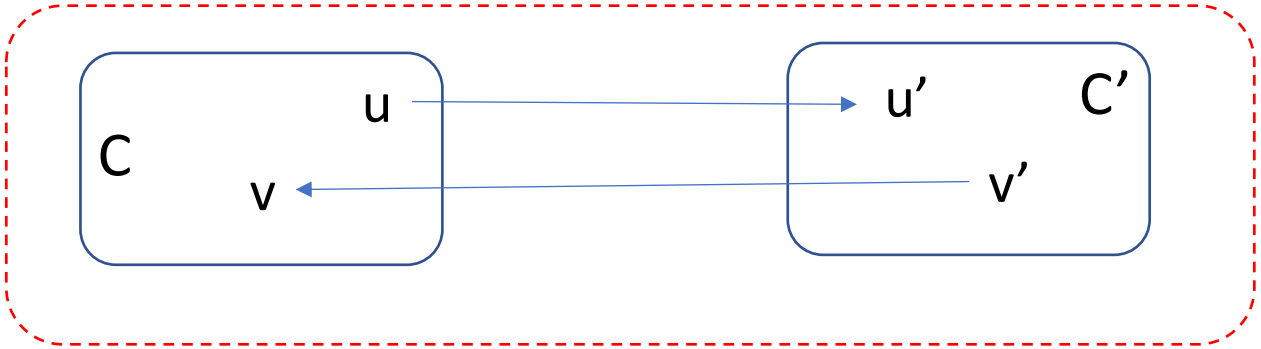
Graphs – Strongly Connected Components (SCC)

Example:



Graphs – Strongly Connected Components (SCC)

- G^{SCC} is a DAG.
- More formally, let C and C' be distinct SCC's in G, let $u, v \in C, u', v' \in C'$, and suppose there is a path $u \to u'$ in G. Then there cannot also be a path $v' \to v$ in G.
- Proof:
  - Suppose there is a path $v' \to v$ in G. Then there are path $u \to u' \to v'$ and $v' \to v \to u$ in G. Therefore, u and v' are reachable from each other, so they are not in separate SCC's.



Graphs – Strongly Connected Components (SCC)

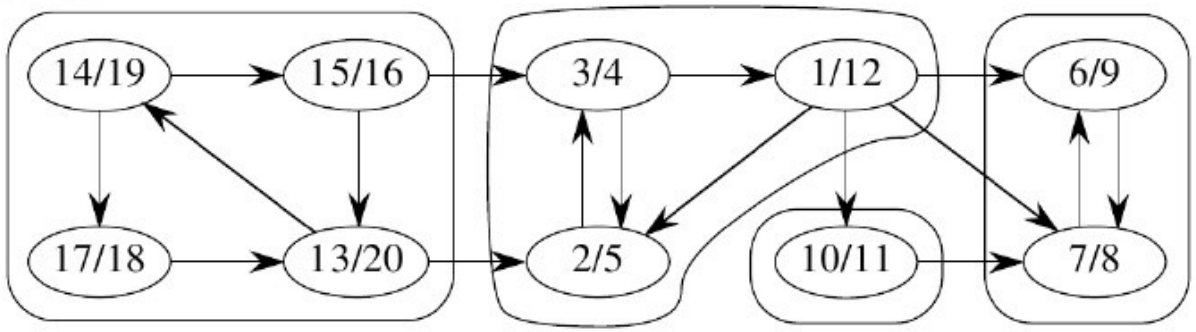
SCC(G):

1. Call DFS(G) to compute finishing times u.f for all u.
2. Compute G^T.
3. Call DFS(G^T), but in the main loop, consider vertices in order of decreasing u.f (as computed in the first DFS).
4. Output vertices in each tree of the depth-first forest formed in the second DFS as a separate SCC.

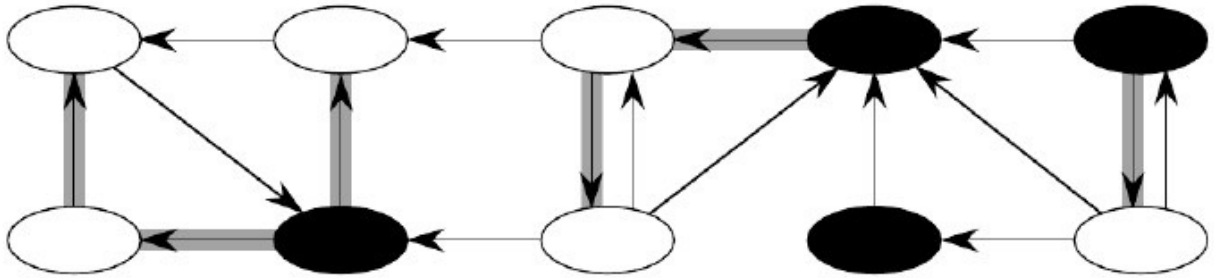
Graphs – Strongly Connected Components (SCC)

Example:

1. Call DFS.



2. Compute G^T.
3. Call DFS (roots blackened)



Time: $\Theta(V + E)$.

Graphs – Strongly Connected Components (SCC)

Idea:

- We are considering vertices in the second DFS in decreasing the order of finishing times from the first DFS.
- We are visiting the vertices of the component graph in topological sort order.
- To prove that it works, we must deal with two notational issues:
  - We will be discussing $u.d$ and $u.f$. These will always refer to the first DFS.
  - We extend the notation for d and f to sets of vertices $U \subseteq V$:
    - $d(U) = \min_{u \in U}\{u.d\}$
    - $f(U) = \max_{u \in U}\{u.f\}$

Graphs – Strongly Connected Components (SCC)

- Let C and C' are distinct SCC's in $G = (V, E)$. If $u \in C$ and $v \in C'$ and if there is an edge $(u, v) \in E$, SCC's are connected as $C \to C'$ but not $C' \to C$!
- DFS will allow $V' \in C'$ to be fully explored before all $V \in C$.

$$\therefore f(C) > f(C')$$

- If $d(C) < d(C')$, all vertices in C and C' are not all discovered except $x \in C$.
- If there is a path from $C \to C'$, then all $V' \in C'$ will be explored before x because DFS runs on x.

$$\therefore x.f = f(C) > f(C')$$

Graphs – Strongly Connected Components (SCC)

- If $d(C) > d(C')$, all vertices in C and C' are not all discovered except $y \in C'$.
- If y is the first discovered vertex, all vertices $V' - y \in C'$ are white$\Rightarrow y.f = f(C')$.
- All vertices, $V \in C$, are white @ $y.d$ time.
- If $C \to C'$ exists, then all $V \in C$ are white and they are not reachable from y.
- Any $x \in C$ will be discovered after $f(C') = y.f$.

$$\therefore f(C) > f(C')$$

- Let C and C' are distinct SCC's in $G = (V, E)$. If $u \in C$ and $v \in C'$ and if there is an edge $(u, v) \in E^T$, there is an edge $(v, u) \in E$.

$$\therefore f(C) < f(C')$$

Graphs – Strongly Connected Components (SCC)

- There are no edges from C to C' in G^T, since $f(C) > f(C')$ for all $C' \neq C$.
  - Therefore, DFS will visit only vertices in C.
  - This means that the depth-first tree rooted at x contains exactly the vertices of C.
  - The next root chosen in the second DFS is in SCC C' such that $f(C')$ is maximum over all SCC's other than C. The DFS visits all vertices in C', but the only edges out of C' go to C, which we have already visited.
- Therefore, the only tree edges will be to vertices in C'.
- We can continue the process.
- Each time we choose a root for second DFS, it can reach only
  - Vertices in its SCC – get tree edges to these,
  - Vertices in SCC's already visited in second DFS – get no tree edges to these.
- We are visiting vertices of $(G^T)^{SCC}$ in reverse of topologically sorted order.

Minimum Spanning Trees

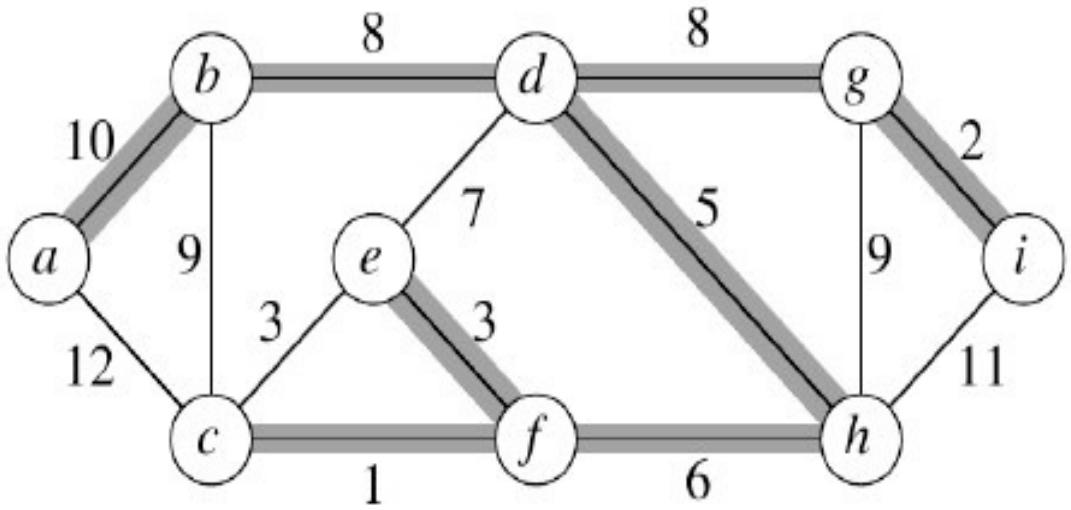
Problem:

- A town has a set of houses and a set of roads.
- A road connects 2 and only 2 houses.
- A road connecting houses u and v has a repair cost $w(u, v)$.
- **Goal**: Repair enough (and no more) roads such that
  - Everyone stays connected: can reach every house from, all other houses, and
  - Total repair cost is minimum.

Model as graph:

- Undirected graph $G = (V, E)$.
- Weight $w(u, v)$ on each edge $(u, v) \in E$.
- Find $T \subseteq E$ such that
  - T connects all vertices (T is a **spanning tree**), and
  - $w(T) = \sum_{(u,v)\in T} w(u, v)$ is **minimized**.
  - $\min\big(w(T)\big)$ is called a **minimum spanning tree (MST).**

Minimum Spanning Trees



- We have more than one MST in this example.
- Replace (e,f) in the MST by (c,e) gives a different MST with the same weight.

Some properties of a MST:

- It has |V|-1 edges.
- It has no cycles.
- It might not be unique.

Minimum Spanning Trees

Building up the solution:

- We will build a set *A* of edges.
- Initially, *A* has no edges.
- As we add edges to *A*, we maintain a loop invariant: *A* is a subset of some MST.
- We add only edges that maintain the loop invariant.
- If *A* is a subset of some MST, an edge (*u*, *v*) is safe for *A* if and only if $A \cup \{(u, v)\}$ is also a subset of some MST ⇒ we will only add safe edges.

Algorithm (GENERIC-MST(G,w))

```
1 A = ∅
2 while (A is not in a spanning tree) do
3   find an edge (u, v) that is safe for A
4   A = A ∪ {(u, v)}
5 return A
```

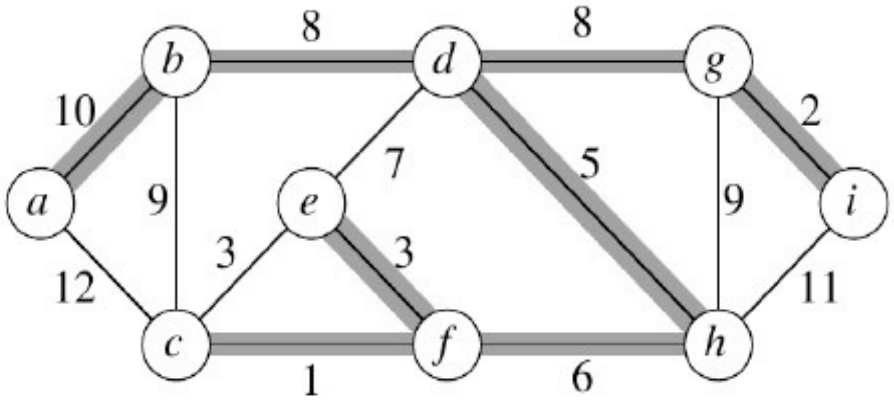
Minimum Spanning Trees

Correctness:

- ❑ Initialization: The empty set trivially satisfies the loop invariants.
- ❑ Maintenance: A remains a subset of some MST, since we add only safe edges.
- ❑ Termination: All edges added to A are in an MST, so when we stop, A is a spanning tree that is also an MST.

Find a safe edge:

- ❑ How do we find safe edges?
- ❑ In this example, the edge (c, f) has the lowest weight of any edge in the graph. Is it safe for $A = \emptyset$?



Minimum Spanning Trees

Intuitively:

- Let $S \subset V$ be any set of vertices that includes c but not f (so that f is in $V - S$).
- In any MST, there has to be one edge (at least) that connects S with $V - S$.
- Why not choose the edge with minimum weight? (would be (c, f) in our case).

Definition:

- Let $S \subset V$ and $A \subseteq E$.
  - A **cut** $(S, V - S)$ is a partition of vertices into disjoint set V and $S - V$.
  - Edge $(u, v) \in E$ **crosses** cut $(S, V - S)$ if one endpoint is in S and the other is in $V - S$.
  - A cut **respects** A if and only if no edge in A crosses the cut.
  - An edge is a **light edge** crossing a cut if and only if its weight is minimum over all edges crossing the cut. For a given cut, there can be > 1 light edges crossing it.

Minimum Spanning Trees

Theorem:

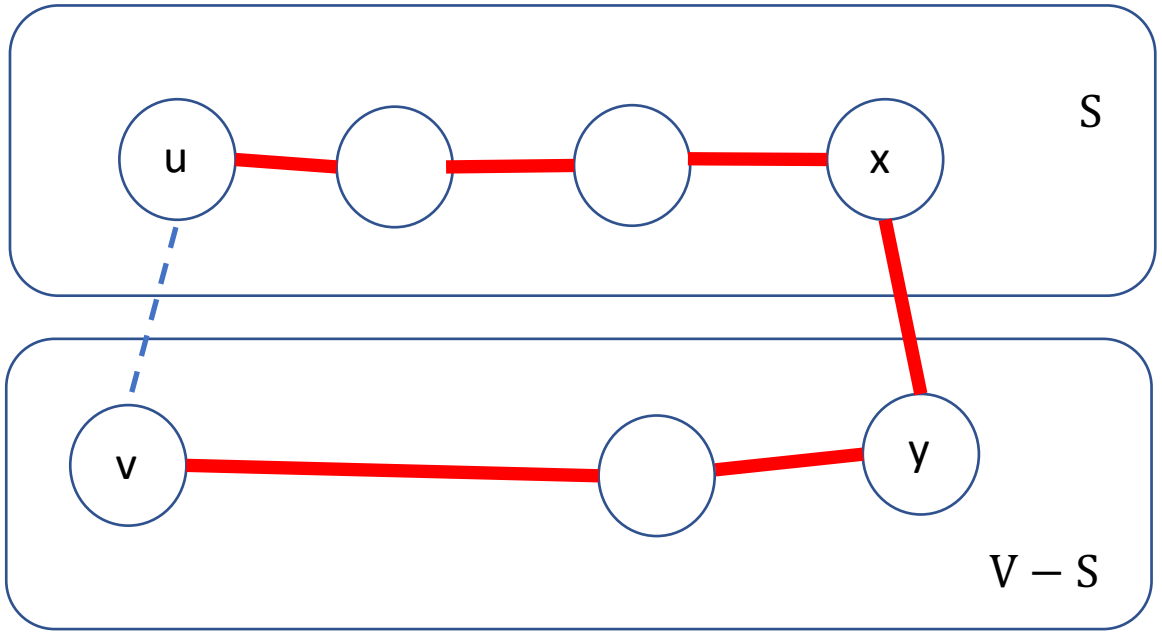
- Let A be a subset of some MST, $(S, V - S)$ be a cut that respects A, and (u, v) be a light edge crossing $(S, V - S)$. Then (u, v) is safe for A.

Minimum Spanning Trees

Proof:

- Let T be an MST that includes A.
- We are done, if T contains (u, v).
- We assume that T does not contain (u, v). We will construct a different MST T' that includes $A \cup \{(u, v)\}$.
- Recall, a tree has a unique path between each pair of vertices.
- Since T in an MST, it contains a unique path p between u and v.
- The path p must cross the cut $(S, V - S)$ at least once. Let (x, y) be an edge of p that crosses the cut. From the way (u, v) is chosen, we must have $w(u, v) \leq w(x, y)$.

Minimum Spanning Trees



Proof:

- Since the cuts respect A, edge (x, y) is not in A.
- To form T' from T:
  - Remove (x, y) . Breaks T into two components.
  - Add (u, v). Reconnects.
  - So $T' = T - \{(x, y)\} \cup \{(u, v)\}$.

Minimum Spanning Trees

- T' is a spanning tree.

$$w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$$

  where $w(u, v) \leq w(x, y)$.
- Since T' is a spanning tree, $w(T') \leq w(T)$, and T is an MST, then T' must be an MST.
- We need to show that (u, v) is safe for A:
  - $A \subseteq T$ and $(x, y) \notin A \Rightarrow A \subseteq T'$.
  - $A \cup \{(u, v)\} \subseteq T'$.
  - Since T' in an MST, (u, v) is safe for A.

Minimum Spanning Trees

In GENERIC-MST:

- A is a forest containing connected components. Initially, each component is a single vertex.
- Any safe edge merges two of these components into one.
- We can consider each component as a tree.
- Since an MST has exactly $|V| - 1$ edges, the for-loop iterates $|V| - 1$ time $\Rightarrow$ After adding $|V| - 1$ safe edges, we are down to just one component.

Minimum Spanning Trees

Corollary:

If $C = (V_C, E_C)$ is a connected component in the forest, $G_A = (V, A)$ and (u, v) is a light edge connecting C to some other component in G_A(i.e., (u, v) is a light edge crossing the cut $(V_C, V - V_C)$), then (u, v) is safe for A.

Kruskal's Algorithm

- It leads to Kruskal's algorithm to solve the MST problem.
- Let $G = (V, E)$ is connected, undirected, weighted graph, $w: E \to \mathbb{R}$.
  - Starts with each vertex being its own component.
  - Repeatedly merges two components into one by choosing the light edge that connects them (i.e., the light edge crossing the cut between them.)
  - Scans the set of edges in monotonically increasing order by weight.
  - Uses a disjoint set data structure to determine whether an edge connects vertices in different components.

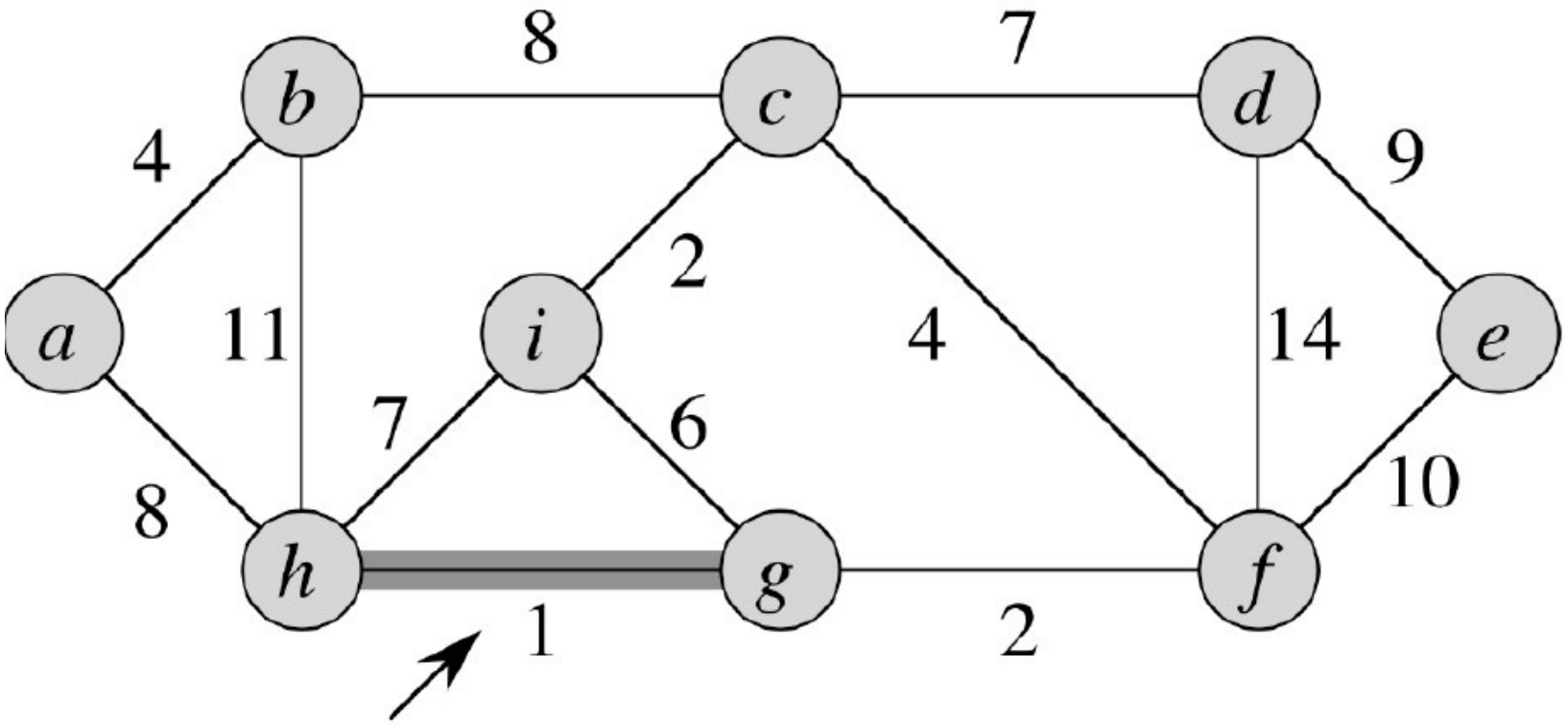
Kruskal's Algorithm

Algorithm (KRUSKAL(G, w))

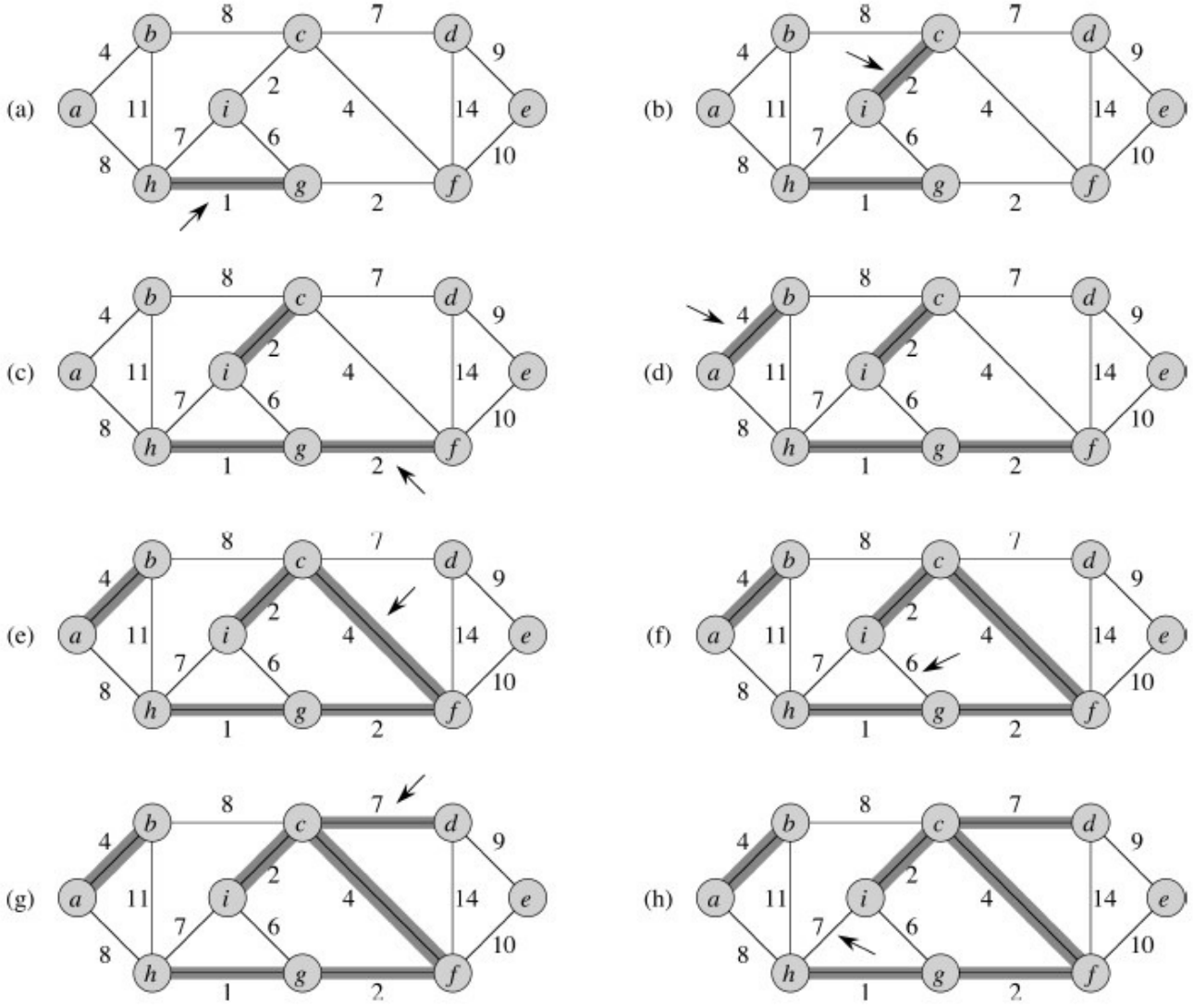
```
(1)  A = ∅
(2)  foreach (vertex v ∈ G.V) do
(3)     MAKE-SET(v)
(4)  od
(5)  sort the edges of G.E into non-decreasing order by weight w.
(6)  foreach ((u, v) taken from the sorted list) do
(7)     if (FIND-SET(u) ≠FIND-SET(v)) then
(8)        A = A ∪ {(u, v)}
(9)        UNION(u, v)
(10)    fi
(11) od
(12) return A
```

MAKE-SET(x) – creates a new set whose only member is x. Since the sets are disjoint, we require that x not already be in some other set.
UNION(x,y) – unites the dynamic sets that contain x and y into a new set that is the union of these two sets.
FIND-SET(x) - returns a pointer to the representative of the (unique) set containing x.

Kruskal's Algorithm

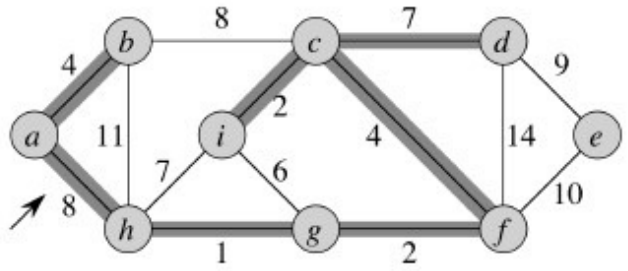


Kruskal's Algorithm

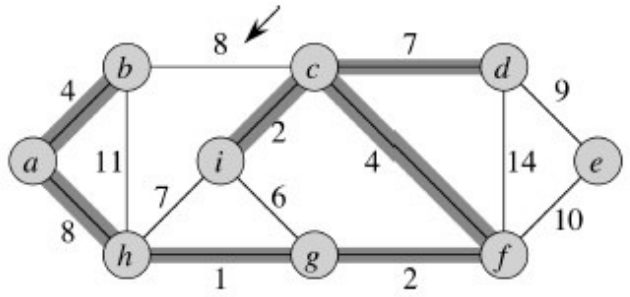


Kruskal's Algorithm

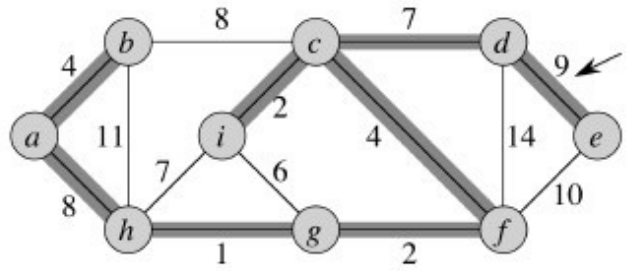
(i)



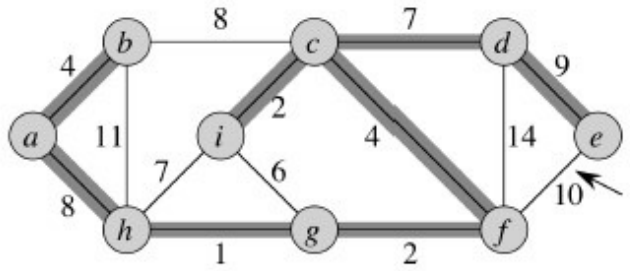
(j)



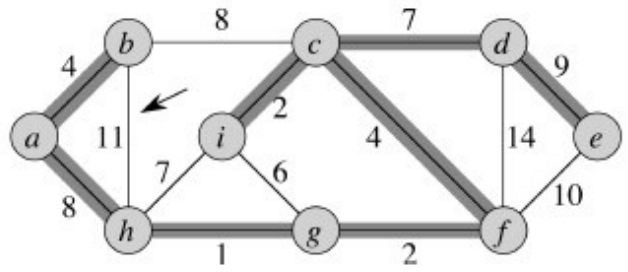
(k)



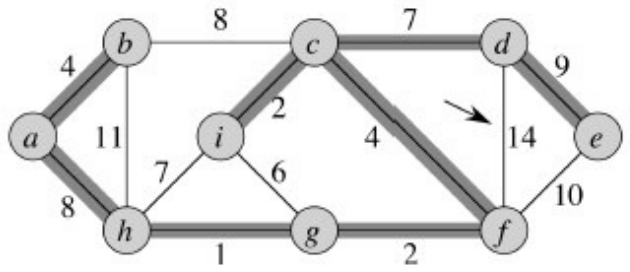
(l)



(m)



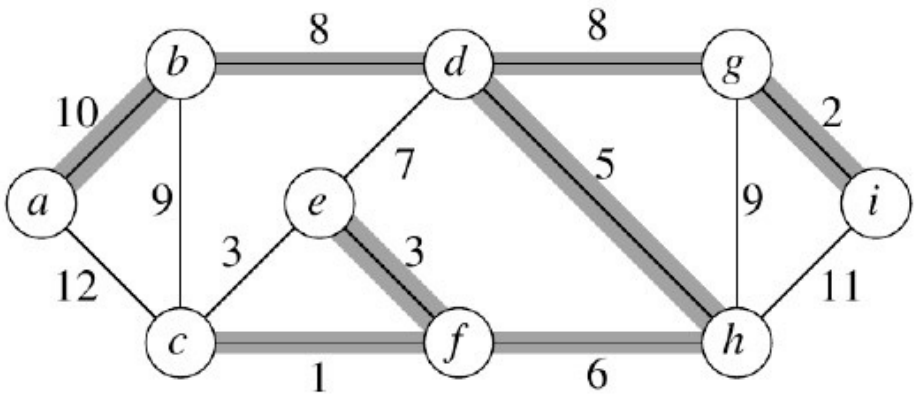
(n)



Kruskal's Algorithm

Example:

- (c, f) : safe
- (g, i) : safe
- (e, f) : safe
- (c, e) : reject
- (d, h) : safe
- (f, h) : safe
- (e, d) : reject
- (b, d) : safe
- (d, g) : safe
- (b, c) : reject
- (g, h) : reject
- (a, b) : safe



If we had examined (c, e) before (e, f). Then (c, e) would have been safe and (e, f) been rejected.

Kruskal's Algorithm

Analysis:

Initialize A: $O(1)$

First for loop: $|V|$ MAKE-SETs

Sort E: $O(E \lg E)$

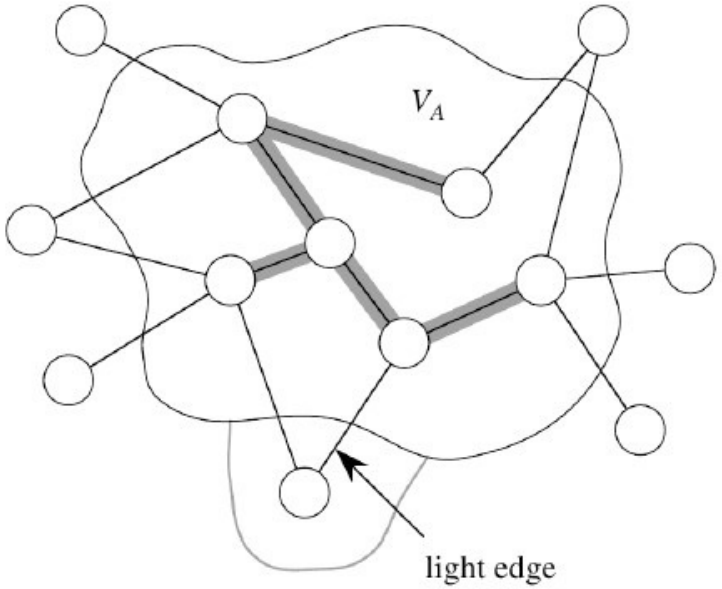
Second for loop: $O(E)$ FIND-SETs and UNIONs

- Assume disjoint-set data structure from Chap. 21: Uses union by rank and path compression: $O((V + E)\alpha(V)) + O(E \lg E)$.
- Since G is connected, $|E| \geq |V| - 1$
  $\Rightarrow O(E\alpha(V)) + O(E \lg E)$.
- $\alpha(|V|) = O(\lg V) = O(\lg E)$.
- Therefore, the total time is $O(E \lg E)$.
- Using $|E| \leq |V|^2 \Rightarrow \lg |E| = O(2 \lg V) = O(\lg V)$.
- Restate $O(E \lg V)$ time for Kruskal algorithm.
  (almost linear $O(E\alpha(V))$, if edges already sorted)

Prim's Algorithm

Prim's algorithm:

- Builds one tree, so A is always a tree.
- Starts from an arbitrary "root" r.
- At each step, we find a light edge crossing cut $(V_A, V - V_A)$, where V_A = vertices that A is incident on. Add this edge to A.



Prim's Algorithm

How to find the light edge quickly?

- Use a priority queue Q:
  - Each object is a vertex in $V - V_A$.
  - Key of v is minimum weight of any edge (u, v), where $u \in V_A$.
  - Then the vertex returned by EXTRACT-MIN is v such that there exists $u \in V_A$ and (u, v) is a light edge crossing $(V_A, V - V_A)$.
  - Key of v is ∞ if v is not adjacent to any vertices in V_A.
- The edges of A will form a rooted tree with root r.
  - r is given as an input to the algorithm, but it can be any vertex.
  - Each vertex knows its parent in the tree by the attribute $v.\pi$ = parent of v. $v.\pi = NIL$ if $v = r$ or v has no parent.
  - As the algorithm progresses, $A = \{(v, v.\pi) : v \in V - \{r\} - Q\}$.
  - At termination, $V_A = V \Rightarrow Q = \emptyset$, so MST is $A = \{(v, v.\pi) : v \in V - \{r\}\}$.

Prim's Algorithm

Algorithm (PRIM(G, w, r))

```
(1)  Q = ∅
(2)  foreach (vertex u ∈ G.V) do
(3)     u.key = ∞
(4)     u.π = NIL
(5)     INSERT(Q, u)
(6)  od
(7)  DECREASE-KEY(Q, r, 0)                    // r.key = 0
(8)  while (Q ≠ ∅) do
(9)     u =EXTRACT-MIN(Q)
(10)    foreach (v ∈ G.Adj[u]) do
(11)       if (v ∈ Q and w(u, v) < v.key) then
(12)          v.π = u
(13)          DECREASE-KEY(Q, v, w(u, v))
(14)    od
(15) od
```

Prim's Algorithm

Algorithm (PRIM(G, w, r))

```
(1)  Q = ∅
(2)  foreach (vertex u ∈ G.V) do
(3)     u.key = ∞
(4)     u.π = NIL
(5)     INSERT(Q, u)
(6)  od
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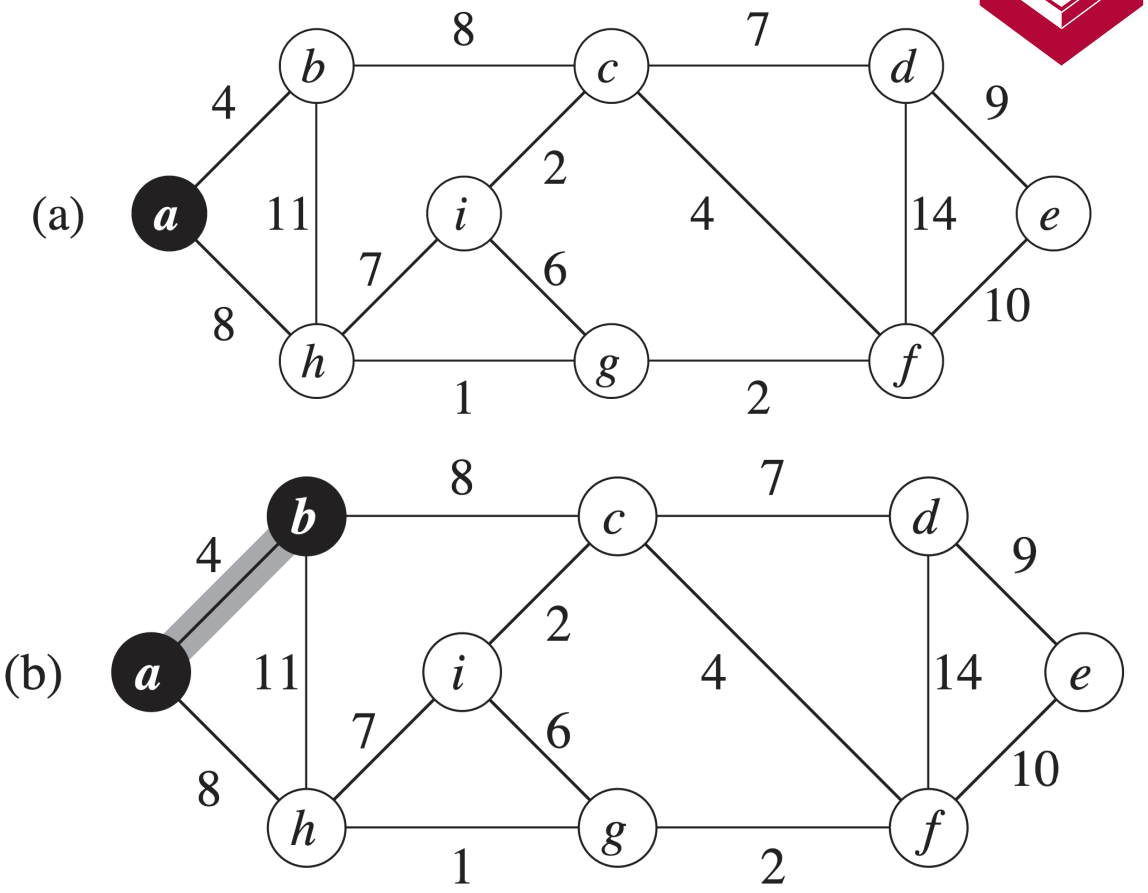
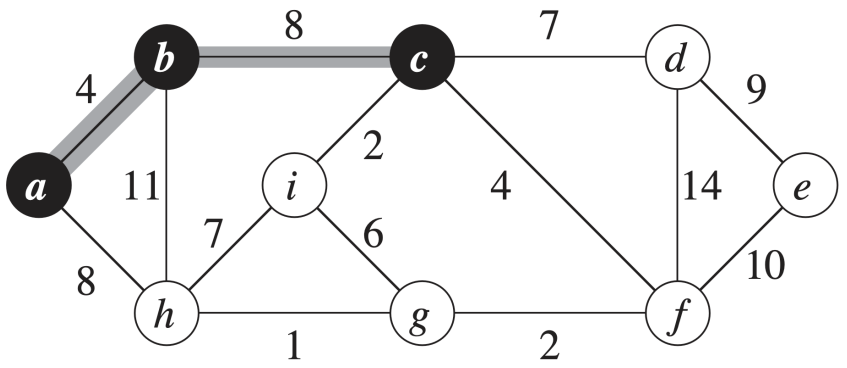
(8)  while (Q ≠ ∅) do
(9)     u =EXTRACT-MIN(Q)
(10)    foreach (v ∈ G.Adj[u]) do
(11)       if (v ∈ Q and w(u, v) < v.key) then
(12)          v.π = u
(13)          DECREASE-KEY(Q, v, w(u, v))
(14)    od
(15) od
```

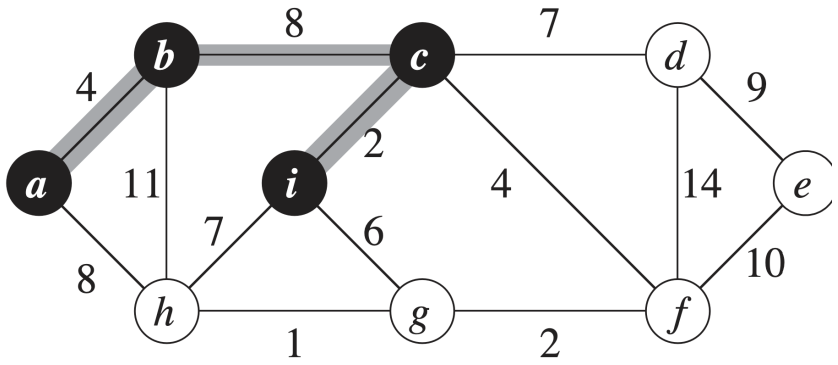


Prim's Algorithm

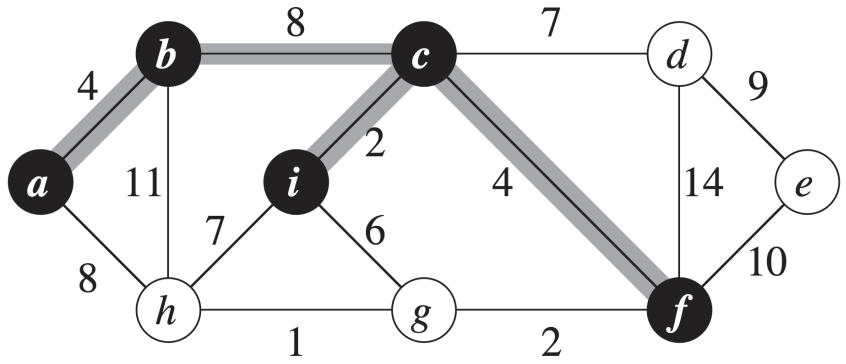
(c)



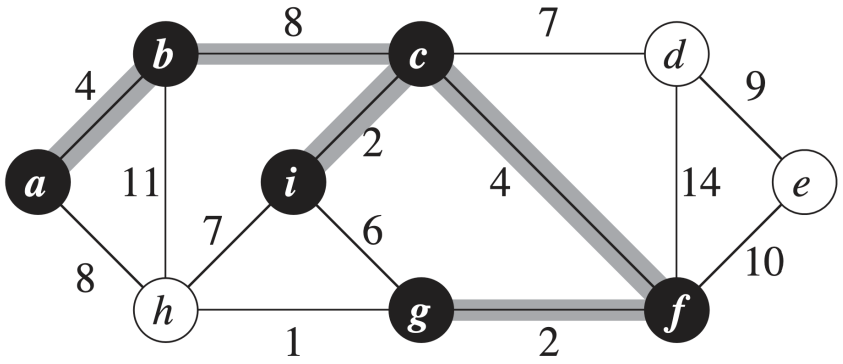
(d)



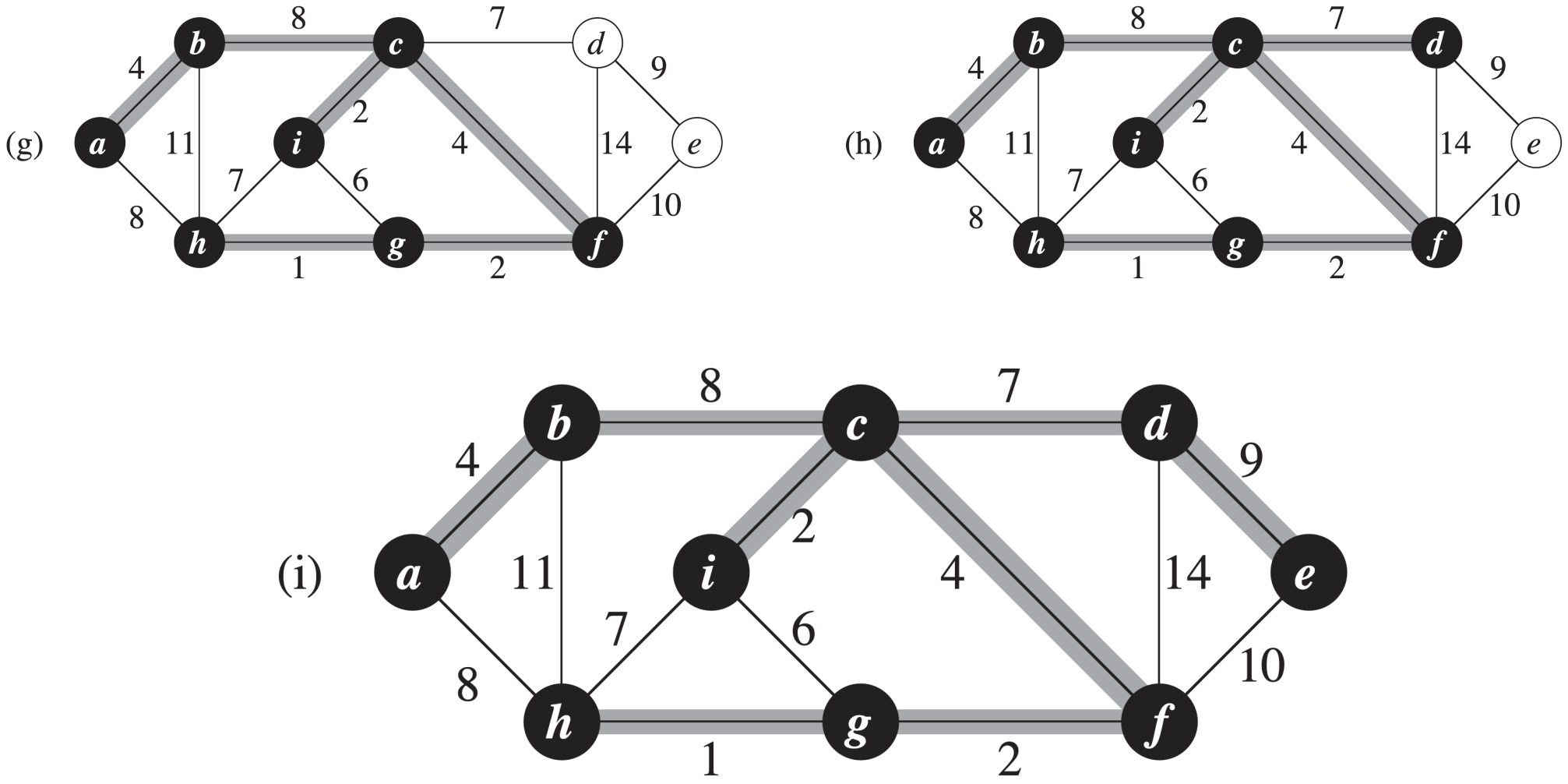
(e)



(f)



Prim's Algorithm



Prim's Algorithm

Analysis: depends on priority queue.

- Suppose Q is a binary heap:
  - Initialize Q and first for loop: $O(V \lg V)$.
  - Decrease key of r: $O(\lg V)$.
  - while loop: $|V|$ EXTRACT-MIN calls $\Rightarrow O(V \lg V) \leq |E|$ DECREASE-KEY calls $\Rightarrow O(E \lg V)$.

  $\Rightarrow$ Total time: $O(E \lg V)$.
- We could do DECREASE-KEY in $O(1)$ *amortized* time.
  Then $\leq |E|$ DECREASE-KEY calls take $O(E)$ time altogether
  $\Rightarrow$ total time becomes $O(V \lg V + E)$.
  (See Fibonacci heaps, Chapter 19)