CS 590 Homework 2 - Recurrences
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Part 1 (20 Points, 4 points each)
Recurrences: Solve the following recurrences using the substitution method. If the provided guess function is not valid, find a new solution. Use Master's Theorem to confirm the solution if applicable. Otherwise, explain the inapplicability.

1. $T(n) = T(n-3) + 3\lg n$. Our guess function $T(n) = O(n\log n)$. Show $T(n) \le cn\lg n$ for some constant $c > 0$. (Note: $\lg n$ is monotonically increasing for $n > 0$)

→ $T(n) = T(n-3) + 3\lg n$
Initial guess f^n $T(n) = O(n\lg n)$
To prove $T(n) \le cn\lg n$ for $c > 0$

Soln :- For $T(n) \le cn\lg n$
$T(n-3) \le c.(n-3)\lg(n-3)$

Substitute $cn\lg n$ in $T(n)$ ∴ $T(n) = c(n-3)\lg(n-3) + 3\lg n$
$= (cn - 3c)\lg(n-3) + 3\lg n$
$cn\lg(n-3) - 3c\lg(n-3) + 3\lg n$

Since we care about the highest term i.e. $cn\lg(n-3)$ and others are constant terms.
and also $\lg(n-3) \simeq \lg n$

∴ $\boxed{T(n) \le cn\lg n}$ Hence Proved

∴ Ans :- Our initial guess function was correct.

2. $T(n) = 4T\left(\frac{n}{3}\right) + n^2$. Our guess function $= O(n^{\log_3 4})$. Show $T(n) \le cn^{\log_3 4}$ for some constant $c > 0$

→ Given $T(n) = 4T\left(\frac{n}{3}\right) + n^2$
Initial guess function $T(n) = O(n^{\log_3 4})$

To prove $T(n) \leq c\, n^{\log_3 4}$ for $c > 0$

Soln :- $T(n) \leq c\, n^{\log_3 4}$

$T(\frac{n}{3}) \leq c(\frac{n}{3})^{\log 4}$

$T(\frac{n}{3}) \leq \frac{c\, n^{\log_3 4}}{4}$

$T(\frac{n}{3}) \leq \frac{c\, n^{\log_3 4}}{4}$

$\Rightarrow T(n) \leq \cancel{4}\left(\frac{c\, n^{\log_3 4}}{\cancel{4}}\right) + n^2$

$T(n) \leq c\, n^{\log_3 4} + n^2$

since $\log_2 4 > 1$

$n^{\log_3 4} > n^2$

$\therefore \boxed{T(n) \leq c\, n^{\log_3 4}}$ → Hence Proved

∴ Ans Our initial guess function proved correct for value $c\, n^{\log_2 4}$

Q3 $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{9}) + n$

Given guess function $T(n) = O(n)$. show $T(n) \leq cn$ for some constant $c > 0$

→ $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n$

Initial Guess function $T(n) = O(n)$

To prove $T(n) \leq c(n)$

Soln :- $T(\frac{n}{2}) \leq c.(\frac{n}{2})$, $T(\frac{n}{4}) \leq c.\frac{n}{4}$, $T(\frac{n}{8}) \leq c.\frac{n}{8}$

$T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + T(\frac{n}{8}) + n$

$T(n) \leq c.\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} + n$

$T(n) \leq c.\frac{7}{8}n + n$

Considering highest exponent term $c.\frac{7}{8}n$ since $\frac{7}{8}n > n$

$\boxed{T(n) \leq c.n}$ Hence Proved

Our Initial guess function proved correct for cn. $c>0$.

Q4. $T(n) = 4T(\frac{n}{2}) + n$. Our initial guess function : $T(n) = O(n^2)$

Show $T(n) \leq C.n^2$ for some constant $c>0$

→ $T(n) = 4T(\frac{n}{2}) + n$

Initial guess function $T(n) = O(n^2)$

To prove $T(n) \leq c.n^2$

Soln :- $T(\frac{n}{2}) \leq c.(\frac{n}{2})^2$

$T(\frac{n}{2}) \leq c.\frac{n^2}{4}$

$T(n) = 4.(c.\frac{n^2}{4}) + n$

$T(n) \leq cn^2 + n$

Since $n^2 > n$

$\therefore \boxed{T(n) \leq cn^2}$ Hence Proved

Our initial guess fn $T(n) = O(n^2)$ proved correct for $c>0$

Q5 $T(n) = 27T(\frac{n}{3}) + n^3$. Our initial guess function $T(n) = O(n^3)$

Show $T(n) \leq cn^3$ for some constant $c>0$

→ $T(n) = 27T(\frac{n}{3}) + n^3$

Initial guess function $T(n) = O(n^3)$

To prove $T(n) \leq c.n^3$

$T\left(\frac{n}{3}\right) \leq c.\left(\frac{n}{3}\right)^3$

$T\left(\frac{n}{3}\right) \leq c. \frac{n^3}{27}$

$T(n) = 27\, c\left(\frac{n^3}{27}\right) + n^3$

$T(n). = c.n^3 + n^3$

$\boxed{T(n) \leq c.n^3}$ Hence Proved.

Our initial guess function proved right for $c>0$ ie. $T(n) = O(n^3)$

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Part 2: Master's Theorem (20 points, 4 points each)
For each of the following recurrences, give an expression for the running time $T(n)$ if the recurrences can be solved with the Master's Theorem. Please provide the case number & constants (ϵ, k) if applicable. Otherwise, indicate that the Master Theorem does not apply

1. $T(n) = 3T\left(\frac{n}{2}\right) + n^2$

$a = 3$ $b = 2$ & $k = 2$ // ($n^2 = k = 2$)

$\log_b a = \log_2 3 = 1.58$ $p = 0$

$\log_b a < k$

$\therefore$ Using case 3 to solve the problem

$T(n) = \theta(n^{\log_b a + \epsilon})$ for $\epsilon > 0$

$\theta(n^k \log^p n)$

$\theta = (n^2 \log^p n)$

$\boxed{T(n) = \theta(n^2)}$ Proved

2 $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

$a = 2^n$ $b = 2$ $k = n$ & $p = 0$

We cannot apply master's Theorem since value of 'a' is dependent on 'n' & n is not a constant value

3 $T(n) = 3T\left(\frac{n}{4}\right) + n \log n$

Given: $a = 3$ $b = 4$ $k = 1$ & $p = 1$

$\therefore \log_b a = \log_4 3 = 0.79$

$\therefore \log_b a < K$

Solve using case 3

$T(n) = \Theta(n^k \log^l n)$

$\therefore \boxed{T(n) = \Theta(n \log n)}$ = proved.

4. $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

Given:- $a = 2$ $b = 2$ $K = 1$ & $l = -1$

$\log_b a = \log_2 2 = 1$

$\log_b a = K$

We need to check $n^{\log_b a} = n$ & $f(n) = \frac{n}{\log n}$ & since $n^{\log_b a}$ &

f(n) does not have a non-polynomial difference between them

$\therefore$ we cannot apply master's theorem

5. $T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$

Given $a = 0.5$ $b = 2$ $K = -1$ $l = 0$

Ans:- We cannot apply master's theorem since 'a' cannot be less than '1'

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