Approximation Schemes for Fully Nonlinear Second Order Equations

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Stochastic Optimal Control

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Outline

Viscosity Solutions and Ellipticity

Let $\Omega \subset \mathbb{R}^d$ be an open set. A continuous function $u \in C(\Omega)$ is called a *viscosity solution* of

$$F(x, u, Du, D^2u) = 0$$

if for every $\varphi \in C^2(\Omega)$ and every point $x_0 \in \Omega$, the following hold:

1. If $u - \varphi$ has a local maximum at x_0, then

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0.$$

2. If $u - \varphi$ has a local minimum at x_0, then

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geqslant 0.$$

Example

Equation $u'' = 0$ has **no** viscosity solution!

Strong Comparison and the Intuition Behind It

Consider

$$G(x, u, Du, D^2 u) \text{ in } \overline{\Omega}.$$

Strong Comparison Principle

If $u \in USC$ — subsolution and $v \in LSC$ — supersolution, then $u \leqslant v$ on $\overline{\Omega}$.

How Numerics Help

Discrete maximum principle = arithmetic average inequality.

Boundary Conditions in the Viscosity Sense
Learning on Examples

Example

$$\begin{cases} u_\varepsilon' - \varepsilon u_\varepsilon''(x) = 1, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. \end{cases}$$

Boundary Conditions in the Viscosity Sense
Useful Definition

Consider the following Dirichlet problem:

$$\begin{cases} H(x, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } \Omega, \\ u_\varepsilon = g & \text{on } \partial\Omega. \end{cases}$$

We can assume that $u_\varepsilon \leqslant g$ on the boundary, and then we find that for all $x \in \partial\Omega$, either

$$u^*(x) \leqslant g(x) \quad \text{or} \quad H(x, u^*(x), D\varphi(x), D^2\varphi(x)) \leqslant 0,$$

or, more compactly,

$$\min\left\{u^* - g,\ H(x, u^*(x), D\varphi(x), D^2\varphi(x))\right\} \leqslant 0.$$

Similarly, for a point of local minimum of $u - \varphi$, corresponding to a future supersolution, we obtain:

$$\max\left\{u_* - g,\ H(x, u_*(x), D\varphi(x), D^2\varphi(x))\right\} \leqslant 0.$$

Boundary Conditions in the Viscosity Sense

Discontinuous Viscosity Solutions

$u \in USC$ is a viscosity **subsolution** of the equation

$$G(x, u, Du, D^2u) = 0 \quad \text{on } \overline{\Omega}$$

if and only if

$$\forall \varphi \in C^2(\overline{\Omega}),\ \text{if } x_0 \in \overline{\Omega} \text{ is a maximum point of } u - \varphi,\ \text{then}$$

$$G_*(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0.$$

$v \in LSC$ is a viscosity **supersolution** of the equation

$$G(x, u, Du, D^2u) = 0 \quad \text{on } \overline{\Omega}$$

if and only if

$$\forall \varphi \in C^2(\overline{\Omega}),\ \text{if } x_0 \in \overline{\Omega} \text{ is a minimum point of } u - \varphi,\ \text{then}$$

$$G^*(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geqslant 0.$$

Notation and Properties

$$S(\rho, x, u^\rho(x), u^\rho) = 0 \text{ in } \overline{\Omega}, \qquad F(x, u, Du, D^2u) = 0 \text{ in } \overline{\Omega}.$$

Examples

1. Heat Equation, Explicit Scheme:

$$S((n+1)\Delta t, j\Delta x, u_j^{n+1}, u_{j+1}^n, u_j^n, u_{j-1}^n) =$$
$$= \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{1}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

2. Heat Equation, Implicit Scheme:

$$S((n+1)\Delta t, j\Delta x, u_j^{n+1}, u_{j+1}^{n+1}, u_j^{n+1}, u_{j-1}^{n+1}) =$$
$$= \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{1}{(\Delta x)^2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right).$$

Notation and Properties

Key Ingredients

1. **Monotonicity:**

$$S(\rho, x, t, v) \leqslant S(\rho, x, t, v), \quad \text{if } u \geqslant v,$$

for any $\rho > 0, x \in \overline{\Omega}, t \in \mathbb{R}$ and $u, v \in B(\overline{\Omega})$.

2. **Consistency:**

$$\varlimsup_{\substack{\rho \to 0+ \\ y \to x \\ \xi \to 0}} S(\rho, y, \varphi(y) + \xi, \varphi + \xi) \leqslant F^* \left(x, \varphi(x), D\varphi(x), D^2\varphi(x) \right),$$

and the opposite inequality for another limit.

3. **Stability:** For all $\rho > 0$ there exists uniformly bounded family of solutions u^ρ.

Main Result

Theorem

Consistency + Monotonicity + Stability

+ Strong Comparison Principle

$\Longrightarrow$ **Locally Uniform Convergence** to the viscosity solution.

Sketch of the Proof (Key Stages)

1. We set:

$$u^*(x) = \overline{\lim_{\substack{\rho\to 0\\ y\to x}}}\, u^\rho(y), \quad u_*(x) = \underline{\lim_{\substack{\rho\to 0\\ y\to x}}}\, u^\rho(y).$$

2. Monotonicity + Constistency $\to$ sub and supersolutions.
3. By the strong comparison result, we have:

$$u^* \leqslant u_* \text{ on } \overline{\Omega}.$$

4. But, by the definition:

$$u_* \leqslant u^* \text{ on } \overline{\Omega}.$$

5. Therefore $u^* = u_*$.
   Uniform convergence is done by 'Dini'.

Rate of Convergence
Godunov's Result

For simplicity, we consider first order equation $F = F(x, u, Du, \cancel{D^2 u})$

Theorem

Let S be monotone and **smooth**, and let

$$F_p(x, u, p) \neq 0$$

at least for one point (x, u, p). Then, there exist numbers $M, c, C \in \mathbb{R}_+$ and $\bar{h}$, such that:

$$ch \leqslant \mathrm{err}(M, h) \leqslant Ch.$$

Idea of proof: first ordered term in the Taylor expansion cannot be cancelled due to monotonicity.

1D Eiconal

This example should be everywhere

Example

Consider a finite difference scheme for

$$\left|\nabla^+ u_h(x)\right| = 1 \quad \text{for } x \in [0,1)_h, \quad \text{and} \quad u_h(0) = u_h(1) = 0.$$

Claim: It is not well-posed.

Heat Equation
Explicit Scheme

$$
\begin{aligned}
S((n+1)\Delta t, j\Delta x, u_j^{n+1}, u_{j+1}^n, u_j^n, u_{j-1}^n) = \\
= u_j^{n+1} - u_j^n - \frac{\Delta t}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).
\end{aligned}
$$

Heat Equation
Implicit Scheme

$$S((n+1)\Delta t, j\Delta x, u_j^{n+1}, u_{j+1}^n, u_j^n, u_{j-1}^n) =$$
$$= \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{1}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

Transport Equation
10 Minutes of the Implementation Excercises

Linear Example

What is a correct discretization?

$$\begin{cases} u_t' - u_x' = 0, \\ u(0,x) = \sin x. \end{cases}$$

Obstacle Equation
Recognized me?

American Option

$$\max\left(\frac{\partial u}{\partial t} - \Delta u,\ u - \psi\right) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

1. With given u^n, we solve

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = 0, \\ w(0, x) = u^n(x). \end{cases}$$

   And then set $u^{n+1/2}(x) = w(x, \Delta t)$.
2. Then, finally

$$u^{n+1} = \inf\left(S(\Delta t)u^n,\ \psi^{n+1}\right) = \inf\left(u^{n+1/2},\ \psi^{n+1}\right),$$

   where S — heat kernel (rollback operator).

Conclusions

For Master Students:

1. The Barles–Souganidis framework can help you easily prove convergence for almost anything you've implemented.
2. However, this convergence is only to the viscosity solution.
3. You cannot accelerate the convergence without losing theoretical guarantees.

For PhD Students:

1. It took Barles and Souganidis four years to establish this result, even though all the definitions were introduced by them.
2. Good notation solves everything.

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Thank you for your attention!