Home Work 11

Ques 1. Prove that the following problem is in NPC: Given an undirected graph $G=(V,E)$, determine whether there is a spanning tree whose degree is not greater than k. That is, whether there is a subgraph $G'(E',V)$, $E' \subset E$, $|E'|=|V|-1$, G' is a connected graph and all its node degrees are less than or equal to k.

Ans. • Given a graph G and a set of edges that form a spanning tree, we can verify in polynomial time if the set of edges form a spanning tree and that spanning tree has degree not greater than k. Hence this is in NP.

• Now, we reduce from Hamiltonian Path Problem. Given a graph G and an instance of Hamiltonian Path problem, we will construct G' as an undirected graph with unit weights on all edges. Then for every set of k nodes in G, we will call the blackbox that solves if there is a minimum spanning tree whose degree is not greater than k. If any of these calls returns a tree such that all nodes in the tree have degree not greater than k, it means that we have found a Hamiltonian Path in G.

Ques2. You are given a directed graph $G=(V,E)$ with weights on its edges $e \in E$. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-Complete.

Ans.

- Given a cycle, it is easy to verify that its weight is 0. Hence the problem is in the NP.
- Next, we use reduction from subset sum problem. We have a set of numbers, positive and negative, we have to decide whether there exists a subset whose sum is exactly 0.
- We construct a graph with $2n$ vertices. For each number a_i the graph contains two vertices: u_i and v_i. From each u_i, there is only one outgoing edge, which goes to v_i and has weight a_i.
  From each v_i, there are n outgoing edges, which go to each u_j and have weights 0.
  Any cycle in this graph have the form $u_1 - v_1 - u_2 - v_2 - \dots - u_k - v_k$. The weight of a cycle is 0, iff the sum of all weights between u_i and its corresponding v_i is 0, iff the sum of all corresponding a_i is 0, iff there is a subset with a sum of 0.

Ques 3. In a certain town, there are many clubs, and every adult belongs to at least one club. The town's people would like to simplify their social life by disbanding as many clubs as possible, but they want to make sure that afterwards everyone will still belong to at least one club.

Formally the Redundant Clubs problem has the following input and output. INPUT:

List of People; list of clubs; list of members of each club; number K.

OUTPUT: Yes if there exists a set of K clubs such that, after disbanding all clubs in this set, each person still belongs to at least one club. No otherwise.

Prove that the redundant clubs problem is NP-Complete.

Ans.

- If we are given a set of k clubs, we can check in polynomial time if each person is a member of another club outside this set. Hence this is in NP.
- We reduce from Set cover problem. We use the set cover's elements as our translated list of people, and make a list of clubs, for each member of the Set cover family.

- Elements of corresponding family are members of each club. We say that $k = F - k_{SC}$, where F is the no. of families in the set cover instance and k_{SC} is the value k from the set cover instance.
- If we have an instance with a cover consisting of k_{SC} subsets, the other k subsets form a solution to the translated Redundant Clubs problem, because each person belongs to a club in the cover.
- Conversely, if we have k redundant clubs, the remaining k_{SC} clubs form a cover. So the answer to the set cover instance is yes if and only if the answer to the translated Redundant clubs instance is yes.

Ques 4. Given a graph $G=(V,E)$ with an even number of vertices as the input, the HALF-IS problem is to decide if G has an independent ~~size~~ set of size $|V|/2$. Prove that HALF-IS is in NP-Complete.

Ans.
- Given a graph $G(V,E)$ and a certifier $S \subset V$, $|S| = |V|/2$, we can verify if no two nodes are adjacent in polynomial time. Hence HALF_IS is in NP.
- Next, we use a reduction of Independent Set problem. Consider a instance of Independent set, which is $A \subset V$, and $|A| = k$ (A is independent set), for a graph $G(V,E)$, such that no two pair of vertices in A are adjacent to each other.
- If $k = |V|/2$, Independent Set reduces to HALF-IS.
- If $k < |V|/2$, we add m new nodes such that $k+m = (|V|+m)/2$. Modified set has even no. of nodes, and all additional nodes are disconnected from each other, they form a subset of independent set. Therefore, the new graph $G'(V',E')$ has an independent set of size $|V'|/2$ if and only if $G(V,E)$ has an independent set of size k.
- If $k > |V|/2$, then again we add $m = |V| - 2k$ new nodes. Connect these new nodes to all other $|V|+m-1$ nodes. Since these m new nodes are connected to every other node, none of them should belong to an independent set. Therefore, the new graph $G'(V',E)$ has an independent set of size $|V'|/2$ if and only if $G(V,E)$ has an independent set of size k.

Ques. There are n courses at USC, each of them requires multiple disjoint time intervals. For example, a course may require the time from 9 am to 11 am and 2 pm to 3 pm and 4 pm to 5 pm (you can assume the no. of intervals of a course is at least 1, at most n). You want to know, given a number k, if its possible to take at least k courses. You cannot choose any two overlapping courses. Prove that the problem is NP-complete, which means that choosing courses is indeed a difficult thing in our life. Use a reduction from the independent set problem.

Ans.

- Given a graph $G(V,E)$ and a set of vertices we can check in polynomial time if no pair of vertices of a different independent set are connected by an edge.
- Next, we use a reduction from independent set problem, consisting of a graph G such that we want to find the largest set of vertices such that no two vertices share an edge.
- Each vertex v is represented by a subject e_v. For each edge (u,v) we say that subject e_u and e_v have overlapping timing. It can be shown that a solution to choosing courses problem gives a solution to the independent set problem.