HomeWork 1

Ques 1. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Answer • I think the above statement is False.

- Consider the preference list,

m_1	m_2	m_3		w_1	w_2	w_3
w_1	w_1	w_2		m_3	m_2	m_1
w_2	w_2 -	w_1 -		m_2	m_1	m_2
w_3 -	w_3	w_3		m_1	m_3	m_3

- when men propose,

w_1	w_2	w_3
~~m_1~~	~~m_3~~	m_1
~~m_2~~	~~m_1~~	
m_3	m_2	

Matchings - (m_1, w_3), (m_2, w_2), (m_3, w_1)

- Here, none of the man is matched to the woman who is ranked first on his preference list.

- When women propose

m_1	m_2	m_3
w_3	w_2	w_1

Matchings are (m_1, w_3), (m_2, w_2) and (m_3, w_1).

Again, none of the men are matched to the woman first on their rank list.

- Hence the the statement is <u>False</u>.

Ques 2. Decide whether you think the following statement is true or false. If it is true, give a short explaination. If it is false, give a counterexample.

True or False? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

Ans. I think the above statement is <u>True</u>.

Proof by contradiction

- Consider, m ranks first in preference list of w. But, in a stable matching S, the returned pair is (m', w).

- This is possible when
  1) w was previously engaged to m but broke her engagement to get engaged to m'. → Contradiction
  (∵ m ranks first in ~~its~~ preference list of w, she will not break her engaged for m').
  2) m never proposed to w.
  ↳ Contradiction
  (∵ w is ranked first in the preference list of m, and men first propose to the woman ranked first on their list.

- Hence, (m', w) is an instability.

- Our assumption that the matching returned is (m', w) is wrong.

- So, if a man ranks a woman first and a woman ranks a man ~~first~~ first in her preference list, ~~they~~ then the pair (m, w) belong to S.

Ques 3. Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation.

For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

Ans. I think the above statement is "True".

consider the preference list,

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_2	m_3	m_2	m_1
w_2	w_2	w_1	m_2	m_1	m_2
w_3	w_3	w_3	m_1	m_3	m_3

when men propose,

w_1	w_2	w_3
~~m_1~~	~~m_3~~	m_1
~~m_2~~	~~m_1~~	
m_3	m_2	

the matchings returned are -
(m_1, w_3), (m_2, w_2), (m_3, w_1)

In the above preference list, all woman are matched to their most prefered man, even though none of the men are matched to their most preferred woman.

Ques 4. Four students a, b, c and d, are rooming in a dormitory. Each student ranks the others in strict order of preference. A roomate matching is defined as a partition of the students into two groups of two roomates each. A roommate matching is stable if no two students who are not roomates prefer each other over their roomates.

Does a stable roomate matching always exist? If yes, give a proof. Otherwise give an example of roomate preferences where no stable roomate matching exists.

Ans. • Consider 4 roomates a, b, c, d.

• Consider the preference list

a	b	c	d
b	c	a	a
c	a	b	b
d	d	d	c

- ~~A~~ roomate matching on the above preference list will have pairs

(i) (a, b)
(c, d)

(ii) (a, c)
(b, d)

(iii) (a, d)
(b, c)

(i) In this pairing (a, b), (c, d),
in the preference list b prefers c over a.
and and c prefers b over d.
↳ Instability.

(ii) In this pairing (a, c), (b, d),
in the preference list a prefers b
over c and b prefers a over d.

(iii) In this pairing (a, d), (b, c),
in the preference list, a prefers c
over d and c prefers a over b.

- Hence, in all ~~these~~ three possible pairings none is a stable match in this preference list.

- Hence, no stable roomate matching exists for this preference list.

Ques 5. Gale and Shapley published their paper on the Stable Matching Problem in 1962; but a version of their algorithm had already been in use for ten years by the National Resident Matching Problem, for the problem of assigning medical residents to hospitals.

Basically, the situation was the following. There were m hospitals, each with a certain number of available positions for hiring residents. There were n medical students graduating in a given year, each interested in joining one of the hospitals. Each hospital had a ranking of the students in order of preference, and each student had a ranking of the hospitals in order of preference. We will assume that there were more students graduating than there were slots available in the m hospitals.

The interest, naturally, was in finding a way of assigning each student to at most one hospital, in such a way that all available positions in all hospitals were filled. (Since we are assuming a surplus of students, there would be some students who do not get assigned to any hospital.)

We say that an assignment of students to hospitals is stable if ~~neigh~~ neither of the following situations arises.

- First type of instability: There are students s and s', and a hospital h, so that
  - s is assigned to h, and
  - s' is assigned to no hospital, and
  - h prefers s' to s.

- Second type of instability: There are students s and s', and hospitals h and h', so that
  - s is assigned to h, and
  - s' is assigned to h', and
  - h prefers s' to s, and
  - s' prefers h to h'.

So we basically have the stable Matching Problem, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

Show that there is always a stable assignment of students to hospitals, and give an algorithm to find one.

Ans. **Step1:**
Input : Preference list of Students
Preference list of hospitals

Output : Matching of students to hospitals.

Step2: Algorithm

Initially all $s \in S$ and $h \in H$ are free
While there is a open position in a hospital

Choose such hospital h

Let s be the highest - ranked student in h's preference list to whom h has not offered

If s is free then
(s, h) is a ~~pair~~ matching and s accepts the offer by h.
free_position_h--
Else ~~to~~ s has offer from h'
If ~~s~~ prefers h' to h then
~~the~~ position of h remains empty
Else ~~to~~ s prefers h to h'
(s, h) becomes a matching
the position by h' becomes free
free_position_h'++
free_position_h--
End if
End if
End while
Return the set S of engaged pairs

Step3 : Proof of correctness

(1)

h_1	h_2 (1)	s_1	s_2	s_3
s_3	s_3	h_1	h_1	h_2
s_1	s_2	h_2	h_2	h_1
s_2	s_1			

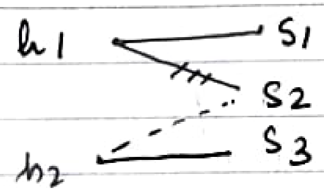
→

s_1	s_2	s_3
h_1	[illegible]	~~h_1~~ h_2

Proof by contradiction :

Consider all hospitals are assigned students (residents) and there exists an instability.

(1)



(2)

i.e, (h_2, s_2)

- First instability → consider, h_2 prefers s_2 and s_2 is not paired up.

Now, did h_2 offer a position to s_2? If s_2 is free it means, h_2 never offered the position to s_2 as all its position are full. Since, it offers the position to highest prefered student, h_2 prefers s_3 over s_2.
← contradiction

Also, it is possible that h_2 offered the position to s_2 but s_2 did not accept, it is possible when s_2 has already accepted offer from its

highest preferred hospital.
It means s_2 cannot be free.
↳ Contradiction!

Hence, first (h_2, s_2) is a instability and ~~cannot~~ was never returned by the algorithm.

• Second instability,

(h_1, s_3) — instability ~~(h_2, s_2)~~

Consider (h_1, s_1), (h_2, s_3) returned by algorithm and assume that h_1 prefers s_3 over s_1 and s_3 prefers h_1 over h_2.

Now, did h_1 offer a position to s_3? In that case, two situations are possible.

(i) s_3 rejected proposal by h_1 since s_3 has offer from ~~to~~ some hospital h' whom s_3 prefers over h_1. If s_3 is matched ~~by~~ to h_2 after end of algorithm it means s_3 prefers h_2 over h' or $h' = h_2$. Hence, it does not prefer h_1 over h_2
↳ contradiction

(ii) s_3 was matched to h_1 and accepted proposal by ~~of~~ another hospital who it prefers over h_1 and refuses the ~~pro~~ offer by h_1. If (h_2, s_3) is returned by the algorithm, it means s_3 prefers h_2 over the hospital it had offer from before. Here, ~~...~~ s_3 prefers h_2 over h_1

Contradiction

Hence, there is a stable ~~matching~~ assignment of student to hospitals always.

Ques 6. For this problem, we will explore the issue of truthfulness in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now ~~to~~ consider a woman w, suppose w prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preference (i.e., by falsely claiming that she prefers m' to m) and running the algorithm with this false preference list, w will end up with a man m'' that she truely prefers to both m and m'? (we can ask the same question for men, but will focus on the case of women for purpose of this question.)

Resolve this question by doing one of the following two things:

(a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or

(b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

Ans. • Consider the preference list,

m	m'	m''	w1	w2	w3
w1	w1	w3	m''	m	m
w3	w3	w1	m	m'	m''
w2	w2	w2	m'	m''	m'

- If we run Gale-Shapley algorithm, the matchings we get are: (m, w_1) (m', w_2) and (m'', w_3)

w_1	w_2	w_3
m	m'	m''

- Consider, ~~we~~ that woman w_1 lies and switches the order of m' and m, both of which are ranked lower in her preference list than her true preference m''.

- New preference list of $w_1 \rightarrow$

w_1
m''
m'
m.

- Now if we run Gale-Shapley algorithm on,

m	m'	m''	w_1	w_2	w_3
w_1	w_1	w_3	m''	m	m
w_3	w_3	w_1	m'	m'	m''
w_2	w_2	w_2	m	m''	m'

w_1	w_2	w_3
$\cancel{m}$		$\cancel{m''}$
m'' $\cancel{m'}$	m'	m

Now the matchings returned are (m, w_3), (m', w_2) and (m'', w_1).

- Here, w1 was matched to her most preferred partner m'' by lying.
- Hence, lying i.e. switching preferences, improved the partner of woman w1.

Ques7. Determine whether the following statement is true or false. If it is true, give a short explaination. If it is false, give a counterexample.

For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm, every man is matched with their most prefered woman.

Ans. I think the above statement is "True".

It is possible ~~with~~ that every man is matched with their most prefered woman if all men prefer different woman first.

- Consider the preference list,

m_1	m_2	m_3
m_1	m_2	m_3
m_3	m_1	m_1
m_2	m_3	m_2

~~w_1~~ w

m_1	m_2	m_3
w_1	w_2	w_3
w_3	w_1	w_1
w_2	w_3	w_2

w_1	w_2	w_3
m_1	m_2	m_3
m_2	m_3	m_2
m_3	m_1	m_1

- If we run G-S algorithm,

w_1	w_2	w_3
m_1	m_2	m_3

the matched pairs are (m_1, w_1), (m_2, w_2), (m_3, w_3).

- Hence, all men are matched to their most preferred woman.

Ques 8. Consider a stable marriage problem where the set of men is given by $M = m1, m2, \ldots, m_N$ and the set of women is $W = w1, w2, \ldots, w_N$. Consider their preference lists to have the following properties:

$$\forall w_i \in W;\ w_i \text{ prefers } m_i \text{ over } m_j \ \forall j > i$$

$$\forall m_i \in M:\ m_j \text{ prefers } w_i \text{ over } w_j \ \forall j > i$$

Prove that a unique stable matching exists for this problem. Note: the $\forall$ symbol means "for all".

Ans. • Input:

Set of Men, $M = m1, m2, \ldots, m_N$

Set of women, $W = w1, w2, \ldots, w_N$.

Preference list of women:

$\forall w_i \in W;\ w_i$ prefers m_i over $m_j \ \forall j > i$

W1	W2	W3	W4
m1	m2	m3	m4
m2	m3	m4	
m3	m4		
m4			

Preference list of men,

m_1	m_2	m_3	m_4
w_1	w_2	w_3	w_4
w_2	w_3	w_4	
w_3	w_4		
w_4			

Output : stable Matching of n men and n women

Algorith proof

consider we run stable matching,

men propose

w_1	w_2	w_3	w_4
m_1	m_2	m_3	m_4

Since men propose to women highest in their list i.e, women with same index as theirs, so all men are paired up with different women.

Proof by contradiction.

consider a pair where $j<i$,

(m_3, w_2) is a

i.e. man m_3 ~~pre~~ is paired with woman w_2.

Now, did m_3 propose to w_2? and get rejected?

It is possible when,

1) w_2 is already engaged to a man she prefers more over m_3.

2) she was engaged to m_3 and broke her engagement to get engaged to a man she prefers over m_3.

~~In both cases, she prefers m2 over m3 a~~

Both the cases are not possible if m_3 ranks higher in ~~w2~~ the preference list of W_2.

Hence, (m_3, w_2) is instability.

- If there are n men and n women,

$M = m_1, m_2, \ldots, M_N$
$W = w_1, w_2, \ldots, W_N$

then (m_n, w_n) will be matched.
~~Since it men will be matched with women~~