Big O Notation: $f(x)$ is $O(g(x))$ if there are constants C & k such that $|f(x)| \le C|g(x)|$, $x > k$

Ex: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

We need to find a c and a k so that $|f(x)| < C|x^2|$ for all $x > k$, $f(x) = x^2 + 2x + 1$, $x^2 = x^2$, $x^2 > 2x$, $x^2 > 1$, $g(x) = 3x^2$, $x^2 = x^2$ always equal, $2x \overset{?}{\le} x^2$ $x^2 > 2x$ when $x > 2$, $1 \overset{?}{\le} x^2$ $x^2 > 1$ when $x > 1$, Let $C = 3$ and $k = 2$, Then $|x^2 + 2x + 1| \le 3x^2$ for all $x > 2$ ∴ by the defn of big O, $x^2 + 2x + 1$ is $O(x^2)$.

Ex: Let $f, g: \mathbb{Z}^+ \to \mathbb{R}$ as defined by: $f(x) = \{2$, for x even; 1, for x odd$\}$. Prove $f(x)$ is $O(g(x))$.

Case 1: x is even, $|2| \le C|3|$, Let $C = 1$, Then $|2| \le 3$ for all even x ∴ by the defn of big O, $f(x)$ is $O(g(x))$

Case 2: x is odd, $|1| \le C|4|$, Let $C = 1$, Then $|1| \le 4$ for all odd x ∴ by the defn of big O, $f(x)$ is $O(g(x))$

Graph: A graph $= (V, E)$ consists of a nonempty set V of vertices and a set E of (directed) edges.

↳ **Simple Graph**: each edge connects 2 different vertices, & no 2 edges connect the same pair of vertices.

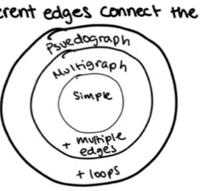
simple graph — simple graph — not a simple graph — not a simple graph

↳ **Multigraphs**: may have multiple edges connecting the same 2 vertices. When m different edges connect the vertices u and v, we say that (u, v) is an edge of multiplicity m.

Multigraph (b, c) is an edge multiplicity 2 — not a multigraph

↳ **Pseudograph**: may include loops, as well as multiple edges connecting the same pair of vertices.

psuedograph — psuedograph



Adjacent Neighbors: 2 vertices in an undirected graph G connected by an edge e. Such an edge e is called incident with the vertices u and v. **Neighborhood**: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$. If $A \subseteq V$, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A.

Degree: The degree of a vertex in an undirected is the number of edges that are incident with it, except that a loop of a vertex contributes 2 to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$. The maximum degree of all of the vertices in a graph designated by max-deg(G).

Ex: $\deg(a) = 3$, $\deg(e) = 3$, $\deg(b) = 4$, $\deg(d) = 2$, max-deg$(G) = 4$

$\deg(a) = 2$ $N(a) = \{b, f\}$; $\deg(b) = 4$ $N(a) = \{a, c, e, f\}$; $\deg(c) = 4$ $N(a) = \{b, d, e, f\}$; max-deg$(G) = 4$

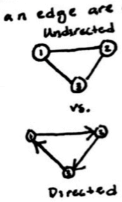
$\deg(d) = 1$ $N(d) = \{c\}$; $\deg(e) = 3$ $N(e) = \{b, c, f\}$; $\deg(f) = 4$ $N(f) = \{a, b, c, e\}$; $\deg(g) = 0$ $N(g) = \{\}$

Handshaking Theorem: If $G = (V, E)$ is an undirected graph, then $2|E| = \sum_{v \in V} \deg(v)$

Directed Graph – a directed graph or digraph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) together with a set E of directed edges (or arcs). Each edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v. The vertex u is called the initial vertex of the edge (u, v) and vertex v is the terminal vertex of this edge. (Note: graphs where the endpoints of an edge are not ordered are set to be undirected graphs.

- In a directed graph, the in-degree of a vertex v, denoted $\deg^-(v)$, is the number of edges which terminate at v. The out-degree of v, denoted $\deg^+(v)$ is the number of edges with v as their initial vertex. (a loop at a vertex contributes 1 to both the in-degree/out-degree

Theorem – Let $G = (V, E)$ be a graph with directed edges. Then, $|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$



Complete Graphs: denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

Ex: K_1 K_2 K_3 K_4 K_5 K_6

↳ **Theorem**: The number of edges in K_n is $\frac{n(n-1)}{2}$

Cycle: A cycle C_n for $n >= 3$ consists of n vertices $v_1, v_2, v_3, \ldots, v_n$, and edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)$.

Ex. C_3 C_4 C_5 C_6

N-dimensional hypercube: Q_n, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between 2 vertices that differ in exactly one bit position.

Ex: Q_1 Q_2 Q_3

Bipartite Graphs: A simple graph G is bipartite if V can be partitioned into 2 non empty disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect 2 vertices in V_1 or V_2.

↳ **Theorem**: A simple graph is bipartite iff it is possible to assign one of 2 colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Ex: V_1 = yellow V_2 = purple

↳ **Complete Bipartite Graph**: A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partioned into 2 subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

$K_{2,3}$ $K_{3,3}$

Subgraph: A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \ne G$.

↳ **Induced Subgraph**: Let $G = (V, E)$ be a simple graph. The subgraph induced $W \subseteq V$ is the graph (W, F), where the edge set F contains an edge in E iff both endpoints are in W.

Ex: Is K_4 a subgraph of K_5 induced by $\{a, b, c, d\}$?

K_4 is a subgraph K_5

$V = \{a, b, c, d, e\}$ $E = \{10$ edges$\}$ — $V = \{a, b, c, d\}$ $E = \{6$ edges$\}$

Union of Graphs: The union of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Adjacency Matrix: $|V| = n = 4$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} \deg(a) = 3 \\ \deg(b) = 2 \\ \deg(c) = 2 \\ \deg(d) = 1 \end{array}$$

terminal (to) vertex; initial (from) vertex

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{array} \quad \begin{array}{l} \deg^+(a) = 3 \\ \deg^+(b) = 2 \\ \deg^+(c) = 1 \\ \deg^+(d) = 3 \end{array}$$

$\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 4$, $\deg^-(d) = 1$

Path: Let n be a nonnegative integer & G a directed graph. A path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is assoc. w/(x_0, x_1), e_2 is assoc. w/(x_1, x_2), & so on, where $x_0 = u$ and $x_n = v$.

↳ **Circuit/Cycle**: A path length greater than 0 that begins & ends in the same vertex.

↳ **Simple**: A path/circuit that does not contain the same edge more than once.

Connected: there is a path between every pair of distinct of the graph.

Cut Vertex: the removal & all of its incident edges produces a subgraph consisting of more connected components.

Cut Edge: the removal of an edge produces a subgraph consisting of more connected components

Vertex/Edge Cut: if a set of vertices/edges taken out together causes the disconnect.

↳ **Theorem**: Every graph except a complete graph has a vertex cut.

*The vertex connectivity $\kappa(G)$ of a graph is the min # of vertices that can be removed from G to disconnect G.

*A directed graph is strongly connected if, for every pair of vertices a and b in the graph, there is a path from $a \to b$ + $b \to a$

*A directed graph is weakly connected if there is a path between every 2 vertices in the underlying undirected graph.

Euler Circuit: a simple circuit containing every edge of G. (path: contains every edge in its path)

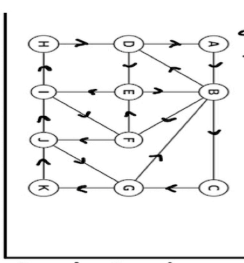
↳ **Theorem**: A connected multigraph w/ at least 2 vertices has an Euler circuit iff each of its vertices has even degree.

↳ **Theorem**: A connected multigraph has an Euler path but not an Euler circuit iff it has exactly 2 vertices of odd degree.

Ex: Euler circuit? ✓ Euler path? ✓ — Euler circuit? x Euler path? ✓ — Euler circuit? x Euler path? x

Hamilton Circuit: a simple circuit that passes through every vertex exactly once. (path: contains every vertex in its path)

Ex: Find an Euler path for the graph below.



Show G and H are isomorphic.

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$

The function f is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of 2 adjacent vertices in H.

$\mathbb{Z}: \{-3, -2, -1, 0, 1, 2, 3\}$ integers, $\mathbb{N}: \{1, 2, 3\}$ natural Nums $(\mathbb{Z}^+)$

Ex: For $n \in \mathbb{Z}^+$, Let $n \ne 1, 3$. Prove that n can be expressed as a sum of 2's and/or 5's using strong induction.

$P(n)$ = "n can be expressed as a sum of 2's and/or 5's" $n \ne 1, 3$

$P(2)$ = "2 can be expressed..." $2 = 2(1)$ $2 = 2$ ✓

$P(4)$ = "4 can be expressed..." $4 = 2 + 2$ $4 = 4$ ✓

$P(5)$ = "5 can be expressed..." $5 = 2 + 3$ $5 = 5$

Assume $P(2), P(4), P(5)$. Show $P(n+1)$.

$P(n+1)$ = "$n+1$ can be expressed..." Since n can be expressed as a sum of 2's and/or 5's, we can substract 2 from n 2 times to get $n - 4$ and add 5 to get $n+1$ ∴ $n+1$ can be

Strong Induction, Zaiwei Recommended Question:
Let an be the sequence defined by $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$.
Prove that $a_n = 3 * 2^{n-1} + 2(-1)^n$ for all $n \in N$.

<u>Base Cases:</u>
- $a_0 = 3/2 + 2 = 5/2$
- $a_1 = 1$, so $1 = 3 * 2^{(1-1)} + 2*(-1)^n = 1$
- $a_2 = 8$, so $8 = 3 * 2^1 + 2(-1)^2 = 8$

Therefore, both base cases have been proven.

<u>Inductive Case:</u>
Assume P(k) is True, try and prove for P(k + 1).
- $P(k + 1) = a_{k-1} + 2a_{k-2} = 3*2^{k-1} + 2*(-1)^k + 2(3*2^{k-2} + 2*(-1)^{k-1})$
  $= 3*2^{k-1} + 2*(-1)^k + 6*2^k * 1/4 - 4*(-1)^k = 3*2^k / 2 + 3*2^k / 2 - 2*(-1)^k$
  $= 3*2^k - 2*(-1)^{k+1} = 3*2^k + 2(-1)^{k+1}$
  - *We have proven P(k + 1) where n = k + 1*

By proving base cases, P(k), and P(k + 1), this theorem is True.

Big O, Zaiwei Recommended Question:
Use the definition of O(g(x)) to show that $f(x) = (x^3 + x^2 + x) / (2x + 10)$ is $O(x^2)$.
Be sure to indicate the values you used for the witnesses.

<u>Determine an easier function by disregarding constant of 10:</u>
- $f_1(x) = (x^3 + x^2 + x) / (2x + 10)$ and $f_2(x) = (x^3 + x^2 + x) / 2x$ where $f_2(x) > f_1(x)$
- Simplify $f_2(x) = x^2 / 2 + x / 2 + 1 / 2$
  - Therefore, $f_2(x) = x^2 / 2 + x / 2 + 1 / 2 <= x^2 + x^2 + x^2 = 3x^2$, thus C = 3

<u>k = 1, which is a point where $3x^2$ is greater than $x^2 / 2 + x / 2 + 1 / 2$:</u>
- $x^2 / 2 <= x^2$, $x <= x^2$, and $1 / 2 <= x^2$ when $x > 1$

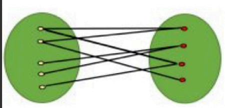
<u>Determine Answer:</u>
- $|x^2 / 2 + x / 2 + 1 / 2| <= |3x^2|$ for C = 3 and x > 1
- $|3x^2| >= |x^2 / 2 + x / 2 + 1 / 2| >= |(x^3 + x^2 + x) / (2x + 10)|$

Therefore, by definition of Big O, f(x) is O(x^2).

Definitions, Recommended by Zaiwei

1. What is a bipartite graph? (Give an example)

Bipartite Graph: A graph whose vertices can be divided into two independent sets, U and V, such that every edge (u, v) either connects a vertex from U to V or a vertex from V to U.

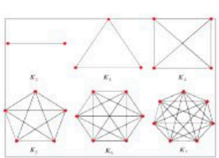


Complete Bipartite Graph: There is an edge from every vertex in U to every vertex in V.

2. What is a Euler circuit, Euler path? (Give an example)
- *Euler Circuit:* A circuit that uses every edge of a Graph exactly once, cannot re-use edges.
- *Euler Path:* A path that uses every edge of a Graph exactly once, cannot re-use edges.

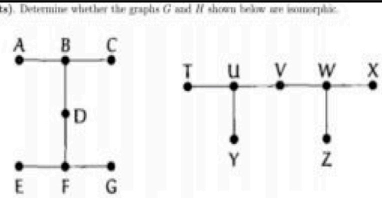
3. What is a Hamilton circuit, Hamilton path? (Give an example)
- *Hamilton Circuit:* A circuit that uses every vertex of a Graph exactly once, cannot re-use vertices.
- *Hamilton Path:* A path that uses every vertex of a Graph exactly once, cannot re-use vertices.

4. What is a connected component?
- *Connected:* An undirected graph is connected if there is a path between every pair of distinct vertices of the graph. There cannot be any openings within the graph. (K1, K2, K3… etc)



- *Connected Component:* A connected sub-graph of G that is NOT a proper sub-graph of another connected sub-graph of G

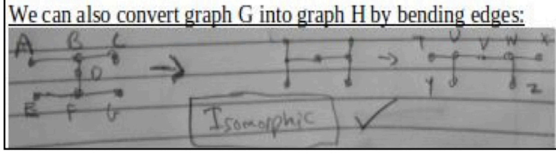
(15 points). Determine whether the graphs G and H shown below are isomorphic.



They are isomorphic because they have the same number of vertices and edges.
<u>Now let's see if the corresponding vertices have the same degree.</u>
- deg(A) = deg(y) = 1, deg(B) = deg(u) = 3, deg(C) = deg(T) = 1, deg(D) = 2 = deg(V), deg(E) = deg(X) = 1, deg(F) = deg(W) = 3, deg(G) = deg(Z) = 1

Therefore, the corresponding degrees are equal to each other so it's isomorphic.

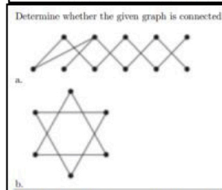
<u>We can also convert graph G into graph H by bending edges:</u>



5. What are cut vertices and cut edges of a graph?
- *Cut Vertex:* If removing it and incident edges, produces a sub-graph of connected component
- *Cut Edge:* If removing it, produces a sub-graph of connected components

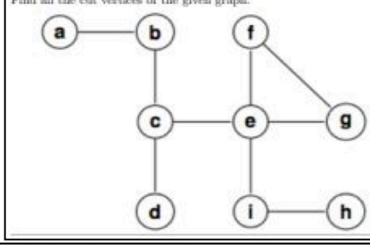
6. What is line (edge) connectivity and what is vertex connectivity?
- Edge Connectivity λ(G): Minimum number of edges to remove for a disconnected graph
- Vertex Connectivity κ(G): Minimum number of vertices to remove for a disconnected graph

Determine whether the given graph is connected.



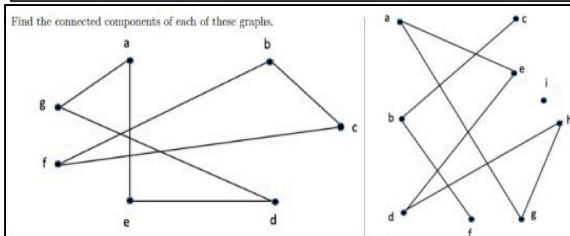
If we can connect one vertex to another vertex through a path, it's a connected graph.
- a) The graph is connected, as any two vertices can be connected by some path
- b) This graph is not connected, as it consists of two 3-vertex groups, and although vertices of one group are connected, there's no path connecting vertices of different "groups"

Find all the cut vertices of the given graph.



The cut vertices are b, c, e, and i. Removing any of these makes a disconnected graph.

Find the connected components of each of these graphs.

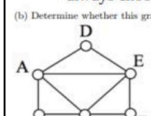


Find the connected sub-graphs of the graph that are made-up within the original graph
- a. The vertices {a, d, e, g} along with edges { {a,e},{a,g},{d,e},{d,g} }, and vertices {b, c, f} along with edges { {b,c}, {b,f}, {c,f} }are the connected components of the graph
- b. The vertices {a, d, e, g, h} along with edges { {a,e},{a,g},{d,e},{d,h},{g,h} }, vertices {b, c, f} along with edges { {b,c}, {b,f} }, and vertex i are the connected components of the graph

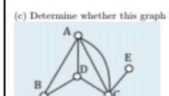
A pendant vertex of a graph is a vertex with degree of 1. Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.
- *Simple Way of Answering:* A pendant vertex is a vertex with a degree of 1, so that means if we try and remove it then it wouldn't really disconnect the graph since it only removes an end-point of the graph, which cutting end-points doesn't cause a disconnected graph.
- *Formal Way of Answering:* Without loss of generality, we can restrict our attention to the component in which the cut edge lies; other components of the graph are irrelevant to this proposition. To fix notation, let the cut edge be uv.
  - When the cut edge is removed, the graph has two components, one of which contains v and the other of which contains u. If v is pendant, then it is clear that the removal of v results in exactly the component containing u, a connected graph. Therefore v is not a cut vertex in this case.
  - On the other hand, if v is not pendant, then there are other vertices in the component containing v, at least one other vertex w adjacent to v. (We are assuming that this proposition refers to a simple graph, so that there is no loop at v.) Therefore when v is removed, there are at least two components, one containing u and another containing w.

Explain the procedure for constructing an Euler circuit in a graph.
- Make sure your graph has either 0 or 2 vertices of odd degree.
- If there are no odd vertices, start at any vertex. Otherwise, start at either of the vertices with odd degree.
- Follow the edges one at a time. Each edge will either be a bridge or a non-bridge. If possible, always choose a non-bridge. Stop when you run out of edges.

(b) Determine whether this graph has an Euler circuit. Explain your answer.



(c) Determine whether this graph has a Hamilton circuit. Explain your answer.



b) Yes. One example is if you start at vertex A, then you can make ADEACEFCBA.
c) No. Vertex E has a degree of 1, which by definition cannot be a Hamilton Circuit. There is no way of leaving E without returning to C. However, we do have Hamilton Paths like ECDAB.

Give an example of a simple graph with with n vertices with $n \geq 3$ that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n - 1)/2$.