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Smooth strictly Arbitrage-free Non-parametric Option Surfaces

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Abstract

We present a simple, numerically efficient and highly flexible non-parametric method to construct representations of option price surfaces $\hat{C}(T, K)$ which are both smooth and strictly arbitrage-free across time and strike. The method can be viewed as a smooth generalization of the widely-known linear interpolation scheme, and retains the simplicity and transparency of that baseline. Call option prices for arbitrary strikes K and some expiry T_j are represented as convex combinations of Black–Scholes call payoffs anchored at quoted strikes K^i with variances V_j with weights q_j

$$\hat{C}(T_j, K) = \sum_{i=0}^N q_j^i \text{Call}(K^i, K, V_j).$$

which are free of arbitrage across time and strike. This is extended to the full model $\hat{C}(T, K)$ to include arbitrary non-quoted expiries. The q_j 's are shown to have a natural interpretation as discrete space transition densities.

Calibration of the model to observed market quotes is formulated as a linear program, allowing bid–ask spreads to be incorporated directly via linear penalties or inequalities, and delivering materially lower computational cost than most of the currently available implied-volatility surface fitting routines.

As a further contribution, we derive an equivalent parameterization of the proposed surface in terms of strictly positive “discrete local volatility” [BR15] variables. This yields, to our knowledge, the first construction of smooth, strictly arbitrage-free option price surfaces while requiring only trivial parameter constraints (positivity). We illustrate the approach using S&P 500 index options.

1 Introduction

This article tackles the problem of fitting a discrete set of observed European option bid-ask prices efficiently and develops a *non-parametric, strictly arbitrage-free, smooth surface* representation of option prices to model prices for a full continuous range of strikes and maturities beyond the observed market quotes. We also present a version of our model which requires only positivity of the input parameters, making our model the first known smooth strictly arbitrage-free model whose parameters are subject to only trivial constraints. As long as bid/ask spreads of option prices are not zero, we can always find a version of our model which fits the market.

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Figure 1 makes the strength of the calibration immediately apparent. In a way, these plots speak more than a 1000 words: Across the full range of strikes and maturities shown, the model curves closely track the observed S&P 500 option data, with deviations that are difficult to discern at the scale of the plots and no evident systematic bias in any particular region of the surface. Notably, this high-quality fit is achieved with negligible runtime—solving a single global linear program over the full cross-section of options across all expiries completes essentially instantaneously.

The data used for illustration in this article are end of day option data sourced from Option Metrics IvyDB.¹. We selected OTM options with at least 100 lots are traded on the day. We also exclude strikes of options whose Vega/ \sqrt{T} is less than 0.1%. Past this, we also only consider expiries with 20 active options. We will publish the data pipeline for this project alongside the code for the model and the notebooks used to generate all graphs in this article.

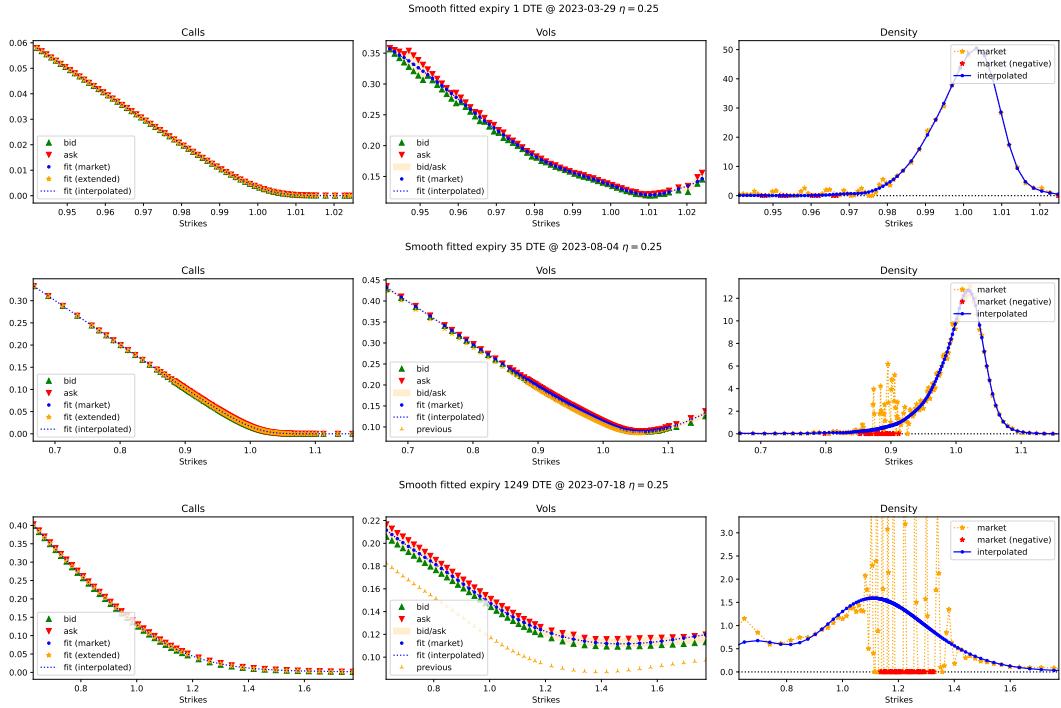


Figure 1: Fit of our model to the market, illustrated for a few DTE's on several days in 2023 using IvyDB OptionMetrics data. The thin bars in the call price and volatility graphs illustrate fit errors, expressed relative to the relevant spread on the right hand axis. (We do see some fitting errors in these charts because of the smoothness we impose; see section 3.1.) The red dots below the zero line in the density graphs represent cases where the market-mid prices have arbitrage.

A convenient feature of our model is that it strictly arbitrage-free and does not require any of the compromises in fit and arbitrage that other currently available approaches rely on: such as Lagrange multipliers, penalties, or reliance on “observations in experiments”. Moreover, its implementation is simple and fast enough to be used in high speed trading environments or as part of machine learning models trained using gradient descent or other heavy numerical methods.

Our approach boils down to writing

$$\hat{C}(T_j, K) := \sum_{i=0}^N q_j^i \text{Call}(K^i, K, V_j) .$$

in terms of martingale densities q_1, \dots, q_M for expiries $T_1 < \dots < T_M$ defined for homogeneous strikes $K_1 < \dots < K_N$ and log-normal variances $V_1 < \dots < V_M$. Here, $\text{Call}(s, k, v)$ is the Black-Scholes call price function for spot s , strike k and variance v . We find the densities q_1, \dots, q_M to match the market bid/ask by linear programming. The method enforces absence of arbitrage in both time and strike space; that means extending our model for call options for arbitrary expiries

¹<https://optionmetrics.com/>

trivially amounts to specifying an increasing interpolation function α such that

$$\hat{C}(T, K) := \alpha_j(T)\hat{C}(T_{j+1}, K) + (1 - \alpha_j(T))\hat{C}(T_j, K) .$$

Our model fits market data surprisingly well and is by construction strictly free of arbitrage and smooth. Our model has a simple interpretation as a discrete jump process multiplied by an underlying martingale process.

In addition to our production model, we also present extensions to other underlying processes than Black-Scholes, and generalize our approach to a theoretically more appealing model which allows specification of different variance curves V_j^i per strike. While this model is theoretically more attractive its calibration is much heavier and its therefore of perhaps academic interest.

Accessible Parametrization

Considering the increased interest in generative models for option prices we also present a parametrization of our model in terms of a parameter set which is subject only to positivity constraints. To this end we express the densities q_1, \dots, q_M in our formula (which are subject to linear but still complicated constraints) with densities generated by "Discrete Local Volatilities" (DLVs) as introduced in [BR15] (which only need to be non-negative). These DLVs together with the forward volatilities to generate the backbone variances only need be positive to generate an arbitrage-free smooth call option price surface.

We note that a model in that space, combined with [BMPW22a, BMPW22b] will yield not only arbitrage-free snapshots, but also discrete time dynamics which are free of dynamic arbitrage. Such a model is subject to further research.

Implementation

We provide a specific implementation strategy and code for both models. For the sake of clarity the discussion in the first parts of the paper are using homogeneous strikes per expiry. However, in our section 4 on implementation (and in code) we allow the use of different strikes per expiry, and permit using model strikes different from market strikes.

Comment for the arxiv pre-print: code is being prepared.

Related Work

Our method shares its approach, numerical efficiency and strict absence of arbitrage with linear interpolation methods [AH11, BR15], but expands upon those by providing smooth fits. Related is also the idea of fitting option prices to Levy processes [CN12] which also generate by construction strictly arbitrage-free smooth surfaces. However, such methods are numerically much more involved. We are not aware of any other method which has smooth non-parametric smooth strictly arbitrage-free option prices with an efficient numerical implementation – this being the motivation for the present paper.

Seemingly similar are *Gaussian mixture models* which have been covered in the literature in the past, chiefly [BMR02] and their earlier [BM00]. Their primary focus is the construction of a local volatility diffusion which mimics the marginal distributions generated by Gaussian mixture models. The central difference, apart from the difference in focus, of these works to ours is that in [BM00, BMR02] the mixing is performed with fixed weights over time over a number of processes which all start at the same point; while our solution works by jumps from one underlying stock price to another: The latter difference can be stated more explicitly, in our work as

$$\hat{C}(T_j, K) := \sum_{i=0}^N q_j^i \text{Call}(\textcolor{teal}{K}^{\textcolor{teal}{i}}, K, V_j)$$

where we used $S_0 = 1$ for spot and variances V_j . Mixture models on the other hand implement

$$\sum_{i=0}^N q_j^i \text{Call}(\textcolor{teal}{1}, K, V_j) .$$

The effect is that we can impose much stronger skew via the weighting functions q_j , at the cost of needing to limit the weights q_j to densities with unit mean. As a result, we are able to fit any grid of arbitrage-free bid/ask prices, while Gaussian mixture models typically fail to do so.

We focus here on non-parametric models which aim to fit a market, rather than on (low-) parametric representations of the option surface which express a view on the market through the lens of intuitive and statistically meaningful parameters. Usually such models exhibit arbitrage – with the exception of the SSVI models presented in the seminal [GJ14]. This is a great tool to express a view on the market through a low-parametric model lens. However, as the authors point out, the most accessible parametrization does not always fit well to observed option markets which means they need to resort to more involved numerical methods to fit their low-parametric model to observed market data while retaining strict absence of arbitrage.

Method	Smooth	Arbitrage-free	Parametric	Linear program
Linear interpolation	✗	✓	No	✓
SSVI	✓	✓	Yes	✗
This paper	✓	✓	No	✓

Table 1: Comparison of option surface construction methods.

American Options

In this article we focus on European options which are quoted for futures and indices. Options on single underlyings and ETFs, in contrast, are typically American options. This is subject to further research.

Structure of the Article

We start in Section 2 with a review of the concept of absence of arbitrage and its relation to linear interpolation option prices. This section also serves to settle notation and ideas. Section 3 then covers the main two proposed call price surface models where we also present some illustrative numerical results. We also reflect on generalization and representation as dynamic processes. Section 4 covers practical implementation. The code will be made available online.

2 Arbitrage-Free Call Prices

We start by fixing some notions and conventions that we will frequently need throughout the paper. For this we first briefly recall the Black–Scholes formula in the form we will frequently use in the paper: With $s > 0$ denoting spot, $k \geq 0$ strike and $v \geq 0$ (total) variance², the formula for European call options is given in terms of the standard normal distribution function \mathcal{N} as

$$\text{Call}(s, k, v) := s \mathcal{N}(d_+) - k \mathcal{N}(d_-) \quad \text{with} \quad d_{\pm} = \frac{-\ln(k/s) \pm \frac{1}{2}v}{\sqrt{v}}. \quad (1)$$

In this article we discuss construction of smooth arbitrage-free European option price surfaces for an equity denoted by S . The following assumptions hold throughout the paper unless otherwise stated. We will assume that the dynamics of the equity S can be written in the form $S_T = F_T Z_T$ where F is the forward curve³ and where Z is a martingale with unit expectation. We also assume that the equity does not default and does not otherwise reach zero⁴. The deterministic discount factors are given by DF_T . The key point is that in this setting if $\mathcal{C}(T, \mathcal{K}) \equiv \text{DF}_T \mathbb{E}[(S_T - \mathcal{K})_+]$ is a surface of call prices with expiries T and cash strikes \mathcal{K} , then we can convert these into “pure” call prices on the martingale Z by setting $\mathbb{E}[(Z_T - K)_+] \equiv C(T, K) := \mathcal{C}(T, F_T K) / (\text{DF}_T F_T)$.

²Many standard settings denote variance v as $v = \sigma^2 T$ with maturity $T \geq 0$ and volatility $\sigma \geq 0$.

³The assumption $S_T = F_T Z_T$ implies that dividends are proportional; alternative cash dividend treatments and handling of simple credit risk which still allow conversion back to “pure” option prices are covered in [Bue08].

⁴We therefore exclude models which can diffuse into zero such as the CEV model given as $dS_t = \sqrt{S_t} dW_t$.

A similar transformation is applied to obtain pure put prices, which we then convert into pure call prices by put-call-parity. With these preparations, we can henceforth focus or attention on "pure" call prices, i.e. those written on the underlying martingale process:

Definition 2.1 (Arbitrage Free Call Price Surfaces). We call a function $C : [0, \infty)^2 \rightarrow [0, \infty)$ an *arbitrage-free call price function* (or "*surface*") if there is a probability measure \mathbb{P} and a positive \mathbb{P} -martingale $(Z_T)_{T \in [0, \infty)}$ with $Z_0 = 1$ a.s., such that⁵

$$C(T, K) = \mathbb{E}_{\mathbb{P}} [(Z_T - K)^+] .$$

In this case we call Z the *martingale representation* of C .

We will by convention consider derivatives of call prices in K as right hand derivatives⁶. Next, we state a theorem that formulates a list of shape requirements on a surface, which guarantee to yield an arbitrage free call price surface in the sense of definition 2.1.

Theorem 2.2. A call price surface $C : [0, \infty)^2 \rightarrow [0, \infty)$ arbitrage-free iff

1. The market has unit expectation: $C(T, 0) \equiv \mathbb{E}[Z_T] = 1$ for all T ;
2. zero is unattainable (i.e. no atom at 0): $\partial_K C(T, 0) \equiv -1$;
3. call prices ultimately reach zero: $\lim_{K \uparrow \infty} C(T, K) = 0$,
4. call prices are convex: $\partial_{KK} C(T, K) \geq 0$; and
5. call prices are increasing in time: $\partial_T C(T, K) \geq 0$.

As a consequence we also have:

6. $C(T, K) \geq (1 - K)_+$ for any K, T ; as a direct consequence of 1. & 5. and
7. $\partial_K C(T, K) \in [-1, 0]$ for any K, T .

Remark 2.3. Several of the above conditions are well-known in the context of related literature. In several related settings, strict absence of arbitrage statements are not always fully complete: the boundary conditions 1. and 3. are frequently omitted, and/or negativity of the derivative 7. is not required.

Our set of conditions for theorem 2.2 are minimal equivalent conditions for the existence of a positive true martingale in sense of definition 2.1. A full proof is delegated to the appendix page 20ff. \triangleleft

Remark 2.4. To ensure positivity, we have imposed condition 2. rather than the commonly used weaker condition $\partial_K C(T, 0) \geq -1$.⁷ This requirement is justified as we consider an asset reaching zero outside a default event undesirable, and point to [Bue08] on how to incorporate "sudden" default risk in our framework. \triangleleft

Definition 2.5 (Smooth call price function). We call a call price function *smooth* (in strike direction) if $\partial_{KK} C(T, K) > 0$ for all $K > 0$.

⁵This is an application of the "First Fundamental Theorem of Asset Pricing" [DS06] which posits the equivalence to absence of arbitrage if and only if a martingale Z exists.

⁶i.e. $\partial_K C(T, K) := \partial_K^+ C(T, K) := \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} (C(T, K + \epsilon) - C(T, K))$; see the proof of theorem 2.2 in the appendix on page 20 why this is the natural definition when working with distributions.

⁷For fixed T and $K > 0$, we have, $\partial_K C(T, K) = \mathbb{E}[\partial_K (Z_T - K)^+] = -\mathbb{P}(Z_T > K)$. Taking the right-limit as $K \downarrow 0$ yields $\partial_K C(T, 0) = -\mathbb{P}(Z_T > 0) = -(1 - \mathbb{P}(Z_T = 0))$.

2.1 Discrete Market Data

In this section we recall results from [BR15] which we also use to fix notation and ideas for the remainder of the paper.

Assume now we are given a homogeneous grid of strikes $0 = K^0 < K^1 < \dots < 1 < \dots < K^N$ and expiries $0 = T_0 < T_1 < \dots < T_M$ with market call prices C_0^0, \dots, C_M^N where C_j^i denotes the price of the call expiring in T_j with strike K^i . Define for $i = 0, \dots, N - 1$

$$dC_j^i := \frac{C_j^{i+1} - C_j^i}{K_j^{i+1} - K_j^i} \quad (2)$$

and set $dC_j^N := 0$.

Theorem 2.6. *A market of call prices C_j^i is arbitrage-free in the sense that there exists a positive (discrete state) martingale Z with unit expectation which satisfies $C_j^i = \mathbb{E}_{\mathbb{P}}[(Z_{T_j} - K^i)^+]$ iff*

1. *the market has unit expectation: $C_j^0 = 1$;*
2. *zero is unattainable: $dC_j^0 = -1$ which with the above and (2) implies $C_j^1 = 1 - K^1$,*⁸
3. *call prices ultimately reach zero: $C_j^N = 0$;*
4. *call prices are convex: $dC_j^i \leq dC_j^{i+1}$; and*
5. *call prices are increasing in time, $C_j^i \leq C_{j+1}^i$.*

As a consequence we also have

6. $C_j^i \geq (1 - K^i)^+$ and
7. $dC_j^i \in [-1, 0]$.

Remark 2.7. We use homogeneous strikes across expiries here to avoid cluttering our discussion with more indices than necessary; however in section 4 on implementation (and in our code) we allow for a different set of strikes per expiry. \triangleleft

Proof. See [BR15] for a constructive proof of the existance of a martingale Z with these marginal distributions in the more general case of non-homogeneous grids.⁹ It is also shown there that the corresponding martingale Z only has mass in K^1, \dots, K^N . \square

Remark 2.8. A common practise¹⁰ when working with real market data to satisfy the conditions on the strikes in the previous theorem is to consider actual market strikes $K^2 < \dots < 1 < \dots < K^{N-1}$ with calls $C_j^i > (1 - K^i)^+$ and $1 > C_j^2 > \dots > C_j^{N-1} > 0$ which satisfy $-1 < dC_j^2 \leq \dots \leq dC_j^{N-2} < 0$, and then to insert the missing data as follows:

1. $K^0 := 0$ with $C_j^0 := 1$;
2. $K^1 \ll K^2$ with $C_j^1 := 1 - K^1$ such that $-1 = dC_j^0 < dC_j^1 < dC_j^2$ for all j ; and
3. $K^N \gg K^{N-1}$ with $C_j^N = 0$ such that $dC_j^{N-2} < dC_j^{N-1} < dC_j^N := 0$ for all j .

⁸Equation (2) (for $i = 0$), and condition $dC_j^0 = -1$ gives $(C_j^1 - C_j^0)/(K^1 - K^0) = -1$, hence $C_j^1 = C_j^0 - (K^1 - K^0)$. Using $C_j^0 = 1$ (condition 1.), and the fact that $K^0 = 0$, gives $C_j^1 = 1 - (K^1 - 0) = 1 - K^1$.

⁹The main difference for non-homogeneous grids where the strikes differ per expiry is the condition that call prices increase in time: for the non-homogeneous grid any call price $C_j^i = C(T_j, K^i)$ must exceed the linear interpolation of the call prices of the earlier expiry.

¹⁰A viable strategy is as follows: first, to find K^0 let K_j^* be the strike where the intrinsic value intersects with the line from (K^2, C_j^2) through (K^1, C_j^1) , i.e. it satsfies $C_j^1 + dC_j^1(K_j^* - K_1) = 1 - K_j^*$. It is therefore the biggest strike we can insert with a call price equal to intrinsic value such that the call price remain convex. Then set $K_0 := 1/10 \min_j K_j^*$.

To determine K^N find per expiry the strike $K_j^\#$ where the line from (K^{N-2}, C_j^{N-2}) through (K^{N-1}, C_j^{N-1}) intersects with zero, i.e. $C_j^{N-1} + dC_j^{N-1}(K_j^\# - K^{N-1}) = 0$. Then define $K^N := 1.5 \max_j K_j^\#$.

□

Given a discrete state martingale Z as per above, let us denote with slight abuse of notation by $Z_j := Z_{T_j}$ the corresponding discrete state and discrete time martingale Markov chain. Given strikes $K = (K^1, \dots, K^N)'$ and a Markov chain $(Z_j)_{j=0, \dots, M}$ with $\mathbb{P}(Z_j = K^i) =: p_j^i$, define the transition probabilities

$$P_{j|j-1}^{i|\ell} := \mathbb{P}\left[Z_j = K^i \mid Z_{j-1} = K^\ell\right], \quad i, \ell \in \{1, \dots, N\}.$$

Let $P_{j|j-1} \in \mathbb{R}^{N \times N}$ denote the matrix with entries $(P_{j|j-1})_{i,\ell} = P_{j|j-1}^{i|\ell}$. (Thus columns correspond to the conditioning state ℓ .)

Definition 2.9 (Martingale transition operator). A matrix $P \in \mathbb{R}^{N \times N}$ is called a *(discrete) martingale transition operator* with respect to strikes $K = (K^1, \dots, K^N)'$ if

1. it is column-stochastic: $P \geq 0$ and $1' \cdot P = 1'$;
2. it satisfies the martingale property: $K' \cdot P = K'$,

where we used “.” to denote matrix/vector multiplications. Equivalently, for each conditioning index $\ell \in \{1, \dots, N\}$, the above properties translate into

$$\sum_{i=1}^N P^{i|\ell} = 1, \quad \text{and} \quad \sum_{i=1}^N K^i P^{i|\ell} = K^\ell.$$

Definition 2.10 (Martingale Density). A sequence $p = (p_j)_{j=1, \dots, M}$ of densities $p_j \in \mathbb{R}_{\geq 0}^N$ defined over strikes $0 < K^1, \dots, K^N$ with unit mean is called a *martingale density* if there exist¹¹ martingale transition operators $P_{j|j-1} \in \mathbb{R}^{N \times N}$ such that

$$p_j = P_{j|j-1} p_{j-1}$$

written out as

$$p_j^{ij} = \sum_{i_{j-1}=1}^N P_{j|j-1}^{i_{j-1}|i_j} p_{j-1}^{i_{j-1}}.$$

Remark 2.11. The martingale density p for an arbitrage-free market of call prices C is given as

$$p_j^i := dC_j^i - dC_j^{i-1} \geq 0. \quad (3)$$

for $i = 1, \dots, N$ with dC_j^i defined in (2). Theorem 3.5 on page 13 shows how to construct respective transition operators. □

2.2 Linear Arbitrage-Free Call Price Surfaces

We now recall the well-known result that linear interpolation in strike and expiry yields an arbitrage-free – but not smooth – interpolator, to motivate our later results.

Definition 2.12 (Linear Interpolation). The linear interpolator in strikes K for expiry T_j is

$$\bar{C}_j(K) := \sum_{i=0}^{N-1} \left\{ dC_j^i \underbrace{(K - K^{i+1})}_{\leq 0} + C_j^{i+1} \right\}_{K^i \leq K < K^{i+1}}.$$

with the trivial $\bar{C}_0(K) := (1 - K)^+$. To interpolate in time without causing calendar arbitrage, any increasing interpolation along fixed strikes will be sufficient. Assume therefore that $\alpha_j : [T_j, T_{j+1}] \rightarrow [0, 1]$ defined for $T \in [T_j, T_{j+1}]$ is increasing with $\alpha_j(T_j) = 0$ and $\alpha_j(T_{j+1}) = 1$. Then let

$$\bar{C}(T, K) := \bar{C}_{j+1}(K) \alpha_j(T) + (1 - \alpha_j(T)) \bar{C}_j(K). \quad (4)$$

The function $\bar{C}(T, K)$ is then arbitrage-free.

¹¹As pointed out in [Bue06] transition operators are by no means unique for given marginal distributions.

Remark 2.13. A naive choice for the time-interpolation functions α_j is linear interpolation

$$\alpha_j(T) := \frac{T - T_j}{T_{j+1} - T_j} . \quad (5)$$

A common alternative is to interpolate in implied Black-Scholes ATM variance. To this end let W_j be the Black-Scholes implied variance for $C_j(1)$. For $T \in [T_j, T_{j+1})$ use for example linear interpolation

$$W(T) = W_j + \frac{W_{j+1} - W_j}{T_{j+1} - T_j}(T - T_j) .$$

This yields the interpolation function

$$\alpha_j(T) := \frac{\text{Call}(1, 1, W(T)) - C_j(1)}{C_{j+1}(1) - C_j(1)} .$$

Instead of piecewise linear interpolation a production setting would use a C^1 monotone spline [FC80]¹² of ATM variances to increase smoothness. \square

Proposition 2.14. *It holds that*

$$\bar{C}_j(K) = \sum_{i=1}^N p_j^i (K^i - K)^+ . \quad (6)$$

with the market martingale density p as defined in (3).

Proof. We notice that

$$C_j^{i+1} = (C_j^{i+1} - C_j^{i+2}) + (C_j^{i+2} - C_j^{i+3}) + \cdots + (dC_j^{N-1} - dC_j^N) = \sum_{\ell=i+1}^{N-1} dC_j^\ell (\underbrace{K^\ell - K^{\ell+1}}_{\leq 0})$$

hence

$$\begin{aligned} \bar{C}_j(K) &= \sum_{i=0}^{N-1} \mathbb{1}_{K^i \leq K < K^{i+1}} \left\{ dC_j^i (\underbrace{K^i - K^{i+1}}_{\leq 0}) + \sum_{\ell=i+1}^{N-1} dC_j^\ell (\underbrace{K^\ell - K^{\ell+1}}_{\leq 0}) \right\} \\ &= \sum_{i=0}^{N-1} dC_j^i ((K^i - K)^+ - (K^{i+1} - K)^+) \end{aligned} \quad (7)$$

$$\begin{aligned} &= \underbrace{dC_j^0 (K^0 - K)^+}_{=0} - dC_j^{N-1} (K^N - K)^+ + \sum_{i=1}^{N-1} \left(\underbrace{(dC_j^i - dC_j^{i-1})}_{\geq 0} \right) (K^i - K)^+ \\ &= \sum_{i=1}^N \left(\underbrace{(dC_j^i - dC_j^{i-1})}_{=p_j^i \geq 0} \right) (K^i - K)^+ = (6) . \end{aligned} \quad (8)$$

where we used that we had defined $dC_j^N := 0$.

The fact that p_j is non-negative and sums up to 1 is clear. To show that p_j has unit expectation follows from $\bar{C}_j(0) = 1$. \square

Proposition 2.15. \bar{C} is arbitrage-free.

Proof.

1. Equation (7) shows $\bar{C}_j(0) = \sum_{i=0}^{N-1} dC_j^i (K^i - K^{i+1}) = 0 - (-1) = 1$.
2. With $\partial_K (K^i - K)^+|_{K=0} = \mathbb{1}_{K>K^0}$ we also see $\partial_K \bar{C}_j(K)|_{K=0} = dC_j^0 = -1$.

¹²Monotone splines are implemented in scipy's PchipInterpolator.

3. $\lim_{K \uparrow K} \bar{C}_j(K) = 0$ is satisfied by construction since $C_j(K^N) = 0$.
4. Equation (6) shows that \bar{C}_j is convex.
5. \bar{C} is increasing in T by construction (4). Note that a convex combination of two convex functions remains convex.

□

3 Smooth Arbitrage-Free Call Price Surfaces

In this section we present the main results of this article: While linear interpolation is a simple and powerful tool construct arbitrage-free call price surfaces, it essentially represents a state-discrete underlying process. Indeed, \bar{C} is not smooth with $\partial_{KK}\bar{C}(T, K) = 0$ for all $K \notin \{K^1, \dots, K^N\}$ or $T \notin \{T_1, \dots, T_M\}$. The density $p(T, K)$ for the full surface can only have mass in (T, K^i) .

In most practical applications we aim to have a smooth representation of call price surfaces which allows pricing any option at any strike and expiry. Linear interpolation might be mathematically valid, but it is also extreme: indeed, linear interpolation of a set of arbitrage-free market prices provides "the most expensive interpolation scheme available" [Bue06]. That means it is not helpful for inferring insights on values of options between quoted strikes and expiries. From a practical perspective a relatively sparse linear interpolator does not lend itself well to the use of continuous state methods, chiefly Dupire's famous local volatility [D⁺94], given on a smooth grid as

$$\sigma(T, K)^2 := 2 \frac{\partial_T C(T, K)}{K^2 \partial_{KK} C(T, K) dT} .$$

(In [BR15] we presented with "discrete local volatilities" an alternative to Dupire's local volatility which works with large time steps and discrete market data.)

We now present an alternative to interpolate a set of arbitrage-free market call prices with smooth functions such that the resulting surface is itself also smooth. We start with the following reference model which performs surprisingly well despite its simplicity. We managed to fit it to a wide range of SPX market data. Section 4 on page 16 discussess implementation.

Theorem 3.1 (Smooth Call Prices). *Let $Y = (Y_1, \dots, Y_M)$ be a positive martingale with unit mean. Let q be a martingale density as defined in definition 2.10 and let*

$$\hat{C}_j(K) := \sum_{i=1}^N q_j^i \mathbb{E}[(K^i Y_j - K)^+] . \quad (9)$$

Using (4) let

$$\hat{C}(T, K) := \alpha_j(T) \hat{C}_{j+1}(K) + (1 - \alpha_j(T)) \hat{C}_j(K) .$$

Then \hat{C} is a smooth call price function.

Moreover, if $q = p$, then $\hat{C}_j \geq \bar{C}_j$ with equality if $Y = 1$. In this case \hat{C} reduces to linear interpolation.

The discrete-time martingale representation for \hat{C} is given by

$$Z_j := X_j Y_j \quad (10)$$

in the sense that $\mathbb{E}[(Z_j - K)^+] = \hat{C}_j(K)$ where $X \sim q$.

Proof of the theorem. Strictly speaking (10) is sufficient to prove that \hat{C} is arbitrage-free. For clarity with still provide a brief proof:

1. $\hat{C}_j(0) = \sum_i q_j^i K^i \mathbb{E}[Y_j] = 1$ because q_j and Y_j has unit mean.
2. $\partial_K \hat{C}_j(0) = - \sum_i q_j^i \mathbb{P}[K^i Y_j > 0] = -1$ because $Y_j > 0$ and m_j has unit mean.
3. $\lim_{K \uparrow \infty} \hat{C}_j(K) = 0$ because $\mathbb{E}[Y_j] = 1$ and dominated convergence applies.

4. $\partial_{KK} \hat{C}_j(K) > 0$ by construction for $K > 0$.
5. To show that call prices are increasing in time, consider

$$\begin{aligned}
\sum_{i=1}^N q_{j+1}^i \mathbb{E}[(K^i Y_{j+1} - K)^+] &\stackrel{(*)}{\geq} \sum_{i=1}^N q_{j+1}^i \mathbb{E}[(K^i Y_j - K)^+] \\
d\mathbb{P}^Y \stackrel{\equiv}{=} Y d\mathbb{P} \quad \mathbb{E}^Y \left[\sum_{i=1}^N q_{j+1}^i (K^i - K/Y_j)^+ \right] \\
&\stackrel{(**)}{\geq} \mathbb{E}^Y \left[\sum_{i=1}^N q_j^i (K^i - K/Y_j)^+ \right] \\
&= \sum_{i=1}^N q_j^i \mathbb{E}[(K^i Y_j - K)^+]
\end{aligned}$$

where $(*)$ follows because Y is a martingale, and $(**)$ follows as before because m is a martingale density which means, in particular,

$$\sum_{i=1}^{N+1} q_{j+1}^i (K^i - K)^+ \geq \sum_{i=1}^{N+1} q_j^i (K^i - K)^+$$

for all K . Applying the linear expectation operator $\mathbb{E}^Y[\cdot]$ preserves the inequality. \square

Remark 3.2. Our approach is a special case of the following concept: let $X = (X_t)_t$ and $Y = (Y_t)_t$ be two independent martingales. Then, $Z_t := X_t Y_t$ is again a martingale and has call prices

$$\mathbb{E}[(Z_t - K)^+] = \mathbb{E}[(X_t Y_t - K)^+] = \int_0^\infty \mathbb{E}[(x Y_t - K)^+] \mathbb{P}[X_t = x] dx ,$$

mirroring (10). It is worth noting that any classic arbitrage-free pricing model Y can be used for our model. Heston's model [Hes93] for example provides a rich surface with pronounced skew which can be used as a background martingale. \triangleleft

3.1 Production Model

For our proposed production model whose implementation we will discuss in section 4 we propose a very simplistic structure which still fits market data remarkably well: To this end we notice that if Y is log-normal with variances $0 < V_1 < \dots < V_M$ then we can write (9) as

$$\hat{C}_j(K) := \sum_{i=1}^N q_j^i \text{Call}(K^i, K, V_j) \tag{11}$$

where $\text{Call}(s, k, v)$ denotes the Black-Scholes call price defined in (1) with spot s , strike k and variance v .

Proposition 3.3 (Interpolation). *For any arbitrage-free set of market mid prices with non-zero spread to ask prices there exist a smooth representation (11) which is strictly arbitrage-free.*

Proof. This is a trivial consequence of the fact that

$$\sigma \rightarrow \hat{C}_j^{(\sigma)}(\cdot) := \sum_{i=1}^N p_j^i \text{Call}(K^i, \cdot, \sigma^2 T_j)$$

is a smooth map with $\hat{C}_j^{(\sigma)} \downarrow \bar{C}_j$ uniformly as $\sigma \downarrow 0$. \square

In practise that means that our production model is given the form

$$\hat{C}_j(K) := \sum_{i=1}^N q_j^i \text{Call}(K^i, K, \eta V_j) \quad (12)$$

where $\eta \in [0, 1)$ controls the smoothness of our model and where $0 < V_1 < \dots < V_M$ are input ATM variances from the market. e then solve for q to fit the model within market bid/ask prices globally (it is one fit across all expiries jointly, see section 4.2).

Essentially η allows controlling the desired smoothness in (3.3): if it is set to zero, and if the market is arbitrage-free, then our model has a solution within bid/ask and will be equivalent to the linear model discussed in section 2.2. By increasing η we can then smoothen our fit. Figures 2, 3 and 4 illustrate three different cases of fits with $\eta = 0$, $\eta = 0.25^2 \approx 0.06$ and $\eta = 0.5^2 = 0.25$. The latter is our recommended default setting.

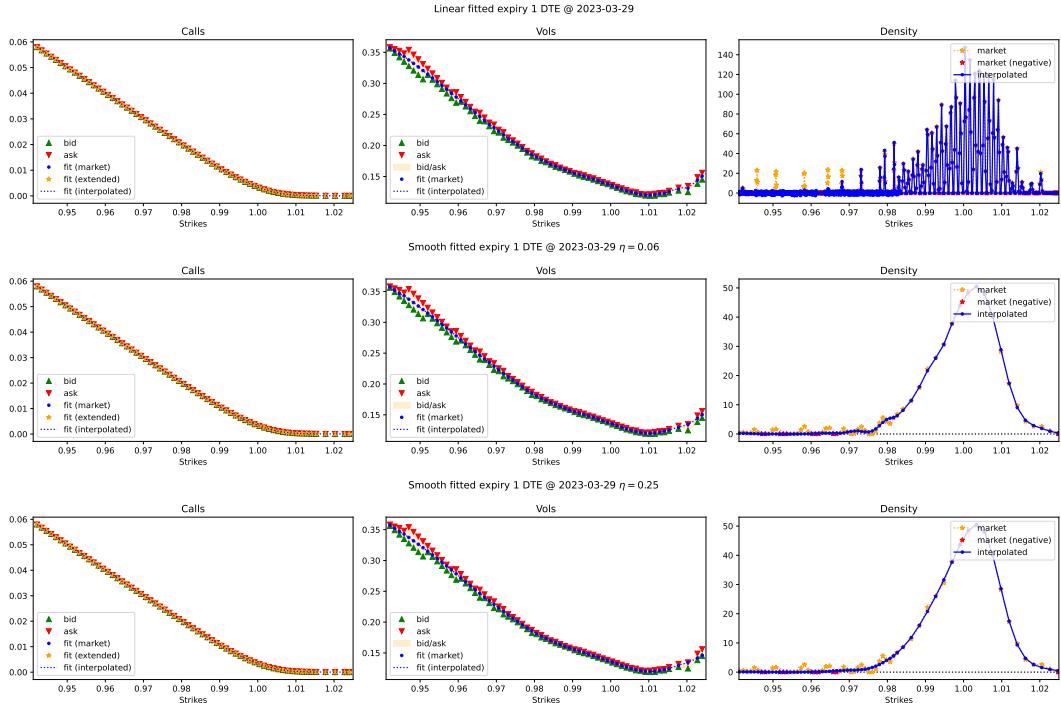


Figure 2: Fit of our model with different smoothness parameters η to 1 day to expiry options on SPX. The right hand side shows the density which is much smoother for the case $\eta = 0.25$ while actually retaining a decent fit: the biggest fitting error is 40% of vol spread.

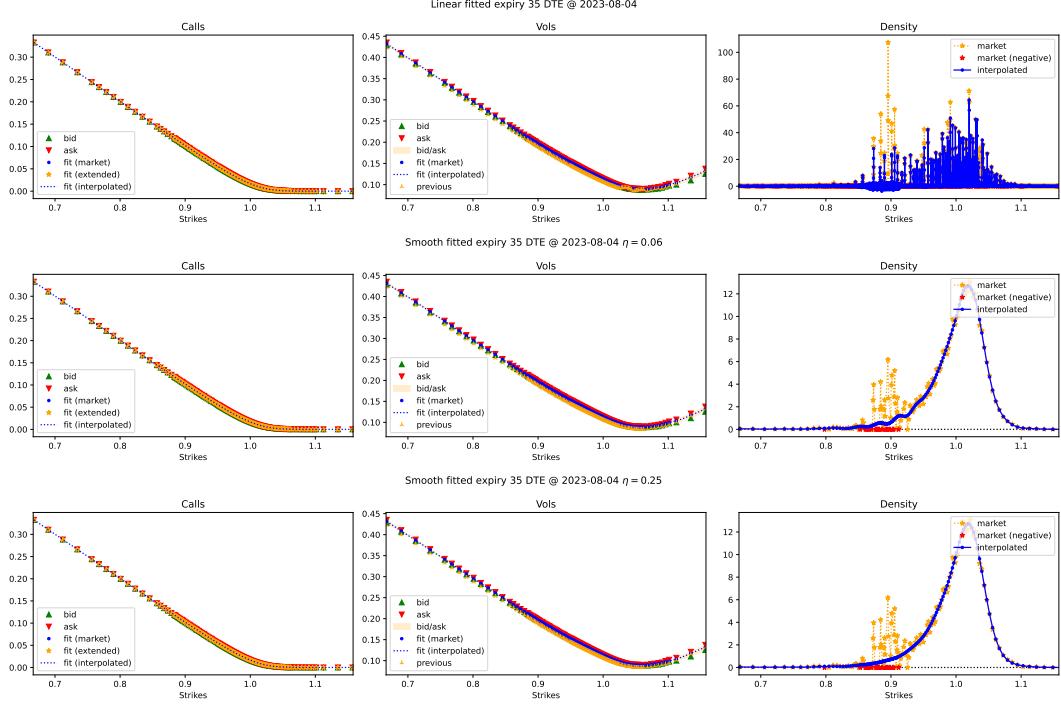


Figure 3: Fit of our model with different smoothness parameters η to SPX options with 35 days to expiry. In this case $\eta = 0.06$ fits the market within bid/ask while imposing sufficient smoothness on the model.

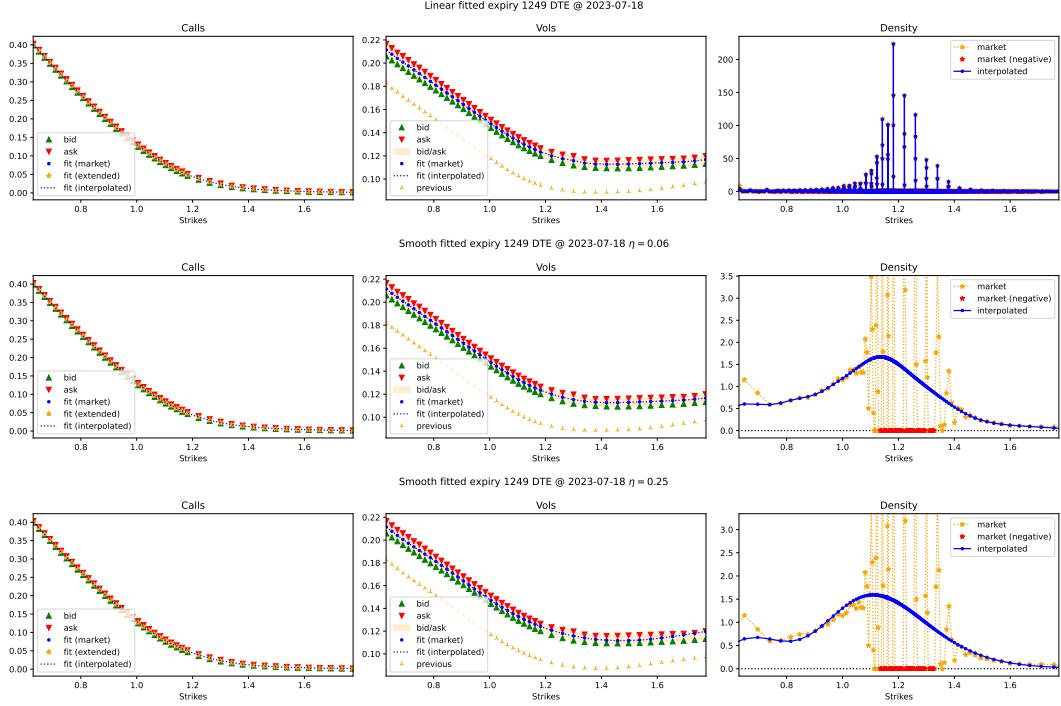


Figure 4: Fit of our model with different smoothness parameters η to SPX options with 1249 days to expiry. This expiry has sparse strikes which leads to the unnatural bump on the right wing for the linear model. This is rectified by the smoother model with $\eta = 0.25$.

Remark 3.4 (Generalized Interpolation in Strike). The proof of theorem 3.1 that the function $K \mapsto \hat{C}_j(K)$ is arbitrage-free in sense of satisfying 1-4 of definition 2.2 remains valid if we replace the

common Y with a martingale by strike:

$$\hat{C}_j(K) := \sum_{i=1}^N q_j^i \mathbb{E}[(K^i Y_j^i - K)^+] .$$

In the log-normal case translates to using different variances per strike

$$\hat{C}_j(K) = \sum_{i=1}^N q_j^i \text{Call}(K^i, K, V_j^i) . \quad (13)$$

We have provided an implementation of this idea in our code base which uses for V_j^i the market bid variance at strike K_j^i for expiry T_j . However, in this case we have not found a numerically efficient condition on the V 's to ensure absence of arbitrage in time for all strikes. For example it is not sufficient that the two expiries are in order at the market strikes as the synthetic example in figure 5 shows.

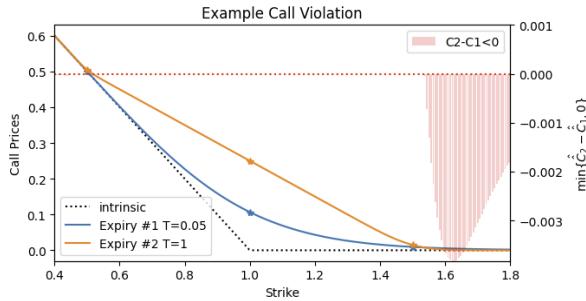


Figure 5: Example of the idea (13) where the call prices at two expiries are increasing in the market strikes $K = (0.5, 1, 1.5)$ but not at the extrapolated strikes $K \geq 1.6$ (the error is plotted at the right hand axis). The example uses constant log-normal volatilities (0.05, 1.2, 0.05) for the three strikes for both expiries. The densities are $q_1 = (0, 1, 0)$ and $q_2 = (0.5, 0, 0.5)$.

In section 3.3 we present a different, formally correct generalization to different Y_j^i 's per strike and expiry; however it is numerically much less efficient than our production model presented here. \triangleleft

3.2 Efficient Parametrization

Fitting the model (9) given variances $0 < V_1 < \dots < V_M$ means finding a martingale density with marginals q_j such that the model prices are within bid/ask spread of observed market data. This can easily be implemented using linear programming as is shown in section 4 which also covers the case of different strikes per expiry (which is the case in practise).

However, if we aim to use our parametrization in generative models then its representation in terms of a martingale density is not convenient as the marginals q_1, \dots, q_M have to satisfy the linear constraints of definition 2.10. For such applications we propose parameterizing the martingale density in terms of the *discrete local volatilities* introduced in [BR15]. This parametrization has been used to great effect in multi-asset arbitrage-free option surface simulators in [WWP⁺21], albeit on discrete grids of strikes. With our approach here such generative model is now able to generate smooth arbitrage-free option surfaces.

The following theorem is from [BR15] with notations aligned:

Theorem 3.5 (Transition Operators from Discrete Local Volatilities). *Let $\Sigma_j^i \geq 0$ for $i = 2, \dots, N-1$ and $j = 1, \dots, M$ be a surface of input discrete local volatilities. Let*

$$\gamma_j^{i+} := \frac{1}{\frac{1}{2}(K^{i+1} - K^{i-1})} \frac{1}{K^{i+1} - K^i} \quad \text{and} \quad \gamma_j^{i-} := \frac{1}{\frac{1}{2}(K^{i+1} - K^{i-1})} \frac{1}{K^i - K^{i-1}}$$

and define

$$w_j^{i\pm} := \frac{1}{2} (\Sigma_j^i K^i)^2 (T_j - T_{j-1}) \gamma_j^{i\pm} \quad (14)$$

and $w_j^{1\pm} = w_j^{N\pm} := 0$. The the inverse $Q_{j|j-1}$ of the tri-band matrix

$$Q_{j|j-1}^{-1} := \begin{pmatrix} 1 & -w_j^{2-} & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 + w_j^{2-} + w_j^{2+} & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & -w_j^{2+} & \ddots & -w_j^{(N-1)-} & & \\ \dots & \dots & \dots & \dots & 1 + w_j^{(N-1)-} + w_j^{(N-1)+} & 0 \\ \dots & \dots & \dots & \dots & -w_j^{(N-1)+} & 1 \end{pmatrix} \in \mathbb{R}^{N \times N} \quad (15)$$

is a martingale transition operator.

Starting in q_0 we then obtain the corresponding martingale densities

$$q_j := Q_{j|j-1} \cdot q_{j-1} .$$

Moreover, the transition operator $Q_{j|j-1}$ reproduces discrete input call prices C_1^1, \dots, C_M^N in the sense that $p_j = Q_{j|j-1} p_{j-1}$ iff we set

$$\Sigma_j^i := \sqrt{2 \frac{\Theta_j^i}{K^{i2} \Gamma_j^i}} \quad \text{with} \quad \Theta_j^i := \frac{C_j^i - C_{j-1}^i}{T_j - T_{j-1}} \quad \text{and} \quad \Gamma_j^i := \frac{dC_j^i - dC_{j-1}^{i-1}}{\frac{1}{2}(K^{i+1} - K^{i-1})} \quad (16)$$

for $j = 1, \dots, M$ and $i = 2, \dots, N-1$ with $0/0 := 0$. In this case

$$w_j^{i\pm} = \frac{C_j^i - C_{j-1}^i}{p_j^i} \bar{\gamma}_j^{i\pm} \quad \text{with} \quad \bar{\gamma}_j^{i+} := \frac{1}{K^{i+1} - K^i}, \quad \bar{\gamma}_j^{i-} := \frac{1}{K^i - K^{i-1}} . \quad (17)$$

We present the proofs to both statement in appendix A.2 on page 21

Remark 3.6. The tri-band matrix $Q_{j|j-1}^{-1}$ can be written in terms of a vector $\Sigma_j = (0, \Sigma_j^2, \dots, \Sigma_j^{N-1}, 0)$ as follows:

$$Q_{j|j-1}^{-1} = E + \Omega \Sigma_j^2 (T_j - T_{j-1})$$

(where the right hand multiplication of the matrix with the vector is column-wise) for pre-computed matrices

$$\Omega := \begin{pmatrix} 0 & -\omega_j^{2-} & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \omega_j^{2-} + \omega_j^{2+} & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & -\omega_j^{2+} & \ddots & -\omega_j^{(N-1)-} & & \\ \dots & \dots & \dots & \dots & \omega_j^{(N-1)-} + \omega_j^{(N-1)+} & 0 \\ \dots & \dots & \dots & \dots & -\omega_j^{(N-1)+} & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

with

$$\omega_j^{i+} := \frac{(K^i)^2}{(K^{i+1} - K^{i-1})(K^{i+1} - K^i)} \quad \text{and} \quad \omega_j^{i-} := \frac{(K^i)^2}{(K^{i+1} - K^{i-1})(K^i - K^{i-1})} .$$

□

Essentially the statement is that in order to simulate a smooth arbitrage-free option surface, fix (or also simulate) increasing $0 < V_1 < \dots < V_M$ and a $(N-2) \times M$ grid of discrete non-negative local volatilities Σ_j^i . They have the natural scale of annualized volatilities (indeed, as the grid density increases the discrete local volatilities approach Dupire's continuous time local volatility). The only constraint on the discrete local volatilities is non-negativity, making them unique suitable for simulation and descriptive statistics. This is subject to further work.

See [BR15] for the case where the strikes are not homogeneous across expiries.

3.3 Generalization

The model (9) shines due to its simplicity. From a practical point of view it is entirely sufficient for fitting observed market data within bid/ask spreads.

For completeness we present now an extension which provides a more advanced version which features a martingale Y per strike. We focus on a log-normal version for clarity, but other driving models Y can be utilized. We view this model primarily as a conceptual result; in practice the production model already captures observed surfaces to high accuracy.

Theorem 3.7 (Iterative Smooth Call Prices). *Let $dV_j^i \geq 0$ be forward Black-Scholes implied variances for $i = 1, \dots, N$ and $j = 1, \dots, M$. Let q_1, \dots, q_M be marginal densities with unit mean (e.g. they do not have to be martingale densities). Define iteratively*

$$\tilde{c}_1(K; v) := \sum_{i_1=1}^N q_1^{i_1} \text{Call}(K^{i_1}, K; dV_1^i + v) \quad (18)$$

and

$$\tilde{c}_j(K; v) := \sum_{i_j=1}^N q_j^{i_j} K^{i_j} \tilde{c}_{j-1}\left(\frac{K}{K^{i_j}}; dV_j^{i_j} + v\right) \quad (19)$$

and then

$$\tilde{C}_j(K) := \tilde{c}_{j+1}(K, 0) \quad \text{and} \quad \tilde{C}(T, K) := \alpha_j(T) \tilde{C}_{j+1}(K) + (1 - \alpha_j(T)) \tilde{C}_j(K) \quad (20)$$

with α such as defined in (4).

Then, \tilde{C} is a smooth arbitrage-free option surface.

The proof is trivial observing that this pricing scheme is realized by

$$Z_j := \prod_{\ell=1}^j X_\ell Y_\ell^{X_\ell} \quad (21)$$

with $X_\ell \sim q_\ell$ and $Y_\ell^i := \exp\left(\sqrt{dV_\ell^i} N_\ell^i - \frac{1}{2} dV_\ell^i\right)$ where N_ℓ^i is iid standard normal.

Proposition 3.8. *We may write*

$$\tilde{C}_j(K) := \sum_{i_j=1}^N q_j^{i_j} \left(\sum_{i_{j-1}, \dots, i_1} q_{j-1}^{i_{j-1}} \cdots q_1^{i_1} \text{Call}(K^{i_j} \cdots K^{i_1}, K; dV_j^{i_j} + \cdots dV_1^{i_1}) \right). \quad (22)$$

This formula is used in our implementation in section 4.3 on page 19 to iteratively fit the model to market data from the earliest expiry towards the latest expiry (the implementation allows for different strikes per expiry; we omitted this here for sake of notational clarity).

As pleasing as this result is theoretically it is numerically more expensive to implement than our simple (9) as we need to calculate the full tensor structure on the right hand side of (22) while iterating forward.

Remark 3.9 (Extension to Martingale Densities). Equation (21) shows that our representation means that the jump process X_j is independent of X_{j-1} . We assumed this for computational efficiency, but it is noteworthy that we can also apply our idea to more general martingales X as follows: assume q_1, \dots, q_M is a martingale density with transition operators $Q_{j|j-1}$. Define then

$$\check{C}_j(K) := \sum_{i_j=1}^N K^{i_j} \left(\sum_{i_{j-1}, \dots, i_1} Q_{j|j-1}^{i_j|i_{j-1}} Q_{j-1|j-2}^{i_{j-1}|i_{j-2}} \cdots Q_{1|0}^{i_1|0} \text{Call}(K^{i_j}, K; dV_j^{i_j} + \cdots dV_1^{i_1}) \right). \quad (23)$$

Then, \check{C} has the martingale representation

$$Z_j = X_j \prod_{\ell=1}^j Y_\ell^{X_\ell}.$$

□

It is notable that this approach has a similar expression in a classic diffusion setting:

Remark 3.10 (Generalization to a Diffusion Setting). Assume that

$$\frac{dY_t^x}{Y_t^x} = u_t(x)dW_t \quad \text{and} \quad \frac{dX_t}{X_t} = v(X_t)dB_t$$

are both positive martingales; then

$$\frac{dZ_t}{Z_t} = u_t(X_t)dW_t$$

is a well-defined martingale. We then have

$$d(X_t Z_t) = X_t u_t(X_t) Z_t dW_t + Z_t v(X_t) X_t dB_t = (X_t Z_t) (u(X_t) dW_t + v(X_t) dB_t)$$

which shows that XY is also a positive martingale (albeit not Markov, and correlated to X). This idea can be trivially extended to processes with jumps. \triangleleft

4 Implementation

We now discuss the practical implementation of fitting our model variants to observed market data. We will start with the simplest variant and work our way to the fully consistent model of theorem 3.7. For both our production and generalized model we provide the ability to specify different strikes per expiry.

4.1 Homogeneous Strikes - Toy Model

We now present an algorithm on how to fit our smooth parametrization to a discrete set of homogeneous market call prices. To illustrate the general mechanics we start by providing a very simple algorithm for the exact problem stated in theorem 3.1. In the the following section 4.2 we present a more practical approach which handles inhomogenous strikes etc.

We will use (11): to this end, assume that we have specified Black-Scholes implied variances V_j which are increasing in expiries, $V_j \geq V_{j-1}$ and let $\eta \in [0, 1)$ be a variance factor, e.g. $\eta = 0.25$. Define the matrix $\mathbf{C}_j \in \mathbb{R}^{(N-2) \times N}$ for $j = 1, \dots, M$ as

$$\mathbf{C}_j^{\ell,i} := \text{Call}(K^i, K^\ell, \eta V_j) . \quad (24)$$

for $\ell = 2, \dots, N-1$ and $i, \ell = 1, \dots, N$. Let also

$$\mathbf{U}^{\ell,i} := (K^i - K^\ell)^+ .$$

The candidate call prices based on m for strikes K at expiry T_j are per (11) given in terms of a density q_j as

$$\mathbf{c}_j := \mathbf{C}_j \cdot q_j \in \mathbb{R}^{N-2}$$

where as before " \cdot " denotes the classic matrix/vector product.

The fitting problem for our model can then be written as the linear program

$$\left\{ \begin{array}{ll} \text{Variables:} & \\ q_1, \dots, q_M \text{ with } q_j \in \mathbb{R}^N & \text{Marginal densities} \\ \text{Slack variables:} & \\ \mathbf{c}_j := \mathbf{C}_j \cdot q_j \in \mathbb{R}^{N-2} & \text{Prices at market strikes} \\ \mathbf{u}_j := \mathbf{U}_j \cdot q_j \in \mathbb{R}^N & \text{Call prices for } q_j \\ \text{Constraints:} & \\ q_j \geq 0, \ 1' \cdot q_j = 1, \ K' \cdot q_j = 1 & \text{Marginal density with unit mean} \\ \mathbf{u}_j \geq \mathbf{u}_{j-1} & \text{Martingale condition} \\ \text{Objective:} & \\ \inf q_1, \dots, q_M : \sum_j |w_j \cdot (\mathbf{c}_j - \mathbf{c}_j)| & \text{Weighted fit to market} \end{array} \right. \quad (\text{SMP})$$

Our example implementation we provided computes the tensor expressions in every iteration. This means the algorithm has quadratic execution time and is in its current form not suitable for production use.

4.2 Production Model

In practice the strikes in the market are not constant in time, in particular not when normalized by the forward into "pure" strikes. When using real market data, it is also sometimes necessary to add additional strikes when market strikes are far away from each other (e.g. after filtering by minimum quoted volumes). The code we used to test our model adds strikes whenever market strikes are further apart than some maximum dx , and adds additional strikes outside the observed strike range.

Therefore, assume first that we are given boundary strikes $0 < K^{\min} < 1 < K^{\max}$ which lie outside any observed market prices. Assume further that we are given for every expiry j :

- **Market strikes** $0 < K^{\min} \ll k_j^1 < \dots < 1 < \dots < k_j^{n_j} \ll K^{\max}$ with observed market call mid-prices C_j^ℓ .
 - **Model strikes** $0 < K^{\min} = K_j^1 < \dots < K_j^{N_j} = K^{\max}$ which may or may not include the market strikes k_j .
 - **Weights** $w_j \in \mathbb{R}_{\geq 0}^{n_j}$ typically the inverse of the prevailing bid/ask spread or the inverse of Vega to approximate a fit in implied volatilities.
 - **Implied Variances** we also assume we have an increasing set of implied variances $0 < V_1 < \dots < V_m$ and associated scaling factor $\eta \in [0, 1)$, for example $\eta = 0.25$. If η is zero, then the model becomes linear. Figure 6 in page 18 illustrates the impact of η .

We pre-compute matrices $\mathbf{G}_j \in \mathbb{R}^{n_j \times N_j}$, $\mathbf{U}_j \in \mathbb{R}^{N_j, N_j}$ and $\mathbf{R} \in \mathbb{R}^{N_{j+1}, N_j}$ as

$$\begin{aligned} \mathbf{C}_j^{\ell,i} &:= \text{Call}(K_j^i, k_j^\ell, \eta V_j) . \\ \mathbf{U}_j^{\ell,i} &:= \text{Call}(K_j^i, K_j^\ell, \eta \omega V_j) \text{ and} \\ \mathbf{R}_j^{\ell,i} &:= \text{Call}(K_{j-1}^i, K_j^\ell, \eta \omega V_{j-1}) . \end{aligned} \quad (25)$$

The variable $w \in \{0, 1\}$ is explained below.

Our linear program then becomes:

Variables:	q_1, \dots, q_M with $q_j \in \mathbb{R}^{N_j}$	Marginal densities
Slack variables:		
$\mathbf{c}_j := \mathbf{C}_j \cdot q_j \in \mathbb{R}^{n_j}$	Prices at market strikes	
$\mathbf{u}_j := \mathbf{U}_j \cdot q_j \in \mathbb{R}^{N_j}$	Prices at model strikes	
$\mathbf{r}_j := \mathbf{R}_j \cdot q_{j-1} \in \mathbb{R}^{N_j}$	Previous model prices at current strikes	(MDL)
Constraints:		
$q_j \geq 0, \mathbf{1}' \cdot q_j = 1, \mathbf{K}' \cdot q_j = 1$	Marginal density with unit mean	
$\mathbf{u}_j \geq \mathbf{r}_j$	Martingale condition (*)	
Objective:		
$\inf q_1, \dots, q_M : \sum_j w_j \cdot (C_j - \mathbf{c}_j) $	Weighted fit to mid	

In order to ensure absence of arbitrage in time condition (*) must be satisfied with $\omega = 0$ as this ensures that the discrete strike marginal densities constitute a martingale density.¹³ However, from a practical and numerical perspective it is more natural to impose the price condition in observed price space, which is the case for $w = 1$.

¹³As explained in [BR15] for two densities m_1 and m_2 to be in convex order of inhomogeneous strikes K_j^i it is sufficient that the call prices computed with m_1 for strikes K_2 are below the call prices computed with m_2 for strikes K_2 . This is because using m_2 for computing call prices is akin to linear interpolation which is such the "most expensive" interpolation, i.e. gives the highest possible call prices.

Remark 4.1 (Bid/Ask Spreads). Actual markets do not have a mid-price to fit to; instead we observe bid B_j^i and ask prices A_j^i which are a positive spread $A_j^i - B_j^i$ apart.

To incorporate bid/ask's, there are a number of trivial approaches to account for bid/ask spreads.

1. Normalize the fitting error by scaling by the observed bid/ask spread i.e.

$$w_j^i := \frac{1}{A_j^i - B_j^i} . \quad (26)$$

2. enforce the fitted prices to be within bid/ask by adding the constraints $B_j \leq C_j \leq A_j$. This yields the additional linear constraints

$$\left\{ \begin{array}{l} \text{Additional Constraints:} \\ \max\{C_j - A_j, 0\} = 0 \\ \max\{B_j - C_j, 0\} = 0 \end{array} \right.$$

3. Increase the fitting penalty when the fitted price is outside bid/ask.

$$\left\{ \begin{array}{l} \text{Alternative Objective:} \\ \inf_{q_1, \dots, q_M} : \sum_j w_j (\epsilon |C_j - c_j| + \max\{C_j - A_j, 0\} + \max\{B_j - C_j, 0\}) \end{array} \right.$$

This is our recommended default setting with $\epsilon = 1E - 8$ while using weights (26).

All of these fit into a linear or quadratic programming framework. \triangleleft

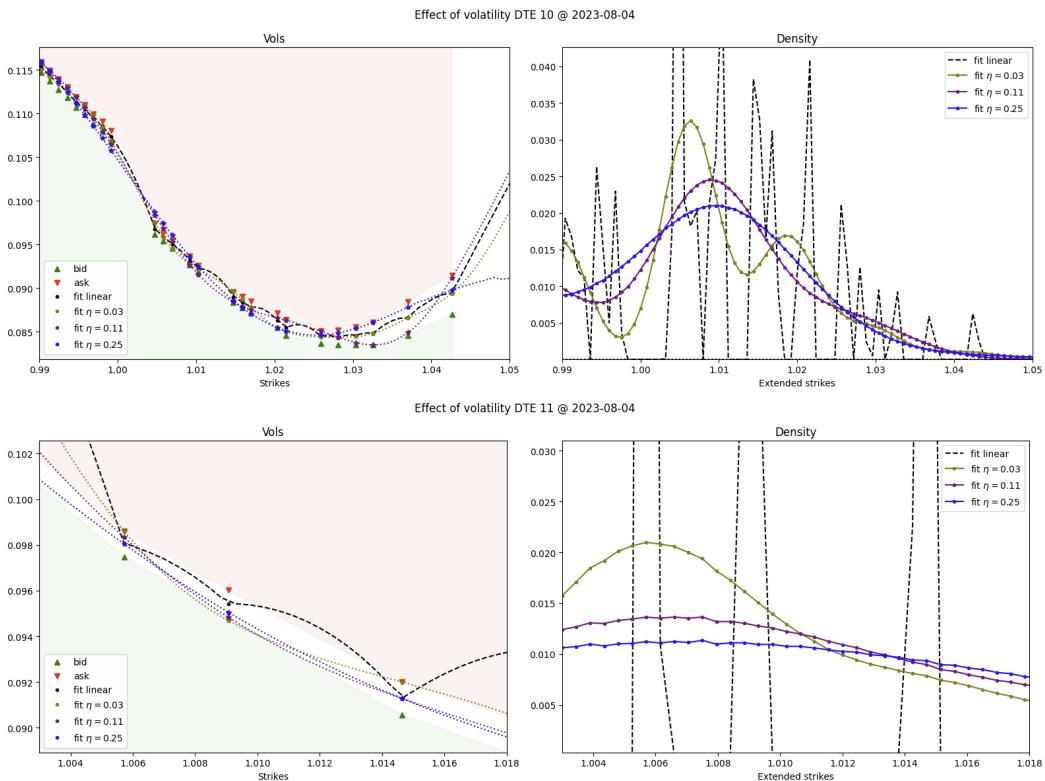


Figure 6: These two examples illustrate the impact of the smoothness parameter η in a zoomed in view on a fit. The left hand graphs show the implied volatility bid/ask, and the model fits. The right hand graph shows the density produced by each model setting. "Linear" refers to the case $\eta = 0$. The implied variances V_j used in the model are ATM bid implied volatilities. The lower example in particular illustrates well the balancing act between the usefulness of a smooth surface vs a good fit: the left hand volatility graph show that the black dashed line from the linear model fits perfectly at the mid point within bid/ask at market strikes, as expected. However, the dashed line also shows the effect of linear interpolation being "too expensive" between market strikes as the implied volatilities bump upwards. In contrast, the models with non-zero smoothness provide much more natural fits the market. The smoothest model has some fitting error, but also provides a more natural interpolation across strikes.

4.3 Generalized Model

The final step is to implement fitting for our generalized model as described in theorem 3.7: Equation (21) on page 15 can be written as

$$\tilde{C}_j(K) := \sum_{i_j=1}^{N_j} q_j^{i_j} \left(\sum_{i_{j-1}, \dots, i_1=1}^{N_{j-1}, \dots, N_1} q_{j-1}^{i_{j-1}} \cdots q_1^{i_1} \cdots K_1^{i_1} \text{Call}\left(K_j^{i_j} K_{j-1}^{i_{j-1}}; K_j^\ell; dV_j^{i_j} + \cdots + dV_1^{i_1}\right) \right).$$

That means that in addition to the input assumptions of the previous section we will also assume that for each j there is a vector $dV_j^1, \dots, dV_j^{N_j}$ of model incremental variances each corresponding to one of the log-normal increments Y_j^i in theorem 3.7.

We will now present an iterative fitting scheme which fits each step using linear programming going forward along expiries.

Define the tensors

$$\begin{aligned} \mathbb{V}_j &:= dV_j \oplus \cdots \oplus dV_1 = (dV_j^{i_j} + \cdots + dV_1^{i_1})_{i_j, \dots, i_1} \in \mathbb{R}^{N_j \times \cdots \times N_1} \\ \mathbb{K}_j &:= K_j \otimes \cdots \otimes K_1 = (K_j^{i_j} \cdots K_1^{i_1})_{i_j, \dots, i_1} \in \mathbb{R}^{N_j \times \cdots \times N_1} \\ \mathbb{Q}_j &:= q_j \otimes \cdots \otimes q_1 = (q_j^{i_j} \cdots q_1^{i_1})_{i_j, \dots, i_1} \in \mathbb{R}^{N_j \times \cdots \times N_1} \end{aligned}$$

and then

$$\begin{aligned} \mathbf{C}_j &:= \text{Call}(\mathbb{K}_j, k_j, \eta \mathbb{V}_j) \in \mathbb{R}^{n_j \times N_j \times \cdots \times N_1} \\ \mathbf{U}_j &:= (\mathbb{K}_j - K_j)^+ \in \mathbb{R}^{N_j \times N_j \times \cdots \times N_1} \\ \mathbf{R}_j &:= (\mathbb{K}_{j-1} - K_j)^+ \in \mathbb{R}^{N_j \times N_{j-1} \times \cdots \times N_1} \end{aligned}$$

where indexing is as implied by the dimensionality of each tensor.

We note that provided \mathbb{Q}_j the inner product

$$\mathbf{c}_j := \mathbf{C}_j \cdot \mathbb{Q}'_j \in \mathbb{R}^{n_j}$$

gives the model prices for the market strike at the j th expiry. However, \mathbb{Q}_j contains products of all the vectors q_j such that above is no longer linear or quadratic.

Iterative Linear Programming: we therefore propose the following scheme: assume that we have found q_1, \dots, q_{j-1} and therefore \mathbb{Q}_{j-1} , and that we are now looking to fit the marginal density for the j th expiry, $q_j \in [0, 1]^{N_j}$.

$$\left\{ \begin{array}{ll} \text{Variables:} & \\ q_j \in \mathbb{R}^{N_j} & \text{Marginal density} \\ \\ \text{Slack variables:} & \\ \mathbf{c}_j := (\mathbf{C}_j \cdot \mathbb{Q}'_{j-1}) \cdot q_j \in \mathbb{R}^{n_j} & \text{Prices at market strikes} \\ \mathbf{u}_j := (\mathbf{U}_j \cdot \mathbb{Q}'_{j-1}) \cdot q_j \in \mathbb{R}^{N_j} & \text{Prices at model strikes} \\ \mathbf{r}_j := (\mathbf{R}_j \cdot \mathbb{Q}'_{j-1}) \in \mathbb{R}^{N_j} & \text{Previous model prices at current strikes} \\ \\ \text{Constraints:} & \\ q_j \geq 0, \mathbf{1}' \cdot q_j = 1, \mathbf{K}' \cdot q_j = 1 & \text{Marginal density with unit mean} \\ \mathbf{u}_j \geq \mathbf{r}_j & \text{Martingale condition} \\ \\ \text{Objective:} & \\ \inf q_j : |w_j \cdot (C_j - \mathbf{c}_j)| & \text{Weighted fit to market} \end{array} \right. \quad (\text{GNR})$$

5 Conclusion

We have presented a smooth strictly arbitrage-free parametrization of an option price surface and shown how to efficiently fit such model – and its simpler variant – to market data, illustrating the broad applicability of the method. We have also shown how to incorporate bid/ask spread into our

fit. Our models can be written in terms of "volatility" parameters which only have to be positive in order to make the model smooth and strictly arbitrage-free. We presented an interpretation as a dynamic process to aid intuition behind our approach.

Our approach is a natural extension of linear option surface fitting whenever smoothness is critical to the application.

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A Appendix

A.1 Absence of Arbitrage

We present a self-contained proof for theorem 2.2 on page 5 which we restate here for convenience:

Theorem A.1 (Theorem 2.2). *A continuous call price surface $C : [0, \infty)^2 \rightarrow [0, \infty)$ arbitrage-free iff*

1. *The market has unit expectation: $C(T, 0) \equiv \mathbb{E}[Z_T] = 1$ for all T ;*
2. *zero is unattainable: $\partial_K C(T, 0) \equiv -1$ since then $\mathbb{P}[Z_T = 0] = -1$;*
3. *call prices ultimately reach zero: $\lim_{K \uparrow \infty} C(T, K) = 0$,*
4. *call prices are convex: $\partial_{KK} C(T, K) \geq 0$; and*
5. *call prices are increasing in time: $\partial_T C(T, K) \geq 0$.*

To prove this statement, we first recall per expiry the following proposition 2.3 from [Bue06]:

Proposition A.2. *A function $c : [0, \infty) \mapsto [0, 1]$ is arbitrage-free in the sense that there exist a random variable Z with unit mean such that $c(K) = \mathbb{E}[(Z - K)^+]$ iff*

1. *c is decreasing,*
2. *c is convex, and*
3. *$c(0) = 1$, $\lim_{K \uparrow \infty} c(K) = 0$ and $c'(0) := \lim_{\delta \downarrow 0} \frac{1}{\delta} c(\delta) \geq -1$.*

Moreover, if $c'(0) = -1$ then Z is positive.

Proof. Since c is convex and decreasing, its right-hand derivative c' exists and is right-continuous and non-decreasing with $c' \geq -1$. Since $c(0) = 1$ and $\lim_{K \uparrow \infty} c(K) = 0$ we also conclude that c is non-negative.

Let $f(K) := 1 + c'(K)$, which is a positive, right-continuous and non-decreasing function and which therefore implies the existence of some positive σ -finite measure via $\tilde{\nu}[(a, b)] := f(b) - f(a)$ cf. [AB06] theorem 9.47 pg. 354. The properties $\lim_{K \uparrow \infty} c(K) = 0$ and $c(K) \geq 0$ imply that $\lim_{K \uparrow \infty} c'(K) = 0$, hence we obtain that $\nu^* := \tilde{\nu}[(0, \infty)] = \lim_{K \uparrow \infty} f(K) - f(0) = 0 - c'(0) \in [0, 1]$. The proof is complete by defining $\nu(A) := (1 - \nu^*) \delta_0(A) + \tilde{\nu}(A) = (1 + c'(0)) \delta_0(A) + \tilde{\nu}(A)$ for measurable sets $A \subseteq \mathbb{R}_0^+$. This measure has unit expectation as a result of the assumption $c(0) = 1$.

We note in particular that iff $c'(0) = -1$ then ν has mass only in the positive numbers, confirming positivity of Z . \square

¹⁴<https://vectorinstitute.ai/partnerships/current-partners/>

To account for term structure we refer to the following historical result from [Kel72]:

Theorem A.3 (Kellerer 1972). *Let $\mathcal{M} = (\mu_T)_{T \in \mathcal{T}}$ be a set of probability measures with unit expectation. Then, a Markov martingale $Z = (Z_T)_{T \in \mathcal{T}}$ with marginal distributions μ_T exists if and only if \mathcal{M} is in Balayage-order, that is*

$$\mu_T \preceq \mu_U$$

for all $T, U \in \mathcal{T}$ with $T < U$.

Proof of theorem 2.2. In light of proposition A.2 and theorem A.3 it remains to show that 5. in theorem 2.2 implies that $\mathbb{E}[f(Z_T)] \leq \mathbb{E}[f(Z_U)]$ for any $T < U$ and all convex f .

For a convex function f

$$f(x) = f(0) + f'(0)x + \int_0^x (x - K)^+ \mu(dK)$$

where $\mu[(a, b)] := f'(b) - f'(a)$ is the measure implied by f . In a generalized sense¹⁵ this can be written as

$$f(x) = f(0) + f'(0)x + \int_0^x (x - K)^+ f''(K) dK$$

where $f''(K) \geq 0$. Taking expectations proves the statement. \square

A.2 Transition Operators with Discrete Local Volatilities

We present a proof for theorem 3.5 on page 13: we start with the following statement which is essentially from [AH11]:

Theorem A.4 (Construction of Transition Matrices). *Assume that $M \in \mathbb{R}^{N \times N}$ is a square matrix with columns adding up to 1, and all whose off-diagonal elements are non-positive.*

Then, its inverse exists, is non-negative, and its columns add up to 1; in other words $M^{-1} \in \mathbb{R}^{N \times N}$ is a probability matrix.

Moreover, if M has the martingale property with respect to $K \in \mathbb{R}^N$ in the sense that $K'M = K'$ then M^{-1} has the martingale property with respect to K , too, in the sense that $K' = K'M^{-1}$.

Proof. A matrix M with the above properties is a Z-matrix, i.e. $M^{i,i} > 0$ and $M^{i,\ell} \leq 0$ for $i \neq \ell$. Moreover, its columns add up to 1, which means it is a non-singular M-matrix, c.f. [BP94], chapter 6: equivalent classification I_{29} of non-singular M-Matrices.

Hence, its inverse exists and is non-negative, c.f. [BP94], chapter 6: equivalent classification N_{38} of non-singular M-Matrices.

Finally, $1'M = 1'$ implies $1' = 1'M^{-1}$, i.e. the columns of M^{-1} add up to 1, too. Similarly, if $K'M = K'$, then $K' = K'M^{-1}$. \square

Proof of theorem 3.5. We show that $Q_{j|j-1}$ defined in (15) on page 14 is indeed a transition operator.

- The inverse satisfies the conditions of theorem (A.4) hence $Q_{j|j-1}$ is a transition matrix.
- We now show $K' \cdot Q_{j|j-1} = K'$ by showing $K' = K' \cdot Q_{j|j-1}^{-1}$. Let

$$U_j^i := \frac{(\sum_{i,j} K^i)^2 (T_j - T_{j-1})}{K^{i+1} - K^{i-1}} .$$

such that

$$w_j^{i+} = \frac{U_j^i}{K^{i+1} - K^i} \quad \text{and} \quad w_j^{i-} = \frac{U_j^i}{K^i - K^{i-1}} .$$

¹⁵There is a countable number of extremal points (c.f. [Bue06]) x_k such that $f''(K) \in \mathbb{R}_{\geq 0}$ for $K \notin \{x_k\}_k$ and $f''(x_k) = p_k \delta_{x_k}$ for $p_k \in \mathbb{R}_{\geq 0}$ and δ_\cdot denotes the Dirac measure.

For $i = 2, \dots, N - 1$ this yields

$$\begin{aligned}
(K' \cdot Q_{j|j-1}^{-1})^i &= -K^{i-1} \frac{U_j^i}{K^i - K^{i-1}} + K^i + K^i \left(\frac{U_j^i}{K^i - K^{i-1}} + \frac{U_j^i}{K^{i+1} - K^i} \right) \\
&\quad - K^{i+1} \frac{U_j^i}{K^{i+1} - K^i} \\
&= K^i + U_j^i \left(\frac{K^i - K^{i-1}}{K^i - K^{i-1}} + \frac{K^i - K^{i+1}}{K^{i+1} - K^i} \right) \\
&= K^i.
\end{aligned}$$

This next step is to show that a discrete local volatility as defined in (16) reprises the input call prices. To do so we re-iterate the proof of section 4.3.2 in [BR15] and show that with Σ as defined above $Q_{j|j-1}^{-1} p_j = p_{j-1}$. We first note that

$$\Sigma_{i,j}^2 := \frac{2}{(K^i)^2} \left(\frac{C_j^i - C_{j-1}^i}{(T_j - T_{j-1})} \right) / \left(\frac{p_j^i}{\frac{1}{2}(K^{i+1} - K^{i-1})} \right)$$

hence w defined in (14) can be written as (17). For $i \in \{2, \dots, N - 1\}$ we therefore get

$$\begin{aligned}
(Q_{j|j-1}^{-1} \cdot p_j)^i &= -p_j^{i-1} w_j^{(i-1)+} + p_j^i (1 + w_j^{i-} + w_j^{i+}) - p_j^{i+1} w_j^{(i+1)-} \\
&= p_j^i - (C_j^{i-1} - C_{j-1}^{i-1}) \bar{\gamma}_j^{(i-1)+} + (C_j^i - C_{j-1}^i) (\bar{\gamma}_j^{i-} + \bar{\gamma}_j^{i+}) - (C_j^{i+1} - C_{j-1}^{i+1}) \bar{\gamma}_j^{(i+1)-} \\
&\stackrel{(*)}{=} p_{j-1}.
\end{aligned}$$

The last equation $(*)$ follows from

$$\begin{aligned}
C_j^{i-1} \bar{\gamma}_j^{(i-1)+} - C_j^i (\bar{\gamma}_j^{i-} + \bar{\gamma}_j^{i+}) + C_j^{i+1} \bar{\gamma}_j^{(i+1)-} &= (C_j^{i+1} \bar{\gamma}_j^{(i+1)-} - C_j^i \bar{\gamma}_j^{i+}) - (C_j^i \bar{\gamma}_j^{i-} - C_j^{i-1} \bar{\gamma}_j^{(i-1)+}) \\
&= dC_j^{i+1} - dC_j^i \\
&= p_j^i.
\end{aligned}$$

For $i = 1$ recall that $p_j^1 = \frac{C_j^2 - C^1}{K_2 - K_1}$ with $C^1 = 1 - K^1$ hence

$$(Q_{j|j-1}^{-1} \cdot p_j)^1 = p_j^1 - (C_j^2 - C_{j-1}^2) \frac{1}{K^2 - K^1} = \frac{C_j^2 - C^1}{K_2 - K_1} - (C_j^2 - C_{j-1}^2) \frac{1}{K^2 - K^1} = p_{j-1}^1.$$

Similarly, for $i = N$ we have $p_j^N = \frac{C^{N+1} - C_j^N}{K_{N+1} - K_N}$ with $C^{N+1} = 0$ such that

$$(Q_{j|j-1}^{-1} \cdot p_j)^N = p_j^N - (C_j^N - C_{j-1}^N) \frac{1}{K_{N+1} - K_N} = \frac{C^{N+1} - C_j^N}{K_{N+1} - K_1} - (C_j^N - C_{j-1}^N) \frac{1}{K_{N+1} - K_N} = p_{j-1}^N.$$

□

References

- [AB06] Charalambos Aliprantis and Kim Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, 06 2006.
- [AH11] J. Andreasen and B. Huge. Volatility interpolation. *Risk*, pages 86–89, 2011.
- [BM00] Damiano Brigo and Fabio Mercurio. A mixed-up smile. *Risk*, 13(9):123–126, 2000.
- [BMPW22a] Hans Buehler, Phillip Murray, Mikko S. Pakkanen, and Ben Wood. Deep hedging: learning to remove the drift. *Risk*, February 2022.

- [BMPW22b] Hans Buehler, Phillip Murray, Mikko S. Pakkanen, and Ben Wood. Deep hedging: Learning to remove the drift under trading frictions with minimal equivalent near-martingale measures. <https://arxiv.org/abs/2111.07844>, 2022.
- [BMR02] Damiano Brigo, Fabio Mercurio, and Francesco Rapisarda. Lognormal-mixture dynamics and calibration to market volatility smiles. *International Journal of Theoretical and Applied Finance*, 05, 06 2002.
- [BP94] Abraham. Berman and Robert J. Plemmons. *Nonnegative matrices in the mathematical sciences / Abraham Berman, Robert J. Plemmons*. Classics in applied mathematics ; 9. Society for Industrial and Applied Mathematics, Philadelphia, 1994.
- [BR15] Hans Buehler and Evgeny Ryskin. Discrete local volatility for large time steps (extended version). <https://papers.ssrn.com/abstract=2642630>, 2015.
- [Bue06] Hans Buehler. Expensive martingales. *Quantitative Finance*, 6(3):207–218, 2006.
- [Bue08] Hans Buehler. Volatility and dividends - volatility modelling with cash dividends and simple credit risk. <https://papers.ssrn.com/abstract=1141877>, 2008.
- [CN12] R. Carmona and S. Nadtochiy. Tangent levy market models. *Finance and Stochastics*, 16:63–104, 2012.
- [D⁺94] Bruno Dupire et al. Pricing with a smile. *Risk*, 7(1):18–20, 1994.
- [DS06] Freddy Delbaen and Walter Schachermayer. A general version of the fundamental theorem of asset pricing (1994). In *The Mathematics of Arbitrage*, pages 149–205. Springer, 2006.
- [FC80] F. N. Fritsch and R. E. Carlson. Monotone piecewise cubic interpolation. *SIAM Journal on Numerical Analysis*, 17(2):238–246, 1980.
- [GJ14] Jim Gatheral and Antoine Jacquier. Arbitrage-free svi volatility surfaces. *Quantitative Finance*, 14(1):59–71, 2014.
- [Hes93] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2):327–343, 1993.
- [Kel72] Hans G. Kellerer. Markov-komposition und eine anwendung auf martingale. *Mathematische Annalen*, 198:99–122, 1972.
- [WWP⁺21] Magnus Wiese, Ben Wood, Alexandre Pachoud, Ralf Korn, Hans Buehler, Phillip Murray, and Lianjun Bai. Multi-asset spot and option market simulation. <https://arxiv.org/abs/2112.06823>, 2021.