Statistics_ Assignment_ 3

5.34 $E(x) = \left[\frac{-1}{8} - \frac{0.5}{4} + \frac{0.5}{2} + \frac{1}{8} \right]$

$= \frac{-1-1+2+1}{8} = \frac{1}{8} = 0.125$

$E(y) = \frac{-2}{8} - \frac{1}{4} + \frac{1}{2} + \frac{2}{8}$

$= \frac{-1}{4} - \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4} = \frac{+1}{4}$

$= 0.25$

$E(xy) = \frac{2}{8} + \frac{0.5}{4} + \frac{0.5}{2} + \frac{2}{8} = \frac{4}{8} + 0.5\left(\frac{1}{4} + \frac{1}{2}\right)$

$= \frac{1}{2} + 0.5\left(\frac{1+2}{4}\right) = \frac{1}{2} + 0.5\left(\frac{3}{4}\right)$

$= 0.875$

$E(x^2) = \frac{1}{8} + \frac{0.25}{4} + \frac{0.25}{2} + \frac{1}{8} = 0.4375$

$E(y^2) = \frac{4}{8} + \frac{1}{4} + \frac{1}{2} + \frac{4}{8} = 1.75$

$V(x) = E(x^2) - [E(x)]^2$

$= 0.4375 - 0.015625 = 0.421875$

$$V(Y) = E(Y^2) - [E(Y)]^2$$
$$= 1.75 - 0.0625 = 1.6875$$

⇒ Covariance:

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 0.875 - (0.125)(0.25) = 0.875 - 0.03125$$

$$= 0.84375$$

⇒ Correlation coefficient:

$$\rho_{xy} = \frac{\text{cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$= \frac{0.84375}{\sqrt{(0.42185)(1.6875)}} = \frac{0.84375}{\sqrt{0.421875 \times 1.6875}}$$

$$= \frac{0.84375}{0.84375} = 1$$

5.65

$\mu = 2 \quad \sigma = 0.1$

$E(X_1) = 2 \qquad V(X_1) = 0.1$

$E(X_2) = 2 \qquad V(X_2) = 0.1$

$E(\text{Total}) = E(X_1) + E(X_2)$

$= 2 + 2 = 4$

$\sigma(\text{Total}) = \sqrt{(\sigma(X_1))^2 + (\sigma(X_2))^2} = \sqrt{0.1^2 + 0.1^2}$

$= \sqrt{0.01 + 0.01} = \sqrt{0.02} = 0.141421$

a) Mean of the total thickness of two halves is 4.

~~Variance~~ of the total thickness of is (Standard Deviation) = 0.141421.

b) $P(T > 4.3) = P\left(\frac{T - E(\text{Total})}{\sigma(\text{Total})} > \frac{4.3 - 4}{0.141421}\right)$

$= P(Z > 2.12132) = 1 - P(Z < 2.12132)$

$= 1 - 0.983414 = 0.016586$

7.7

$\mu = 2500$

$\sigma = 50$

$n = 5$

$\cancel{\sigma_x = \frac{50}{\sqrt{2500}}}$

$$\sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{50}{\sqrt{5}} = \frac{50}{2.236} = \cancel{22.36}\ 22.421$$

$$P(2499 < X < 2510) = P\left(\frac{2499 - 2500}{22.421} < \frac{X - 2500}{22.421} < \frac{2510 - 2500}{22.421}\right)$$

$$= P\left(\frac{-1}{22.421} < Z < \frac{10}{22.421}\right) = P(-0.0446 < Z < 0.446)$$

$$= \Phi(0.446) - \Phi(-0.0446)$$

$$= 0.673645 - 0.480061 = 0.193584 \simeq 0.193$$

7.36 a) Prove $\hat{P} = \frac{x}{n}$ is an unbiased estimator of p.

Assuming an event K indicates k^{th} event as either success (1) or failure (0).

$$\mu_K = 1(p) + 0(1-p)$$
$$= p$$

Total number of success X is the sum of all the events of E_K.

$$E(X) = \mu_1 + \mu_2 + \cdots \mu_k + \cdots \mu_n$$

$$= p + p\ p + \cdots + p = np$$

$$E(\hat{p}) \quad E(\hat{P}) = \frac{1}{n} E(X) = \frac{1}{n}(np) = p$$

$\therefore$ $\hat{P} = \frac{x}{n}$ is an unbiased estimator of p.

b) Variance of each event E_i is

$$V(E_i) = E(E_i^2) - [E(E_i)]^2$$

Assume $E(E_i^2) = 1^2(p) + 0^2(1-p) = p$

$$V(E_i) = p - p^2 = p(1-p)$$

Variance of $x = V(E_1) + V(E_2) + V(E_3) + \cdots + V(E_n)$

$= np(1-p)$

Given $\hat{p} = \frac{x}{n} \Rightarrow V(\hat{p}) = V\left(\frac{x}{n}\right) = \frac{1}{n^2} np(1-p)$

Standard Error:

$$S.E \text{ of } \hat{p} = \sqrt{V(\hat{p})} = \sqrt{\frac{p(1-p)}{n}}$$

7.24

$$E(\hat{\theta}_1) = E\left[\frac{x_1+x_2}{2}\right] = \frac{1}{2}\left[E(x_1) + E(x_2)\right]$$

$$E(\hat{\theta}_1) = \frac{1}{2}(\mu+\mu) = \mu$$

$$E(\hat{\theta}_2) = E\left(\frac{x_1+3x_2}{4}\right) = \frac{1}{4}\left[E(x_1) + 3E(x_2)\right]$$

$$E(\hat{\theta}_2) = \frac{\mu}{4} + \frac{3\mu}{4} = \mu$$

So, both $E(\hat{\theta}_1)$ and $E(\hat{\theta}_2)$ are unbiased estimator of μ.

Variance of estimators:

$$V(\hat{\theta}_1) = V\left[\frac{x_1+x_2}{2}\right] = \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2$$

$$= \frac{2}{4}\sigma^2 = \frac{1}{2}\sigma^2$$

$$V(\hat{\theta}_2) = V\left[\frac{x_1+3x_2}{4}\right] = \frac{1}{16}\sigma^2 + \frac{9}{16}\sigma^2$$

$$= \frac{10}{16}\sigma^2 = \frac{5}{8}\sigma^2$$

$$V(\hat{\theta}_1) = \frac{1}{2}\sigma^2, \quad V(\hat{\theta}_2) = \frac{5}{8}\sigma^2$$