7.50 $f(x) = c(1+\theta x)$, $-1 \le x \le 1$

a) Find the value of the constant c

PDF : $\int_x f(x) \cdot dx = 1$

so, $\int_{-1}^{1} c(1+\theta x) \cdot dx = c \int_{-1}^{1} (1+\theta x)\, dx$

$$c\left[x + \frac{\theta x^2}{2}\right]_{-1}^{1} = 1$$

$$c\left[\left(1+\frac{\theta}{2}\right) - \left(-1+\frac{\theta}{2}\right)\right] = c\left[1+\frac{\theta}{2}+1-\frac{\theta}{2}\right]$$

$$= c[2]$$

$$2c = 1 \Rightarrow c = \frac{1}{2}$$

b) Moment estimator for θ

$$\mu_1' = \int_{-1}^{1} x\, f(x) \cdot dx$$

$$= \frac{1}{2}\int_{-1}^{1} x + \theta x^2 \cdot dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{\theta x^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left[\left(\frac{1}{2}+\frac{\theta}{3}\right) - \left(\frac{1}{2}-\frac{\theta}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{\theta}{3} - \frac{1}{2} + \frac{\theta}{3}\right] = \frac{1}{2}\left[\frac{2\theta}{3}\right] = \frac{\theta}{3}$$

$$\boxed{\text{Moment estimator for } \theta = \frac{\theta}{3}}$$

c) S.T $\hat{\theta} = 3\bar{X}$ is an unbiased estimator of θ.

Sample estimator of mean $= \bar{x}$

$$\mu_1' = \frac{\theta}{3} \Rightarrow 3\mu_1' = \theta$$

$$\theta = 3\mu_1'$$

$$\theta = 3\bar{X}$$

Hence, $\hat{\theta} = 3\bar{x}$ is an unbiased estimator of θ.

Scanned by CamScanner

7.48 $f(x) = \frac{1}{\theta^2} x e^{-\frac{x}{\theta}}$ $\quad 0 \le x < \infty$

$0 < \theta < \infty$

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \left(\frac{1}{\theta^2} x_1 e^{-\frac{x_1}{\theta}}\right)\left(\frac{1}{\theta^2} x_2 e^{-\frac{x_2}{\theta}}\right) \cdots$$

$$\left(\frac{1}{\theta^2} x_n \theta^{-\frac{x_n}{\theta}}\right)$$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i e^{-\frac{x_i}{\theta}}$$

$$= \left(\frac{1}{\theta^2}\right)^n \prod_{i=1}^{n} x_i e^{-\frac{x_i}{\theta}}$$

Taking log on $L(\theta)$

$$\log(L(\theta)) = n \log \frac{1}{\theta^2} + \log \prod_{i=1}^{n} x_i e^{-\frac{x_i}{\theta}}$$

$$\cancel{= n \log \frac{1}{\theta^2} + \log \prod_{i=1}^{n} x_i + \log e^{-\frac{x_i}{\theta}}}$$

$$= n \log \frac{1}{\theta^2} + \log \prod_{i=1}^{n} x_i + \log \prod_{i=1}^{n} e^{-\frac{x_i}{\theta}}$$

$$= n \log \frac{1}{\theta^2} + \log \prod_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i}{\theta}$$

Scanned by CamScanner

Differentiate it w.r.t θ, $\cancel{\frac{d\log(L(\theta))}{d\theta} = 0}$

$$\frac{d(\log(L(\theta)))}{d\theta} = 0$$

$$= \frac{n}{\frac{1}{\theta^2}} \cancel{\cdot} -2\theta^{-3} + 0 \cancel{\theta} - \sum_{i=1}^{n} x_i (-1 \cdot \theta^{-2})$$

$$= \frac{n}{\theta^{-2}} \cdot (-2\theta^{-3}) + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i$$

$$= n\theta^2(-2\theta^{-3}) + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i$$

$$\Rightarrow -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i = 0$$

$$\frac{1}{\theta^2} \sum_{i=1}^{n} x_i = \frac{2n}{\theta}$$

$$\sum_{i=1}^{n} x_i = \frac{2n \cdot \theta^2}{\theta} = 2n\theta$$

$$\boxed{\hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} x_i}$$

8.4

a) 95% CI $\quad Z_{\frac{\alpha}{2}} = 1.96$

$\alpha = 0.05 \quad Z_{\frac{0.05}{2}} = 1.96$

$n = 10 \quad \bar{x} = 1000 \quad s = 20$

$\cancel{\bar{x}\ \mu} \quad \bar{x} - Z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \le \mu \le \bar{x} + Z_{\alpha/2} \frac{s}{\sqrt{n}}$

$1000 - 1.96 \frac{20}{\sqrt{10}} \le \mu \le 1000 + 1.96 \frac{20}{\sqrt{10}}$

$\cancel{1012.396 \le \mu \le}$

$987.60 \le \mu \le 1012.396$

b) 95% C.I $Z_{\frac{0.05}{2}} = 1.96$

$n = 25 \quad \bar{x} = 1000 \quad s = 20$

$$\bar{x} - Z_{\alpha/2} \frac{s}{\sqrt{n}} \le \mu \le \bar{x} + Z_{\alpha/2} \frac{s}{\sqrt{n}}$$

$$1000 - 1.96(4) \le \mu \le 1000 + 1.96(4)$$

$$992.16 \le \mu \le 1007.84$$

c) 99% C.I $n = 10 \quad \bar{x} = 1000 \quad s = 20 \quad \alpha = \cancel{0.05}\ 0.01$

$Z_{\alpha/2}$ of 99% CI = 2.58

$$\bar{x} - Z_{\alpha/2} \frac{s}{\sqrt{n}} \le \mu \le \bar{x} + Z_{\alpha/2} \frac{s}{\sqrt{n}}$$

$$1000 - 2.58\left(\frac{20}{\sqrt{10}}\right) \le \mu \le 1000 + 2.58\left(\frac{20}{\sqrt{10}}\right)$$

$$983.683 \le \mu \le 1016.317$$

d) $n = 25 \quad \bar{x} = 1000 \quad s = 20 \quad$ 99% CI

$Z_{\frac{0.01}{2}} = 2.58$

$$1000 - 2.58\left(\frac{20}{\sqrt{25}}\right) \le \mu \le 1000 + 2.58\left(\frac{20}{\sqrt{25}}\right)$$

$$989.68 \le \mu \le 1010.32$$

c) Length of the interval decreases with increase in sample size & ~~they~~ increases with the increase of confidence level.

8.11 $\sigma = 3$ 91.6, 88.75, 90.8, 89.95, 91.3

95% two-sided CI

$$\bar{x} = \frac{91.6 + 88.75 + 90.8 + 88.95 + 91.3}{5}$$

$$= 90.28$$

$100(1-\alpha)\% = 95\%$

$(1-\alpha) = 0.95$

$1 - 0.95 = \alpha \Rightarrow \alpha = 0.05 \Rightarrow Z_{\frac{\alpha}{2}} = 1.96$

$$\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$90.28 - 1.96(1.341) \leq \mu \leq 90.28 + 1.96 (1.341)$

$87.65 \leq \mu \leq 2.62 + 90.28$

$87.65 \leq \mu \leq 92.9$

8.13 $\sigma = 0.001$ $n = 15$ $\bar{x} = 74.036$

a) 99% CI

$$\bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq z + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$\frac{\sigma}{\sqrt{n}} = \frac{0.001}{\sqrt{15}} = \frac{0.001}{3.872} = 0.00025$$

$100(1-\alpha)\% = 99\%$ $1-\alpha = 0.99$

$\alpha = 1 - 0.99 = 0.01$

$\frac{\alpha}{2} = 0.005$

$z_{\alpha/2} = 2.576$

$$74.036 - (2.576)(0.00025) \leq \mu \leq 74.036 + (2.576)(0.00025)$$

$$74.036 - 0.000644 \leq \mu \leq 74.036 + 0.000644$$

$$74.0353 \leq \mu \leq 74.0366$$

Scanned by CamScanner

b) Lower Confidence Bound

$$\mu \geq \bar{x} - Z_\alpha \cdot \frac{\sigma}{\sqrt{n}}$$

$\sigma = 0.001$ $\quad \sqrt{n} = \sqrt{15}$ $\quad \frac{\sigma}{\sqrt{n}} = 0.00025$

$\bar{x} = 74.036$ $\quad \cancel{Z_\alpha = \alpha\ 0.001}$

$Z_\alpha = 2.33$

$$\mu \geq 74.036 - 2.33\,(0.00025)$$

$$\mu \geq 74.0354175$$

The lower bound of this confidence level is slightly greater than the one in part a), this is because $Z_{\alpha/2} > Z_\alpha$.

$74.0354175 - 74.0353 = 0.0012825$