CS 611: Theory of Computation

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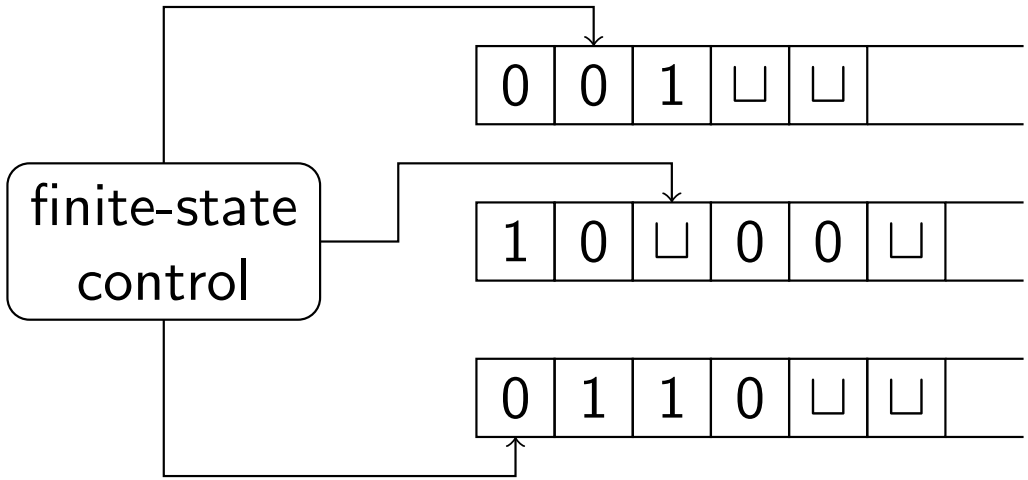
Turing Machines

Formal Definition

A Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ where

- Q is a finite set of control states
- Σ is a finite set of input symbols
- $\Gamma \supseteq \Sigma$ is a finite set of tape symbols. Also, a blank symbol $\sqcup \in \Gamma \setminus \Sigma$
- $q_0 \in Q$ is the initial state
- $q_{\text{acc}} \in Q$ is the accept state
- $q_{\text{rej}} \in Q$ is the reject state, where $q_{\text{rej}} \neq q_{\text{acc}}$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\text{L}, \text{R}\}$ is the transition function. Given the current state and symbol being read, the transition function describes the next state, symbol to be written and direction (left or right) in which to move the tape head.

Multi-Tape Turing Machine



- Input on Tape 1
- Initially all heads scanning cell 1, and tapes 2 to k blank
- In one step: Read symbols under each of the k-heads, and depending on the current control state, write new symbols on the tapes, move the each tape head (possibly in different directions), and change state.

Nondeterministic Turing Machine

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- $Q, \Sigma, \Gamma, q_0, q_{\text{acc}}, q_{\text{rej}}$ are as before for 1-tape machine
- $\delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{\text{L}, \text{R}\})$

Turing Machine

Accept and Reject

On input w a TM M may halt (enter q_{acc} or q_{rej}) or M may run forever ("loop"). So M has 3 possible outcomes for each input w:

- Accept w (enter q_{acc})
- Reject w by halting (enter q_{rej})
- Reject w by looping (running forever)

Turing Machine

TM Recognizers and Deciders

Definition

For a Turing machine M, define $L(M) = \{w \mid M \text{ accepts } w\}$. M is said to accept or recognize a language L if $L = L(M)$.

Definition

A Turing machine M is said to decide a language L if $L = L(M)$ and M halts on every input.

Turing Machine

TM Recognizers and Deciders

Definition

L is Turing-recognizable if $L = L(M)$ for some TM M.

Definition

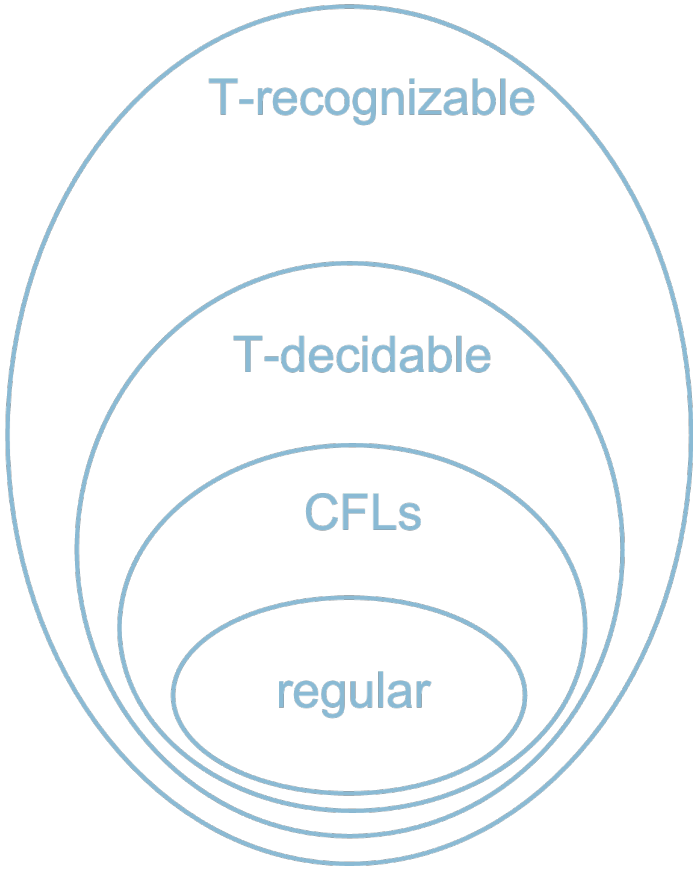
TM M is a decider if M halts on all inputs.

Say that M decides L if $L = L(M)$ and M is decider.

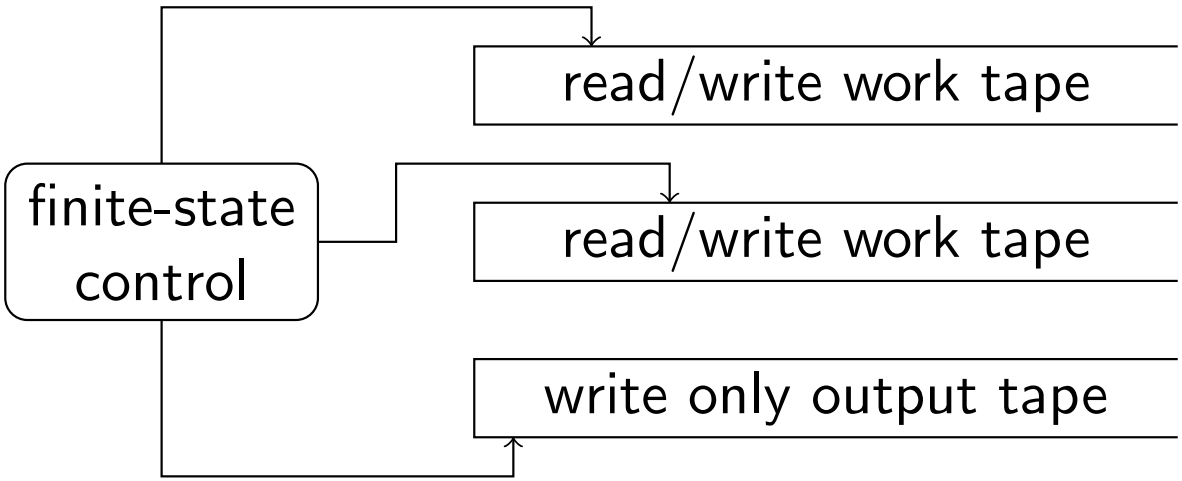
Definition

L is Turing-decidable if TM M is a decider if M halts on all inputs.

Big Picture



Enumerators



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  - Write-only means (a) symbol on output tape does not affect transitions, and (b) tape head only moves right.

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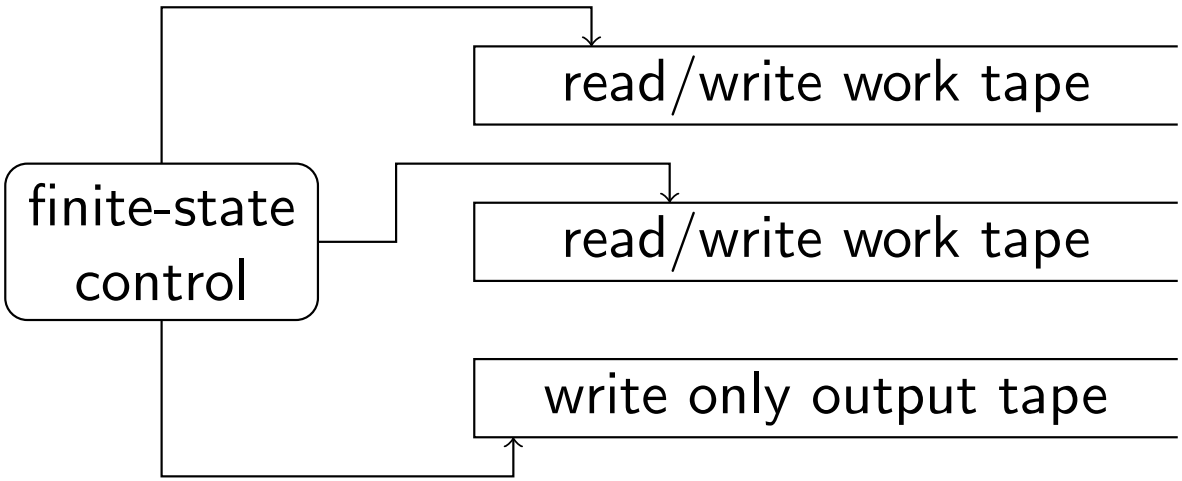

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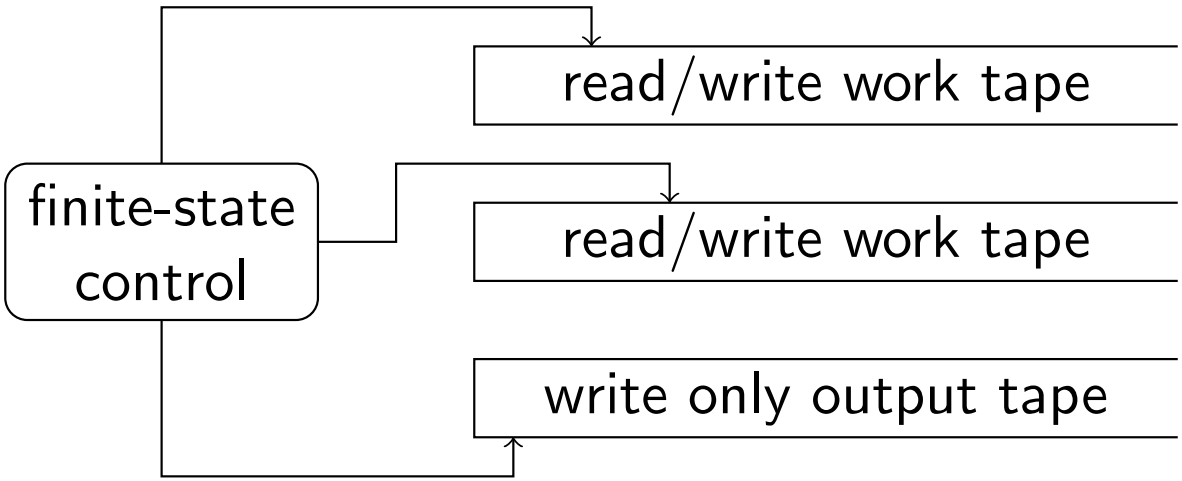

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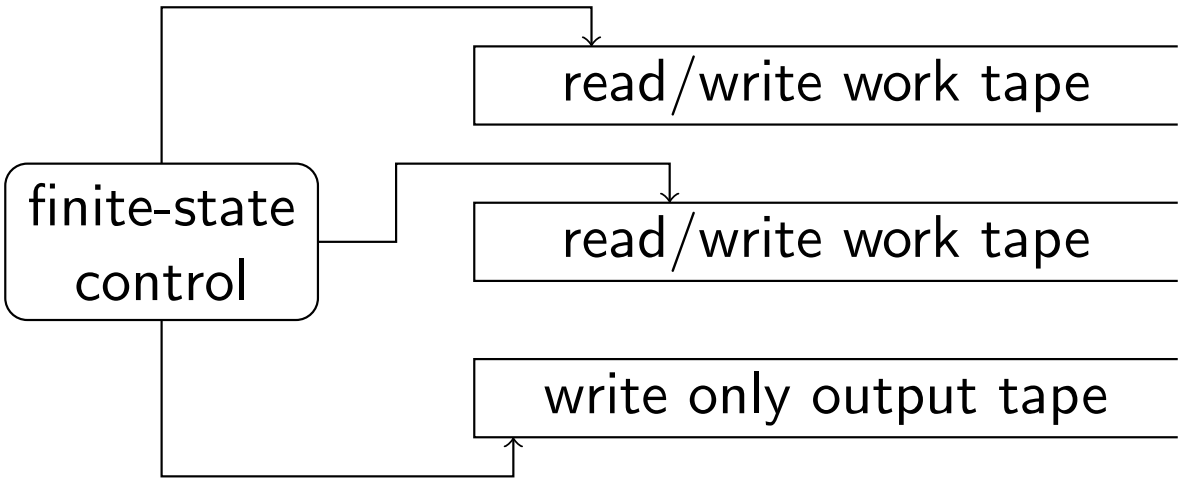

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  - Write-only means (a) symbol on output tape does not affect transitions, and (b) tape head only moves right.
- Intially all tapes blank (no input). During computation the machine adds symbols to the output tape. Output considered to be a list of words (separated by special symbol #)

Recursively Enumerable Languages

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Definition

L is recursively enumerable (r.e.) iff there is an enumerator M such that $L = E(M)$.

Recursively Enumerable Languages and TMs

Theorem

L is recursively enumerable if and only if L is Turing-recognizable.

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Note

Hence, when we say a language L is recursively enumerable (r.e.) then

- there is a TM that accepts L, and
- there is an enumerator that enumerates L.

Recognizers From Enumerators

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Clearly, if $w \in L$, M accepts w, and if $w \notin L$ then M never halts. $\cdots\rightarrow$

Enumerators From Recognizers

Proof (contd).

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Does N enumerate L?

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Enumerators From Recognizers

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for $i = 1, 2, 3 \ldots$ do
    let $w_1, w_2, \ldots w_i$ be the first i strings (in lexicographic order)
    simulate M on w_1 for i steps, then on w_2 for i steps and ...simulate M on w_i for i steps
    if M accepts w_j within i steps then write w_j (with separator) on output tape

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- Enhanced Turing Machine models: TM with 2-way infinite tape, multi-tape TM, nondeterministic TM, probabilistic Turing Machines, quantum Turing Machines ...
- Restricted Turing Machine models: queue machines, 2-stack machines, 2-counter machines, ...

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"Intuitive notion of Algorithms = Turing Machine."

- Connection between the informal notion of algorithm and the precise definition
- Definition of algorithm necessary to resolve Hilbert's tenth problem

Hibert's 10th problem

In 1900 David Hilbert posed 23 problems:

- Give an algorithm for solving Diophantine equations.

Diophantine equations:

Equations of polynomials where solutions must be integers.
Example:

$$3x^2 - 2xy - y^2z = 7 \tag{1}$$

Solution: $x = 1, y = 2, z = -2$

Let
$D = \{p | \text{polynomial} p(x_1, x_2, \ldots, x_k) = 0$ has a solution in integers$\}$
Hilbert's 10th problem: Give an algorithm to decide D.
Matiyasevich proved in 1970: D is not decidable.
Note: D is T-recognizable.

Notation for encodings and TMs

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- If O is some object (e.g., polynomial, automaton, graph, etc.), we write $\langle O \rangle$ to be an encoding of that object into a string.
- If $O_1, O_2, \ldots, O_k$ is a list of objects then we write $\langle O_1, O_2, \ldots, O\_k \rangle$ to be an encoding of them together into a single string.

Notation for writing Turing machines:
We will use high-level English descriptions of algorithms when we describe TMs, knowing that we could (in principle) convert those descriptions into states, transition function, etc. Our notation for writing a TM M is

$$M = \text{“On input } w \tag{2}$$

$$\text{[English description of the algorithm]”} \tag{3}$$

TM - example revisited

TM M recognizing $B = \{a^k b^k c^k | k \geq 0\}$

$M =$ "On input w (4)

1. Check if $w \in a^* b^* c^*$, reject if not (5)
2. Count the number of a, b, c in w (6)
3. Accept if all counts are equal; reject if not. (7)

High-level description is ok. You do not need to manage tapes, states, etc ...