SOFT SYSTEMATIC RESAMPLING FOR ACCURATE POSTERIOR APPROXIMATION AND INCREASED INFORMATION RETENTION IN PARTICLE FILTERING

Praveen B. Choppala, Marcus R. Frean, Paul D. Teal

School of Engineering and Computer Science, Victoria University of Wellington
Email: {praveen, marcus, pault}@ecs.vuw.ac.nz

ABSTRACT

The resampling step in the particle filter results in the loss of information contained in the lower weight particles. While the stochastic resamplers cause further loss by reweighing all particles to the same number, the deterministic resamplers impede the accumulation of information contained in the weights that are increasing over time. The soft resampler overcomes this information loss problem by redistributing the discarded weight among the lower weights. However, the technique does not accurately approximate the posterior because it always discards the lower weight particles located near tails of the cumulative distribution of the posterior density. In this paper, we rectify this problem by proposing a method to stochastically resample these discarded particles.

***Index Terms*—** Particle filter, soft resampling, information retention, accurate posterior approximation

1. INTRODUCTION

For Bayesian estimation in linear Gaussian systems, the Kalman filter [1] provides an optimal estimate of the state of the target and the uncertainty associated with the estimate. For non-linear and non-Gaussian filtering, a closed form solution cannot be derived due to the high dimensional state space limitation. This limitation is overcome using approximate solutions. An efficient Bayesian approximation technique is the particle filter (PF) [2] which approximates the posterior pdf of the target state by a set of weighted particles. The PF operates in two stages – the first, *sequential importance sampling (SIS)* specifies the process of predicting new particles at each time sample and updating their weights. SIS by itself, encounters *degeneracy* [3], a problem in which, after a few iterations, all but one particle have negligible weights. This is overcome in the second stage, the *resampling*, that aims to reduce the variance among the weights [4]. The interest of this paper is in resampling, the purpose of which is to generate a resampled particle set that approximates the posterior more accurately.

Among the *stochastic resamplers* that are widely used in PFs, multinomial resampling [5] operates by first evaluating the cumulative sum of the normalised particle weights and then finding a value of the sum greater than a random sample drawn from $\mathcal{U}(0,1]$. Apart from being computationally expensive, this technique could easily discard particles that are gradually gaining weight [6]. The stratified [7], the residual [8] and the systematic [3, 9] resamplers overcome this problem (with minor differences in their operation) by searching along the cumulative sum after pre-partitioning the space $(0,1]$ into I linearly increasing disjoint sets, where I represents the total number of particles. However, stochastically discarding particles could result in the loss of information contained in the particles that are gradually gaining weight. This could lead to loss of track if the target makes a sharp maneuver towards a low weight particle. Moreover, the stochastic resamplers reset all the weights to the same number. The implication that every resampled particle is equally valid in representing the posterior pdf of the target state is only asymptotically true. A comparison of the stochastic resamplers can be found in [4, 10]. Among the *deterministic resamplers*, partial deterministic resampling [11] proposes to categorise the weights as dominant, moderate and negligible, and while the moderate weights are unchanged, the dominant and the negligible weights are resampled using stochastic resampling. However, the reweighing is now based on the size of the sets but not on the individual weights. Non-proportional reweighing causes loss of potential information contained in the moderate and the negligible particles. The performance of these resamplers in the context of giving importance to lower weights can be seen in [6].

Retaining the information contained in the particles with lower weights is crucial in the detection of very covert targets [12] and in the tracking of highly maneuvering targets, especially when the number of particles is limited. Soft resampling [6] aids in this retention. Here, the particles having weights greater than a threshold are replicated in accordance with their weights and the weights are divided by the number of replications. Once the particle limit I is reached, the redundant particles and the weights are discarded. The information contained in the weights is retained by redistributing the discarded weight among the lower weight particles. However the redundant particles that are located near the tails of the cumulative distribution function (cdf) of the posterior density are discarded all the time. These particles are usually of low weight and lie near the boundary of the particle set. Discarding these particles could prove detrimental especially when the weights of the particles close to the target is very large, see for example [13]. These large weight particles are replicated more times causing many lower weight particles that lie near the edges of the particle set to be eliminated. As a result, the PF approximation is truncated near the tails of the cdf thereby leading to an inaccurate posterior pdf. This problem is less evident in the stochastic resamplers because they resample all the particles with non-zero probability thus ensuring that not every low weight particle is lost all the time.

This paper is an extension and correction to the soft resampling technique and presents a scheme to overcome the problem of discarding the lower weight particles all the time. The main idea of the paper is to first use a modified variant of soft resampling without discarding any particle and then stochastically resample the lower weights using the systematic resampler. Hence, we call our technique "*soft systematic resampling*". The merits of this proposal are that the PF approximation is more accurate and the particles retain more information.

2. PARTICLE FILTERING

In this section, we introduce the notation and briefly describe the PF operation. Target inference is achieved by estimating the state of a target x_k at time k using the noisy sensor data $z_{1:k}$ received until the kth time step. The sequential Bayes' filter facilitates this estimation by using the previous posterior pdf $p(\mathrm{x}_{k-1}|z_{1:k-1})$ to construct $p(\mathrm{x}_k|z_{1:k})$. In the PF approximation, $p(\mathrm{x}_{k-1}|z_{1:k-1})$ is represented by a set of weighted particles as $\{\mathrm{x}_{k-1}^i, w_{k-1}^i\}_{i=1}^I$, where i is the particle index and I is the total number of particles. To obtain the approximation to $p(\mathrm{x}_k|z_{1:k})$, the PF conducts particle prediction by drawing a new set of particles from an importance distribution and updates the weights of the predicted particles. The formulation of these processes can be found in [2, 3].

After a few iterations, the discrepancy between the weights increases, causing degeneracy problems. The solution is to eliminate the particles that have negligible weights and replace them by copies of other particles that have larger weights, i.e., for $i = 1, ..., I$, we sample an index $j(i)$ distributed according to the probability $\Pr\{j(i) = m\} = \{w_k^m\}_{m=1}^I$ and replace $\mathrm{x}_k^i = \mathrm{x}_k^{j(i)}$ and set $w_k^i = 1/I$.

3. SOFT SYSTEMATIC RESAMPLING

In this section, we will first describe the soft resampling procedure and then present our proposed soft systematic resampling scheme.

3.1. Soft resampling

In the soft resampling stage [6], after the SIS, all the I weights are normalised and sorted in descending order. If the weight of a particle x_k^i is greater than $2/I$, the number of replications of x_k^i is

$$C_{\mathrm{x}_k^i} = \max\{1, \lfloor I w_k^i \rfloor\} \tag{1}$$

Based on this rule, the new resampled particles are

$$\{\hat{\mathrm{x}}_k^j\}_{j=\ell}^{C_{\mathrm{x}_k^1}+...+C_{\mathrm{x}_k^i}} = \mathrm{x}_k^i,\ i = 1, ..., I \tag{2}$$

and their new weights are recalculated as

$$\{\hat{w}_k^j\}_{j=\ell}^{C_{\mathrm{x}_k^1}+...+C_{\mathrm{x}_k^i}} = \frac{w_k^i}{C_{\mathrm{x}_k^i}},\ i = 1, ..., I \tag{3}$$

where

$$\ell = \begin{cases} 1 & \text{if } i = 1 \\ C_{\mathrm{x}_k^1} + ... + C_{\mathrm{x}_k^{i-1}} + 1 & \text{otherwise} \end{cases}$$

The number of reweighted particles after this process is $K = \sum_{i=1}^I C_{\mathrm{x}_k^i}$ and usually $K > I$. Since we need only I particles, all the particles after the Ith particle are discarded all the time. The sum weights is preserved by either *renormalising* the first I weights or *redistributing* the discarded weights among the remaining lower weights. The final weighted particle set is then the subset $\{\mathrm{x}_k^i, w_k^i\}_{i=1}^I$ of the set $\{\hat{\mathrm{x}}_k^i, \hat{w}_k^i\}_{i=1}^K$.

3.2. Soft systematic resampling

The soft resampler eliminates the particles that are usually located near the tails of the cdf of the posterior. This results in inaccuracies in approximating the posterior of the target state. In this paper, we propose to overcome this problem by using a systematic resampler [3] that randomly resamples a few lower weight particles so that not every low weight particle is eliminated all the time. The consequential soft systematic resampling in presented here.

After the SIS step, the weights are normalised and sorted in the descending order. For a particle x_k^i, if its weight $w_k^i > 2/I$, x_k^i is replicated

$$C_{\mathrm{x}_k^i} = \max\{1, \lfloor \alpha I w_k^i \rfloor\} \tag{4}$$

times, where $0 < \alpha \leq 1$. Low values for α corresponds to less replications of the larger weights. α can be used to regulate the spread of the large weights, particularly when the likelihood of the particle closest to the target is very large [13]. Based on this principle we obtain the resampled-reweighted particle set $\{\hat{\mathrm{x}}_k^i, \hat{w}_k^i\}_{i=1}^K$ using (2) and (3), and $K = \sum_{i=1}^I C_{\mathrm{x}_k^i}$. Note that the combined weight of the resampled particles for a particular index i is the same before and after resampling, and is independent of α. We then choose N_{low} lower weight particles to stochastically resample, where

$$N_{\text{low}} = \min\Big\{K, \lfloor \beta(K - I) \rceil\Big\} \tag{5}$$

where $\beta \geq 1$ regulates the number of particles to be stochastically resampled. The weighted particle set $\{\hat{\mathrm{x}}_k^i, \hat{w}_k^i\}_{i=K-N_{\text{low}}+1}^K$ is then fed to the systematic resampler after normalising its weights. The number of particles the resampler generates is limited to

$$N_{\text{resamp}} = N_{\text{low}} - (K - I) \tag{6}$$

These N_{resamp} particles will then replace the particles $\{\hat{\mathrm{x}}_k^i\}_{i=K-N_{\text{low}}+1}^I$. This process ensures that all the $K - I$ particles are not eliminated all the time. The resampled particles are then reweighted as

$$\hat{w}_k^i = \frac{\sum_{j=K-N_{\text{low}}+1}^K \hat{w}_k^j}{N_{\text{resamp}}},\ i = K - N_{\text{low}} + 1, ..., I \tag{7}$$

The remaining $K - I$ weighted particles are then discarded. As in section 3.1, the final weighted particle set is then the subset $\{\mathrm{x}_k^i, w_k^i\}_{i=1}^I$ of the set $\{\hat{\mathrm{x}}_k^i, \hat{w}_k^i\}_{i=1}^K$.

4. EVALUATION

In this section, we first analyse the performance of our proposed technique for a simple model in terms of its ability to accurately represent the posterior and to retain more information contained in the lower weights. We use a linear Gaussian model so that comparison can be made with the Kalman filter [1] which is known to give the optimal posterior. We also show the potential of our technique to accurately track targets in linear and non-linear sensing scenarios. In our simulations, we found that the performance of the residual and the stratified resamplers is almost identical to that of the systematic resampler. Hence we only show the results for the latter. Also, the partial deterministic and the soft resamplers exhibit similar performance and hence we only show the latter.

For testing under linear Gaussian scenarios, we use the 2D model in which the state x_k is characterised by the position and velocity of the target. The process model is

$$\mathrm{x}_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathrm{x}_{k-1} + q_{k-1} \tag{8}$$

with $T = 1$ second, the process noise $q_{k-1} \sim \mathcal{N}(0, Q)$ where

$$Q = \begin{bmatrix} 0.95 & 0.2 \\ 0.2 & 0.75 \end{bmatrix}$$

The observation model is $z_k = \mathrm{x}_k + r_k$ where $r_k \sim \mathcal{N}(0, R)$ with R having the variance $\sigma^2 = 0.5$ along each dimension.

Firstly, we test the faithfulness of our technique in representing the posterior in accordance to its Kolomogorov-Smirnov (KS) statistic [14] agreement with the optimal Kalman filter. KS testing has been used previously for other PF problems [15] and has the advantage of providing a reliable measure of the accuracy of the estimate of the posterior. This test has been under-utilised in the literature so far. For KS testing in multi-dimensional Gaussian models [16], we use a different approach. After resampling, we de-correlate and de-mean the particles using the Kalman mean and covariance which will result in a set of uncorrelated particles with zero mean and unit variance. We then conduct the 1D KS test along each dimension separately and take the maximum KS deviation over our dimensions. Fig. 1 shows the KS statistic of various filters for varying numbers of particles. It can be observed that with increasing β our filter exhibits better agreement with the theoretically optimal posterior distribution. Fig. 2 shows the cdfs of the various PFs at a single time sample when the target is making a sharp maneuver. It can be observed that while the soft resampler in Fig. 2(d) does not properly treat the tails of the cdf, the soft systematic resampler in Fig. 2(b) overcomes this problem by stochastically resampling the particles located near the tails of the cdf. For increasing β, the performance of the soft resampler will approach that of the systematic resampler. From Fig. 1 and Fig. 2, we can see that the soft systematic resampler can represent the posterior more accurately than the soft resampler.

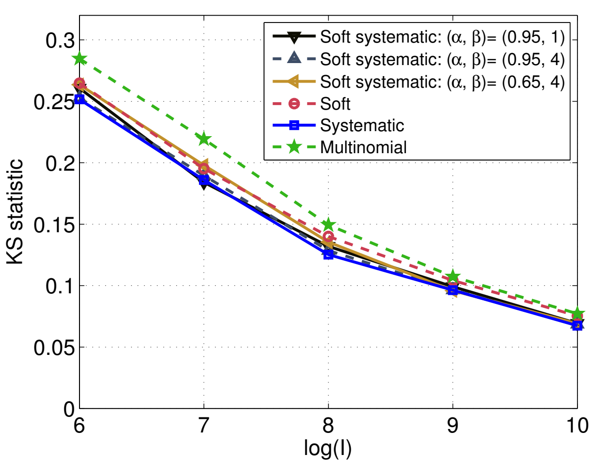
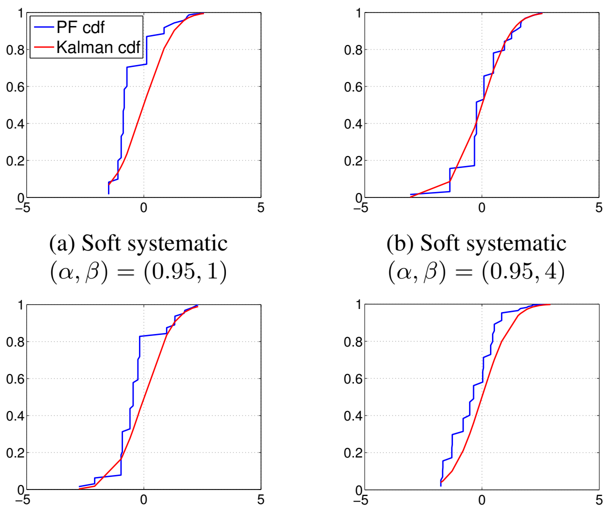


Fig. 1. KS statistic versus the logarithm of the number of the particles. The results are averaged over 20 instances, each having 50 time steps.

Secondly, we analyse our technique in terms of its ability to retain more information over successive time steps and thereby aid in a faster lock. For this test, we initialised the particles with unit covariance and a mean that is 8m away from the actual target. This arrangement results in a particle set that is just near the target and requires retention of the weight information for a faster lock. Fig. 3 illustrates how fast the target is detected by showing the drop in the rmse of the PF estimates for 10 time samples. It can be inferred that by appropriate reweighing and retaining more information contained in the lower weights, our soft systematic resampler can lock onto the targets faster than the systematic resampler that regards all the particles as equally weighted.



(a) Soft systematic $(\alpha, \beta) = (0.95, 1)$

(b) Soft systematic $(\alpha, \beta) = (0.95, 4)$

(c) Systematic.

(d) Soft.

Fig. 2. Cumulative distribution functions for PFs operated using various resamplers. These cdfs correspond to the position of a target. The legend of Fig. 2(a) applies to Fig. 2(b,c,d) also.

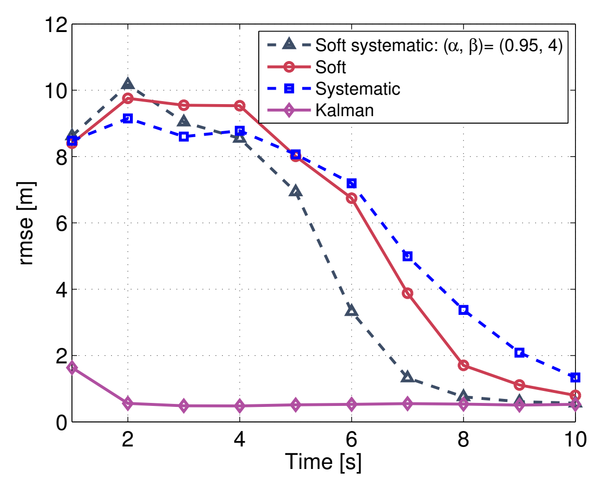


Fig. 3. rmse versus time samples. $I = 1024$. The results are averaged over 100 instances, each having 10 time steps.

Thirdly, as in [6], we test the ability of the soft systematic resampler to detect suddenly appearing covert targets. For this we use the track-before-detect PF technique using the non-linear image sensing model described in Ch.11 of [17]. The target appears in the 6th frame and maneuvers for 20 frames thereafter. Fig. 4 shows the number of frames taken by the filters to lock onto the target for varying numbers of particles. We observed that by using large values of β, the looseness of the particle cloud decreases. As a result, our filter (at $\beta = 4$ and low I) takes longer time to lock the target. However, with sufficient particles, the filter takes advantage of the reweighing scheme that aids in the accumulation of information over successive frames and hence outperforms the soft resampler.

Finally, we test the accuracy of our technique for the 2D linear Gaussian and the image sensing models for varying σ^2. The target is present all the time. The result is shown in Fig. 5 and it can be observed that the systematic resampler performs slightly better than the soft resampler and by increasing β, the performance of our soft systematic resampler approaches that of the systematic resampler. In both cases, our technique tracks the targets with high accuracy. Moreover, in Fig. 5(a), the error curves of the proposed technique are very close to that of the theoretically optimal Kalman filter.

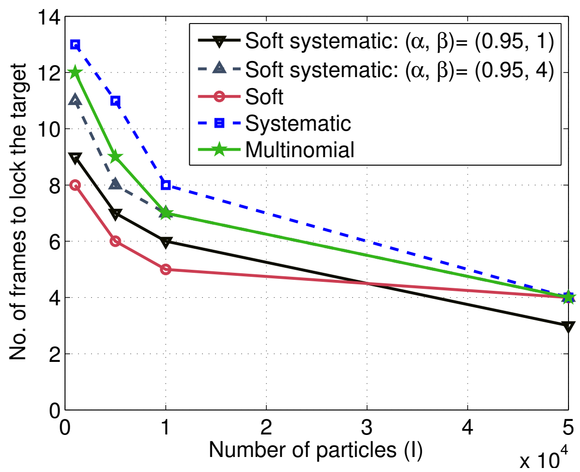
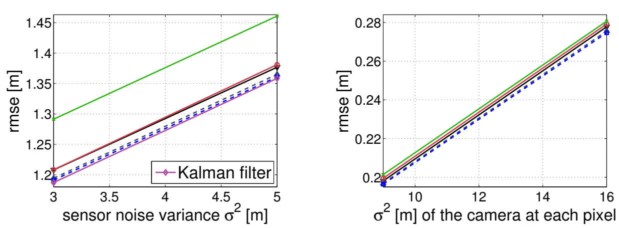


Fig. 4. Number of frames taken from the appearance of a target to lock onto a target (i.e for the rmse to decrease to below 0.1) versus number of particles. At each pixel, $\sigma^2 = 6.25$. The results are averaged over 20 instances, each having 30 time steps.



(a) 2D linear Gaussian model. (b) Image sensing model.

Fig. 5. rmse versus the sensor noise variance. $I = 1024$. The legend for the filters other than the Kalman filter is shown in Fig. 4. The results are averaged over 20 instances, each having 50 time steps (for the linear Gaussian model) and 30 time steps (for the image model).

5. CONCLUSION

In this paper, we extend the idea of soft resampling. The soft resampler always discards the particles located near the tails of the cdf of the posterior thereby resulting in an inaccurate PF approximation. The key innovation in this paper is to resample the lower weight particles using the systematic resampler. We also proposed a reweighing scheme that retains the information contained in the weights and aids in information accumulation over time. The main contribution of the scheme is a more accurate posterior approximation along with more information retention and accumulation over successive observations. Through simulations and the Kolomogorov-Smirnov statistic, we demonstrated the advantage of our proposal in representing the posterior more accurately than the soft resampler while retaining more weight information which then results in a faster lock. We also showed that our technique aids in accurate tracking in linear and non-linear scenarios.

6. REFERENCES

[1] R.E. Kalman, “A new approach to linear filtering and prediction problems,” *J. Basic Engineering*, vol. 82, no. 1, pp. 35–45, 1960.

[2] N.J. Gordon, D.J. Salmond, and A.F.M. Smith, “Novel approach to non-linear/non-Gaussian Bayesian state estimation,” in *Proc. IEE Radar and Signal Process.*, 1993, vol. 140, pp. 107–113.

[3] S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, “A tutorial on particle filters for online nonlinear/non-gaussian Bayesian tracking,” *IEEE Trans. Signal Process.*, vol. 50, no. 2, pp. 174–188, 2002.

[4] R. Douc and O. Cappé, “Comparison of resampling schemes for particle filtering,” in *Proceedings ISPA 4th International Symposium on Image and Signal Processing and Analysis*, 2005, pp. 64–69.

[5] B. Efron and R.J. Tibshirani, “An introduction to the bootstrap,” *Chapman and Hall*, 1994.

[6] P.B. Choppala, P.D. Teal, and M.R. Frean, “Soft resampling for improved information retention in particle filtering,” in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, 2013, vol. 13, pp. 4036–4040.

[7] G. Kitagawa, “Monte Carlo filter and smoother for non-Gaussian nonlinear state space models,” *J. of Computational and Graphical Statistics*, pp. 1–25, 1996.

[8] J. Liu and R. Chen, “Sequential Monte Carlo Methods for Dynamic Systems,” *Journal of the American Statistical Association*, vol. 93, pp. 1032–1044, 1998.

[9] J. Carpenter, P. Clifford, and P. Fearnhead, “An improved particle filter for nonlinear problems,” *IEE Proc. Radar, Sonar and Navigation*, vol. 146, pp. 2–7, 1999.

[10] J.D. Hol, T.B. Schön, and F. Gustafsson, “On Resampling Algorithms for Particle Filters,” in *Proc. Nonlinear Stat. Signal Process. Workshop*, 2006.

[11] M. Bolić, P.M. Djurić, and S. Hong, “Resampling algorithms and architectures for distributed particle filters,” *IEEE Trans. Signal Process.*, vol. 53, no. 7, pp. 2442–2450, 2005.

[12] D.J. Salmond and H. Birch, “A particle filter for track-before-detect,” in *Proc. IEEE American Control Conf.*, 2001, vol. 5, pp. 3755–3760.

[13] X. Zhong, A.B. Premkumar, and A.S. Madhukumar, “Particle filtering for acoustic source tracking in Impulsive noise with Alpha-stable process,” *IEEE J. Sensors*, vol. 13, no. 2, pp. 589–600, 2013.

[14] J. A. Peacock, “Two-dimensional goodness-of-fit testing in astronomy,” *Royal Astronomical Society*, vol. 202, pp. 615–627, 1983.

[15] P.M. Djurić and J. Míguez, “Assessment of Nonlinear Dynamic Models by Kolmogorov–Smirnov Statistics,” *IEEE Trans. Signal Process.*, vol. 58, no. 10, pp. 5069–5079, 2010.

[16] G. Fasano and A. Franceschini, “A multidimensional version of the Kolmogorov-Smirnov test,” *Royal Astronomical Society*, vol. 225, pp. 155–170, 1987.

[17] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*, Artech House Publishers, 2004.