Probabilistic Stability of Dropout Outputs

Problem Statement

A neuron's output during training is modeled as a random variable Z due to dropout. The neuron produces an output Y with probability $q = 1 - p$ (active state) or 0 with probability $p = 0.2$ (inactive state). Assume the active output Y is a continuous random variable with mean μ and variance σ^2. Derive the probability that the neuron's stochastic output Z deviates from its expected value $\mathbb{E}[Z]$ by more than a threshold $\epsilon > 0$.

Solution

1. Define Z:

The stochastic output Z is defined as:

$$Z = \begin{cases} Y, & \text{with probability } q = 1 - p, \\ 0, & \text{with probability } p. \end{cases}$$

2. Expected Value of Z:

The expected value $\mathbb{E}[Z]$ is:

$$\mathbb{E}[Z] = \mathbb{E}[Y] \cdot q + 0 \cdot p = q\mu,$$

where $\mu = \mathbb{E}[Y]$ is the mean of the active neuron's output.

3. Variance of Z:

The variance $\text{Var}(Z)$ is computed as:

$$\text{Var}(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2.$$

First, calculate $\mathbb{E}[Z^2]$:

$$\mathbb{E}[Z^2] = \mathbb{E}[Y^2] \cdot q + 0^2 \cdot p = q\mathbb{E}[Y^2.$$

Using the identity $\mathbb{E}[Y^2] = \sigma^2 + \mu^2$:

$$\mathbb{E}[Z^2] = q(\sigma^2 + \mu^2).$$

Now, substitute into $\mathrm{Var}(Z)$:

$$\mathrm{Var}(Z) = q(\sigma^2 + \mu^2) - (q\mu)^2.$$

Simplify:

$$\mathrm{Var}(Z) = q\sigma^2 + q\mu^2 - q^2\mu^2.$$

$$\mathrm{Var}(Z) = q\sigma^2 + \mu^2 q(1 - q).$$

Thus:

$$\mathrm{Var}(Z) = q\sigma^2 + \mu^2 q(1 - q).$$

4. Chebyshev's Inequality:

By Chebyshev's inequality:

$$P(|Z - \mathbb{E}[Z]| \geq \epsilon) \leq \frac{\mathrm{Var}(Z)}{\epsilon^2}.$$

Substitute $\mathbb{E}[Z] = q\mu$ and $\mathrm{Var}(Z) = q\sigma^2 + \mu^2 q(1 - q)$:

$$P(|Z - q\mu| \geq \epsilon) \leq \frac{q\sigma^2 + \mu^2 q(1 - q)}{\epsilon^2}.$$

5. Example:

Suppose:

- Dropout probability $p = 0.2$, so $q = 0.8$,
- The active neuron's output Y has mean $\mu = 2$ and variance $\sigma^2 = 1$,
- Threshold $\epsilon = 0.5$.

The variance $\mathrm{Var}(Z)$ is:

$$\mathrm{Var}(Z) = q\sigma^2 + \mu^2 q(1-q).$$

$$\mathrm{Var}(Z) = 0.8 \cdot 1 + 2^2 \cdot 0.8(1-0.8).$$

$$\mathrm{Var}(Z) = 0.8 + 4 \cdot 0.8 \cdot 0.2 = 0.8 + 0.64 = 1.44.$$

Using Chebyshev's inequality:

$$P(|Z - q\mu| \geq 0.5) \leq \frac{1.44}{0.5^2} = \frac{1.44}{0.25} = 5.76.$$

Since probabilities cannot exceed 1, this result shows that the probability of deviation is low in practice, but Chebyshev provides a theoretical upper bound.

Conclusion

This derivation illustrates the probabilistic stability of a neuron's output under dropout. By incorporating general random variables for active states, it applies broadly to real-world scenarios, such as neural networks trained with dropout.