Problem 1

a) If a system is in CCF, then it can be written in the following form:

$x^+ = Gx + Hu$, where:

$$G = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}; \quad H = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$ is the system's characteristic polynomial

n is the # of state variables

A system is considered controllable if rank([H GH ⋯ G^{k-1}H]) $\underbrace{\text{equals } n}$

if there exists n linearly independent columns

$$H = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad GH = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -a_1 \end{bmatrix}$$

First Column

one non-zero term

Second Column

two non-zero terms

$$G^2H = G(GH) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -a_1 \end{bmatrix} = \begin{array}{c} \text{Third Column} \\ \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -a_1 \\ -a_2 + a_1^2 \end{bmatrix} \end{array} \left.\vphantom{\begin{matrix} 1 \\ -a_1 \\ -a_2 + a_1^2 \end{matrix}}\right\} \text{Three non-zero terms}$$

$$G^3H = G(G^2H) = \begin{array}{c} \text{Fourth Column} \\ \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -a_1 \\ -a_2 + a_1^2 \\ -a_3 + 2a_1a_2 - a_1^3 \end{bmatrix} \end{array} \left.\vphantom{\begin{matrix} 1 \\ -a_1 \\ -a_2 + a_1^2 \\ -a_3 \end{matrix}}\right\} \text{Four non-zero terms}$$

The four calculated columns are all linearly independent from one another because each column adds an independent dimension to the column space of the $[H \ GH \ \cdots \ G^{k-1}H]$ matrix that the preceding columns did NOT add. Hence, it can be deduced that after n columns, the rank of the matrix will be equal to n. Therefore, a system expressed in controllable canonical form will always be controllable.

b) If we have a system whose transfer function is as follows:

$$\frac{Y(z)}{U(z)} = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n}$$

And if it's expressed in OCF, then the system is as follows:

$$\begin{cases} x^+ = Gx + Hu \\ y = Cx \end{cases}$$

where $G = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix}$; $H = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Additionally, a system is considered observable iff rank$(N) = n$, where:

$$N = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{k-1} \end{bmatrix}$$

$$CG = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \end{bmatrix} \rightarrow 2^{nd} \text{ Row}$$

$$CG^2 = (CG)G = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (a_1^2 - a_2) & -a_1 & 1 & 0 & \cdots & 0 \end{bmatrix} \rightarrow 3^{rd} \text{ Row}$$

The pattern can be generalized such that after n rows:

leading 1's

$$N = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{1,1} & 1 & 0 & \cdots & 0 \\ \alpha_{2,1} & \alpha_{1,2} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha_{n-1,1} & \alpha_{n-2,2} & \cdots & \alpha_{1,n-1} & 1 \end{bmatrix}$$

where $\alpha_{i,j}$ are just some polynomial functions of the constants $a_1, a_2, \ldots, a_n$

rank(N) equals n because each column vector clearly adds an independent dimension to the column space of N that the succeeding column vectors (the column vectors AFTER the column vector of interest) do NOT add. Hence, it can be stated that any system expressed in OCF is therefore observable.

Problem 2

$x^+ = Gx + Hu$

$y = Cx$

If G is diagonal, then $G = \begin{bmatrix} g_1 & 0 & \cdots & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & g_{n-1} & 0 \\ 0 & 0 & \cdots & \cdots & g_n \end{bmatrix}$

A system is considered controllable if rank$(M) = n$.
A system is considered observable if rank$(N) = n$.

$$M = [H \ \ GH \ \cdots \ G^{k-1}H] \ ; \ N = \begin{bmatrix} C \\ CG \\ \vdots \\ CG^{k-1} \end{bmatrix}$$

Since G is diagonal:

$$G^x = \begin{bmatrix} g_1^x & 0 & \cdots & \cdots & 0 \\ 0 & g_2^x & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & g_{n-1}^x & 0 \\ 0 & 0 & \cdots & \cdots & g_n^x \end{bmatrix}$$

For Controllability:

$$H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$ generalized H matrix

Which leads to:

$$M = \begin{bmatrix} h_1 & h_1 g_1 & h_1 g_1^2 & \cdots & h_1 g_1^n \\ h_2 & h_2 g_2 & h_2 g_2^2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ h_n & h_n g_n & h_n g_n^2 & \cdots & h_n g_n^n \end{bmatrix}$$ After n time steps

The rank of M would equal n if and only if all of the values $h_1, \ldots, h_n$ in the H matrix are nonzero. Otherwise, if at least one zero term exists, then the rows of matrix M corresponding to the rows of the zero terms in the H matrix would be ALL zeros, thereby reducing the rank of the matrix M. In this case, the maximum rank of M would be:

$$\text{rank}(M) = n - (\#\text{ of zero terms in } H)$$

Thereby proving that the system is controllable iff # of zero terms in H equals 0.

For Observability:

$C = [c_1, c_2, \ldots, c_n]$

Which leads to:

$$N = \begin{bmatrix} c_1 g_1 & c_2 g_2 & \cdots & c_n g_n \\ c_1 g_1^2 & c_2 g_2^2 & \cdots & c_n g_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ c_1 g_1^n & c_2 g_2^n & \cdots & c_n g_n^n \end{bmatrix}$$

After n time steps

Similar to our analysis for controllability:

$$\text{rank}(N) = n - (\#\text{ of zero terms in } C)$$

Hence, the system is observable iff the number of zero terms in C equals zero. Otherwise, the columns of N corresponding to the columns of the zero terms in C will be ALL zeros. Since the max number of columns of N is always equal to n, the presence of any columns of zeros would reduce the rank of N, thereby causing the system to NOT be observable.

Problem 3

$x^+ = Gx + Hu$; $G = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.7 \end{bmatrix}$; $H = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$

a) $zI - G = \begin{bmatrix} (z-0.5) & -1 \\ -0.5 & (z-0.7) \end{bmatrix}$

$$|zI - G| = (z-0.5)(z-0.7) - 0.5 = z^2 - 1.2z - 0.15$$
$$= (z - 1.314)(z + 0.114) = 0$$

Poles of System: $p_1 = 1.314$, $p_2 = -0.114$

Although p_2 lies within the unit circle, since $|p_1| > 1$, the system is therefore **NOT STABLE.**

b) $u = -Kx + r$, where $K = [k_1, k_2]$

$$HK = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 0.2k_1 & 0.2k_2 \\ 0.1k_1 & 0.1k_2 \end{bmatrix}$$

$(z-(0.5-0.2k_1))(z-(0.7-0.1k_2)) =$

$$zI - (G - HK) = \begin{bmatrix} z-0.5+0.2k_1 & 0.2k_2 - 1 \\ 0.1k_1 - 0.5 & z-0.7+0.1k_2 \end{bmatrix}$$

$$|zI - (G-HK)| = (z-0.5+0.2k_1)(z-0.7+0.1k_2) - (0.2k_2-1)(0.1k_1-0.5)$$
$$= z^2 + (0.2k_1 + 0.1k_2 - 1.2)z + (-0.5+0.2k_1)(-0.7+0.1k_2)$$
$$-(0.2k_2-1)(0.1k_1-0.5)$$

$$z^2 + (0.2k_1 + 0.1k_2 - 1.2)z + 0.35 - 0.05k_2 - 0.14k_1 \cancel{+ 0.02k_1k_2}$$
$$\cancel{- 0.02k_1k_2} + 0.1k_2 + 0.1k_1 - 0.5$$

$$= z^2 + (0.2k_1 + 0.1k_2 - 1.2)z + (-0.04k_1 + 0.05k_2 - 0.15)$$
$$= (z - (0.4 + 0.3j))(z - (0.4 - 0.3j)) = [(z - 0.4)^2 + 0.3^2]$$
$$= z^2 - 0.8z + 0.16 + 0.09 = \underbrace{z^2 - 0.8z + 0.25}_{\text{Desired Char. Poly.}}$$

Equating the coefficients together yields:

$$0.2k_1 + 0.1k_2 = 0.4 \quad \Rightarrow \quad 2k_1 + k_2 = 4$$
$$-0.04k_1 + 0.05k_2 = 0.4 \qquad -k_1 + 1.25k_2 = 10$$

$$\Rightarrow k_1 = -1.4286, \; k_2 = 6.8571$$

Hence, the designed state-feedback controller is as follows:

$$\boxed{u = [-k_1 \;\; -k_2]x + r \quad \text{where } k_1 = -1.4286, \; k_2 = 6.8571}$$

c) $y_d = 1$, $r = y_d/\text{gain}$, $\text{gain} = C(I-G_{CL})^{-1}H$, $C = [1 \; 0]$

Calculated in MATLAB

$\text{gain} = 0.3556 \Rightarrow r = 1/\text{gain} \Rightarrow \boxed{r = 2.8125}$

d) Integral Control will be used:

$u = -K_s x - K_I v(k) + r$, $e = y_d - y$, $v(k+1) = v(k) - e(k)$

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} G & 0 \\ C & I \end{bmatrix}\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} - \begin{bmatrix} H \\ 0 \end{bmatrix}\begin{bmatrix} K_s & K_I \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}} + \begin{bmatrix} Fd + Hr \\ -y_d \end{bmatrix}$$

$$\underbrace{\left\{\begin{bmatrix} 0.5 & 1 & 0 \\ 0.5 & 0.7 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 0.1 \\ 0 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\right\}}_{G^*}\begin{bmatrix} x_1 \\ x_2 \\ v \end{bmatrix}$$

Eigenvalues of G^* must be equal to $0.4 \pm 0.3j$, 0.5

This corresponds to the following gains:

$[k_1 \; k_2 \; k_3] = [1.6518, 5.6964, 1.4062]$

Hence, the state-feedback controller is as follows:

$u = -K_s x - K_I v(k) + r$ where $K_s = [1.6518, 5.6964]$

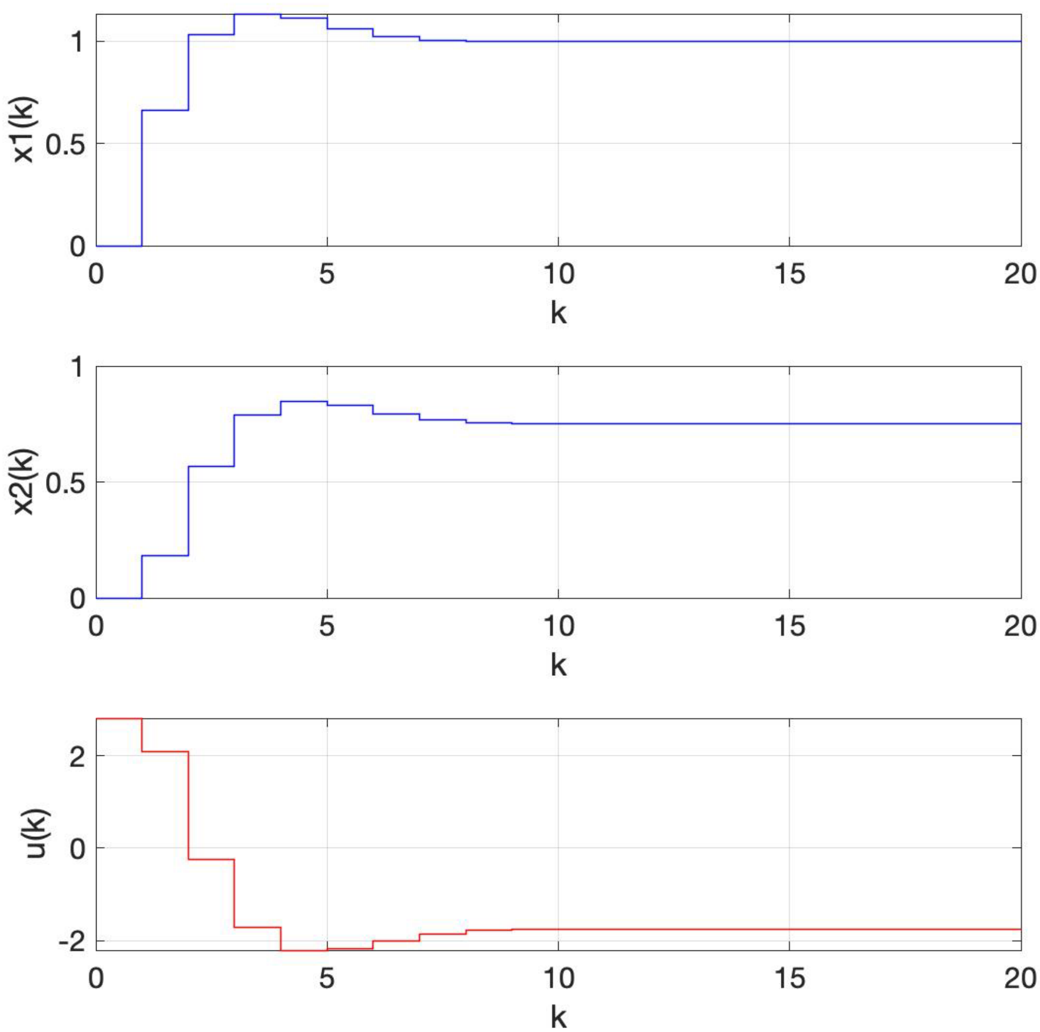
$K_i = 1.4062$

(from part (c)) $\rightarrow r = 2.8125$

(e) $$\begin{bmatrix} x^+ \\ v^+ \end{bmatrix} = G^* \begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{Fd}{yd} + \frac{H}{gain} \\ -1 \end{bmatrix}}_{H^*} yd$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C^*} \begin{bmatrix} x_1 \\ x_2 \\ v \end{bmatrix} \quad ; \quad D^* = 0$$

Simulating the responses for $yd = d = 1$:



MATLAB Code for Problem 3:

```matlab
% Defining State Matrices:
G = [0.5 1; 0.5 0.7];
H = [0.2; 0.1];
C = [1 0];
yd = 1; % Desired output

%%% Part c:
p_des = [(0.4 + 0.3i), (0.4 - 0.3i)]; % Desired poles
K = place(G, H, p_des); % Calculating state feedback controller gains
G_cl = (G - (H * K));
gain = C * inv(eye(2) - G_cl) * H; % Calculating the reference input gain
r = 1./gain; % Calculating the reference input r for tracking control

%%% Part d:
G_mod = [G [0; 0]; C 1]; % Modified G matrix
H_mod = [H; 0]; % Modified H matrix
p_des = [(0.4 + 0.3i), (0.4 - 0.3i), 0.5]; % Desired poles
K_int = place(G_mod, H_mod, p_des); % Calculating state feedback controller
gains

%%% Part e:
d = 1; % Disturbance
F = [0.1; -0.1]; % Disturbance matrix

% Recalculating the state matrices:
G_st = G_mod - (H_mod * K_int);
H_st = [(((d./yd) * F) + (H./gain)); -1];
C_st = [1 0 0];
D_st = 0;

% Linear Simulation:
sys_int = ss(G_st, H_st, C_st, D_st, 1); % Creating a discrete state-space
model with T = 1 second
t_mod = [0:20]; % Time steps
u_mod = ones(1, length(t_mod)); % Desired constant output y_d (y_d = 1 in
this case)

[output, tOut, states] = lsim(sys_int, u_mod, t_mod); % Performing
discretized linear simulation
states = states';
x1_k = states(1, :); % x1(k)
x2_k = states(2, :); % x2(k)
u_k = (-K_int * states) + r; % Calculating u(k)
```

```matlab
% Plotting Results:
figure
subplot(3, 1, 1)
stairs(tOut, x1_k, 'b', 'LineWidth', 0.75) % Plotting x1(k) with ZOH
xlabel('k')
ylabel('x1(k)')
set(gca, 'FontSize', 15)
grid on

subplot(3, 1, 2)
stairs(tOut, x2_k, 'b', 'LineWidth', 0.75) % Plotting x2(k) with ZOH
xlabel('k')
ylabel('x2(k)')
set(gca, 'FontSize', 15)
grid on

subplot(3, 1, 3)
stairs(tOut, u_k, 'r', 'LineWidth', 0.75) % Plotting u(k) with ZOH
xlabel('k')
ylabel('u(k)')
set(gca, 'FontSize', 15)
grid on
```

Problem 4

a) System is in OCF $\Rightarrow$ system is observable

$$\hat{x}^{+} = G\hat{x} + Hu + L(y - C\hat{x}) = (G-LC)\hat{x} + Hu + Ly$$
$$= G_o\hat{x} + Hu + Ly$$

$\Rightarrow G_o = G - LC$

$L = [l_1 \ \ l_2 \ \ l_3]^T$

Observer Poles (eigenvalues of G_o) must be within the unit circle:

$$G = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix} ; \ LC = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} l_1 & 0 & 0 \\ l_2 & 0 & 0 \\ l_3 & 0 & 0 \end{bmatrix}$$

$$G - LC = \begin{bmatrix} (a_1 - l_1) & 1 & 0 \\ (a_2 - l_2) & 0 & 1 \\ (a_3 - l_3) & 0 & 0 \end{bmatrix} = G_o$$

$$zI - (G - LC) = \begin{bmatrix} z + l_1 - a_1 & -1 & 0 \\ l_2 - a_2 & z & -1 \\ l_3 - a_3 & 0 & z \end{bmatrix}$$

$$\det(zI - G + LC) = (z + l_1 - a_1)z^2 + z(l_2 - a_2) + (l_3 - a_3) = 0$$

(Actual Char Eqn.) $= z^3 + (l_1 - a_1)z^2 + (l_2 - a_2)z + (l_3 - a_3) = 0$

Choosing the observer poles to be $p_{1,2,3} = 0$: $z^3 = 0$

(Since Full-Order Observer is deadbeat)

(Desired Char Eqn.)

Comparing Coefficients:

$l_1 - a_1 = 0 \Rightarrow \underline{l_1 = a_1}$

$l_2 - a_2 = 0 \Rightarrow \underline{l_2 = a_2}$

$l_3 - a_3 = 0 \Rightarrow \underline{l_3 = a_3}$

Hence, $\hat{x}^+ = G_o \hat{x} + Hu - Ly$ where:

$$G_o = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad L = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

b) Since $C = [1\ 0\ 0]$, $y = x_1$. Therefore, x_2 and x_3 are the unmeasurable states:

G_{aa} G_{ab} H_a x_a

$$G = \left[\begin{array}{c|cc} a_1 & 1 & 0 \\ \hline a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{array}\right] ; \quad H = \left[\begin{array}{c} b_1 \\ \hline b_2 \\ b_3 \end{array}\right] ; \quad x = \left[\begin{array}{c} x_1 \\ \hline x_2 \\ x_3 \end{array}\right]$$

G_{ba} G_{bb} H_b x_b

$$\hat{z}^+ = G_r z + H_u u + H_y y$$

$$\hat{x}_b = z + Ly$$

$$G_r = G_{bb} - L G_{ab} ; \quad H_u = H_b - L H_a$$

$$H_y = G_r L + G_{ba} - L G_{aa}$$

$$L = [\ell_1\ \ell_2]^T$$

Poles of G_r must be within the unit circle:

$$G_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} [1\ 0] = \begin{bmatrix} -\ell_1 & 1 \\ -\ell_2 & 0 \end{bmatrix}$$

$$zI - G_r = \begin{bmatrix} z+\ell_1 & -1 \\ \ell_2 & z \end{bmatrix}$$

$$\det(zI - G_r) = z^2 + l_1 z + l_2 = 0$$

(Actual Char Eqn.)

Desired Observer Poles : $p_{1,2} = 0 \Rightarrow z^2 = 0$

(Since ROO is deadbeat) (Desired Char Eqn.)

Comparing Coefficients:

$$l_1 = l_2 = 0 \Rightarrow L = [0 \;\; 0]^T$$

$$G_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \; H_u = H_b - \cancel{L H_a}^{\,0} = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

$$H_y = \cancel{G_r L}^{\,0} + G_{ba} - \cancel{L G_{aa}}_{\,0} = \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}$$

$$\boxed{\begin{gathered}\text{Hence, the reduced order observer is as follows:}\\ \hat{z}^+ = G_r z + H_u u + H_y y \\ \hat{x}_b = z + L y \\ L = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \; G_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \; H_u = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} ; \; H_y = \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}\end{gathered}}$$