Problem 1

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}$$

$$\underbrace{y}_{\text{output, not the y-coordinate}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}$$

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} ; \quad A - LC = \begin{bmatrix} -\ell_1 & 1 & 0 \\ -\ell_2 & 0 & 1 \\ -\ell_3 & -\omega^2 & 0 \end{bmatrix}$$

In order to design an effective observer, the eigenvalues of $(A-LC)$ must be in the left-half of the imaginary plane. Choosing the eigenvalues to be $-1, -1 \pm j$:

Char. Eqn. of $(A-LC)$: $\lambda^3 + \ell_1 \lambda^2 + (\ell_2 + \omega^2)\lambda + (\ell_1 \omega^2 + \ell_3) = 0$

Des. Char. Eqn.: $(\lambda+1)(\lambda+1+j)(\lambda+1-j) = 0$

$$(\lambda^2 + 2\lambda + 2)(\lambda+1) = \lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

$\Rightarrow \ell_1 = 3, \ \ell_2 = 4 - \omega^2, \ \ell_3 = 2 - 3\omega^2$

Hence, the observer gains are as follows:

$$\boxed{\Rightarrow L = \begin{bmatrix} 3 \\ 4-\omega^2 \\ 2-3\omega^2 \end{bmatrix} \text{ for } \dot{\hat{x}} = (A-LC)\hat{x} + Bu + Ly}$$

These observer gains would effectively apply to all coordinates (x, y, z) since the same type of sinusoidal perturbation is applied to all three coordinates.

Problem 2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -K \\ 0 & 0 & K \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\omega = \sqrt{2K}$$

a) $u = -kx + r$

$$Bk = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ k_1 & k_2 & k_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 0 & -K \\ -k_1 & -k_2 & K-k_3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$A - Bk - \lambda I = \begin{bmatrix} -\lambda & 0 & -K \\ -k_1 & -k_2-\lambda & K-k_3 \\ 1 & -1 & -\lambda \end{bmatrix}$$

$$|A - Bk - \lambda I| = -\lambda^3 - k_2\lambda^2 + (k_3 - 2K)\lambda + (-Kk_1 - Kk_2) = 0$$

$$\lambda^3 + k_2\lambda^2 + (2K - k_3)\lambda + (Kk_1 + Kk_2) = 0$$

Des. Char. Eq. $= (\lambda + \omega)(\lambda + \omega + \omega j)(\lambda + \omega - \omega j) = 0$

$$(\lambda+\omega)(\lambda^2+2\omega\lambda+2\omega^2) = \lambda^3+3\omega\lambda^2+4\omega^2\lambda+2\omega^3 = 0$$

$$\Rightarrow k_2 = 3\omega;\quad k_3 = 2K-4\omega^2 = \overset{= -3\omega^2}{-6K};\quad Kk_1 + 3\omega K = 2\omega^3$$

$$\Rightarrow k_1 = \omega$$

$$K = [\omega, 3\omega, -3\omega^2]$$

Hence, the controller is as follows:

$$\boxed{u = -\omega x_1 - 3\omega x_2 + 3\omega^2 x_3 + r}$$

Finding C.L.T.F. $Y(s)/R(s)$:

$$\dot{x} = (A-Bk)x + BR,\quad y = Cx$$

$$G(s) = \frac{Y(s)}{R(s)} = C(sI-(A-Bk))^{-1}B + \cancelto{0}{D}$$

Using MATLAB:

$$\boxed{\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{K}{(s^3+3\omega s^2+4\omega^2 s+2\omega^3)} \\ &= \frac{\omega^2}{2(s^3+3\omega s^2+4\omega^2 s+2\omega^3)}\end{aligned}}$$

b) $y_d = (1-\cos(\omega t))$; $\dot{y}_d = \omega\sin(\omega t)$; $\ddot{y}_d = \omega^2\cos(\omega t)$

$\dddot{y}_d = -\omega^3\sin(\omega t)$

$$\frac{R(s)}{Y_d(s)} = \frac{s^3+3\omega s^2+4\omega^2 s+2\omega^3}{K}$$

$$\Rightarrow r(t) = \frac{1}{K}\left(\dddot{y}_d + 3\omega\ddot{y}_d + 4\omega^2\dot{y}_d + 2\omega^3 y_d\right)$$

$$r(t) = \frac{2}{\omega^2}\left(-\omega^3\sin(\omega t)+3\omega^3\cos(\omega t)+4\omega^3\sin(\omega t) - 2\omega^3\cos(\omega t)+2\omega^3\right)$$

$$\boxed{r(t) = 6\omega\sin(\omega t)+2\omega\cos(\omega t)+4\omega}$$

c) $A - LC = \begin{bmatrix} 0 & 0 & -K \\ 0 & 0 & K \\ 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$$A - LC = \begin{bmatrix} -k_1 & 0 & -K \\ -k_2 & 0 & K \\ 1-k_3 & -1 & 0 \end{bmatrix}$$

(A-LC) Char. Eqn.: $-\lambda^3 - k_1\lambda^2 + (Kk_3 - 2K)\lambda + (-Kk_1 - Kk_2) = 0$

$$\lambda^3 + k_1\lambda^2 + (2K - Kk_3)\lambda + (Kk_1 + Kk_2) = 0$$

Des. Char. Eqn.: $(\lambda + 10\omega)(\lambda + 10\omega + 10\omega j)(\lambda + 10\omega - 10\omega j) = 0$

$$(\lambda + 10\omega)(\lambda^2 + 20\omega\lambda + 200\omega^2) = 0$$

$\lambda^3 + 30\omega\lambda^2 + 400\omega^2\lambda + 2000\omega^3 = 0$

$\Rightarrow k_1 = 30\omega$; $k_3 = -798$; $30\omega K + Kk_2 = 2000\omega^3 = 4000\omega K$

$$\Rightarrow k_2 = 3970\omega$$

Hence, the observer is as follows:

$$\boxed{L = \begin{bmatrix} 30\omega \\ 3970\omega \\ -798 \end{bmatrix} \text{ for } \dot{\hat{x}} = (A-LC)\hat{x} + Bu + Ly}$$

d)

$$A = \begin{array}{cc} A_{11} & A_{12} \\ \left[\begin{array}{c|cc} 0 & 0 & -K \\ \hline 0 & 0 & K \\ 1 & -1 & 0 \end{array}\right] & \\ A_{21} & A_{22} \end{array} \; ; \; x = \begin{array}{c} \left[\begin{array}{c} x_1 \\ \hline x_2 \\ x_3 \end{array}\right] \end{array} \begin{array}{l} x_1 \\ x_2 \end{array} \; ; \; B = \left[\begin{array}{c} 0 \\ \hline 1 \\ 0 \end{array}\right] \begin{array}{l} B_1 \\ B_2 \end{array}$$

$$L = [\ell_1 \;\; \ell_2]^T$$

$$A_{22} - LA_{12} = \begin{bmatrix} 0 & K \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & -K \end{bmatrix}$$

$$= \begin{bmatrix} 0 & K \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -K\ell_1 \\ 0 & -K\ell_2 \end{bmatrix} = \begin{bmatrix} 0 & K+K\ell_1 \\ -1 & K\ell_2 \end{bmatrix}$$

$$(A_{22} - LA_{12}) - \lambda I = \begin{bmatrix} -\lambda & K+K\ell_1 \\ -1 & K\ell_2 - \lambda \end{bmatrix}$$

$$\det[(A_{22}-LA_{12}) - \lambda I] = \lambda^2 - K\ell_2\lambda + (K+K\ell_1) = 0$$

Des. Char. Eqn: $(\lambda + 10\omega + 10\omega j)(\lambda + 10\omega - 10\omega j) = 0$

$$\lambda^2 + 20\omega\lambda + 200\omega^2 = 0$$

$-K\ell_2 = 20\omega \Rightarrow \ell_2 = \frac{-40}{\omega}$; $\frac{\omega^2}{2}(\ell_1 + 1) = 200\omega^2 \Rightarrow \ell_1 = 399$

Hence, the reduced second order observer gains are:

$$L_r = \begin{bmatrix} 399 \\ -40/\omega \end{bmatrix}$$

$$A_r = A_{22} - L_r A_{12} = \begin{bmatrix} 0 & 200\omega^2 \\ -1 & -20\omega \end{bmatrix}$$

$$B_u = B_2 - L_r B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B_y = A_r L_r + A_{21} - L_r A_{11} = \begin{bmatrix} -8000\omega \\ 402 \end{bmatrix}$$

$$\dot{z} = A_r z + B_u u + B_y y$$
$$\hat{x}_2 = z + Ly$$

$$\boxed{\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 200\omega^2 \\ -1 & -20\omega \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} -8000\omega \\ 402 \end{bmatrix} y \\ \hat{x}_2 &= z + \begin{bmatrix} 399 \\ -40/\omega \end{bmatrix} y \end{aligned}}$$

e) $G_{closed\,1}(s) = G_{c1}(s) = C(sI - A + BK)^{-1}B$

$G_{closed\,2}(s) = G_{c2}(s) = (I + PF_1F_2)^{-1}PF_1$

$F_1(s) = [1 + K_2(sI - A_r)^{-1}B_\omega]^{-1}$; $F_2(s) = K_2(sI - A_r)^{-1}B_y + K_1 + K_2L_r$

$P = C(sI - A)^{-1}B$

Using MATLAB:

$$\boxed{F_1(s) = \left[\frac{3\omega s + 63\omega^2}{s^2 + 20\omega s + 200\omega^2} + 1\right]^{-1} = \frac{s^2 + 20\omega s + 200\omega^2}{s^2 + 23\omega s + 263\omega^2}}$$

$$\boxed{F_2(s) = \frac{1318\omega s^2 + 1154\omega^2 s + 800\omega^3}{s^2 + 20\omega s + 200\omega^2}}$$

To show that the combined controller/reduced order observer can be represented with the block diagram in the Dynamic Compensation Slides, $G_{c1}(s)$ and $G_{c2}(s)$ must be equal to each other.

$$G_{c1}(s) = \frac{\omega^2}{2(s^3 + 3\omega s^2 + 4\omega^2 s + 2\omega^3)}$$ (Using MATLAB)

Calculating $G_2(s)$:

$$P = C(sI-A)^{-1}B = \frac{\omega^2}{2(s^3+\omega^2 s)}$$ (Using MATLAB)

Using MATLAB to calculate $G_2(s)$:

$$G_{c2}(s) = \frac{\omega^2}{2s^3+6\omega s^2+8\omega^2 s+4\omega^3} = G_{c1}(s)$$

$G_{c2}(s)$ has pole-zero cancellation since the order of $(1+PF_1F_2)^{-1}PF_1$ is initially greater than the order of $G_{c1}(s) = C(sI-A+BK)^{-1}B$ (which is 3).

Hence:

$$G_{c1}(s) = G_{c2}(s) = \frac{\omega^2}{2(s^3+3\omega s^2+4\omega^2 s+2\omega^3)}$$

And therefore the combined controller/reduced order observer can be represented with the block diagram in the Dynamic Compensation slides

Problem 2 MATLAB Code:

```
syms k1 k2 k3 lam w s; % Necessary symbols for calculations

K = 0.5 .* (w.^2); % Setting the expression for K

% Setting up system matrices:
A = [0 0 -K; 0 0 K; 1 -1 0];
B = [0; 1; 0];
C = [1 0 0];

% Calculating gains for full-state feedback:
k = [k1 k2 k3];

ABk = A - (B*k);
ABk_lam = ABk - (eye(3) * lam);
det(ABk_lam); % Characteristic Equation of A - (B * k)

k = [w, 3*w, (-3 .* (w.^2))]; % Gains for full-state feedback calculated
ABk = A - (B*k);

G_s = C * inv((s*eye(3)) - ABk) * B; % Finding the transfer function
Y(s)/R(s)

% Designing the full-state observer:
L = [k1; k2; k3];
A_LC = A-(L*C);
A_LC_eig = (A_LC - (eye(3) * lam));
det(A_LC_eig); % Characteristic Equation of A - (L * C)

% Designing the reduced-order observer:
A11 = 0;
A12 = [0 -K];
A21 = [0; 1];
A22 = [0 K; -1 0];
B1 = 0;
B2 = [1; 0];

L_r = [399; (-40/w)]; % Calculated gains for the reduced-order observer

% System matrices for the reduced-order observer:
A_r = A22 - (L_r * A12);
B_u = B2 - (L_r * B1);
B_y = (A_r * L_r) + A21 - (L_r * A11);
K1 = k(1);
K2 = k(2:3);

% Calculating F1(s) and F2(s):
F1_s = simplify(inv(1 + (K2 * inv(s * eye(2) - A_r) * B_u)));
F2_s = simplify((K2*inv(s * eye(2) - A_r)*B_y) + K1 + (K2 * L_r));

% Calculating Gc1_s and Gc2_s:
Gc1_s = C * inv(s * eye(3) - A + (B * k)) * B;

P = C * inv(s * eye(3) - A) * B;
Gc2_s = simplify(inv(1 + (P * F1_s * F2_s)) * P * F1_s);
```

Problem 3

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30 & 0 & 0 & 20 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & -10 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ -25 \\ 0 \\ 10 \end{bmatrix}$$

$$C = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

a) Full-State Feedback Controller Poles:

$s_{1,2} = -1 \pm i \;;\; s_3 = -1 \;;\; s_4 = -2$

Using MATLAB $\Rightarrow K = [-1.36, -0.1032, -0.016, -0.808]$

Full-State Observer Poles:

$s_{1,2} = -10 \pm i \;;\; s_3 = -10 \;;\; s_4 = -20$

Using MATLAB $\Rightarrow$

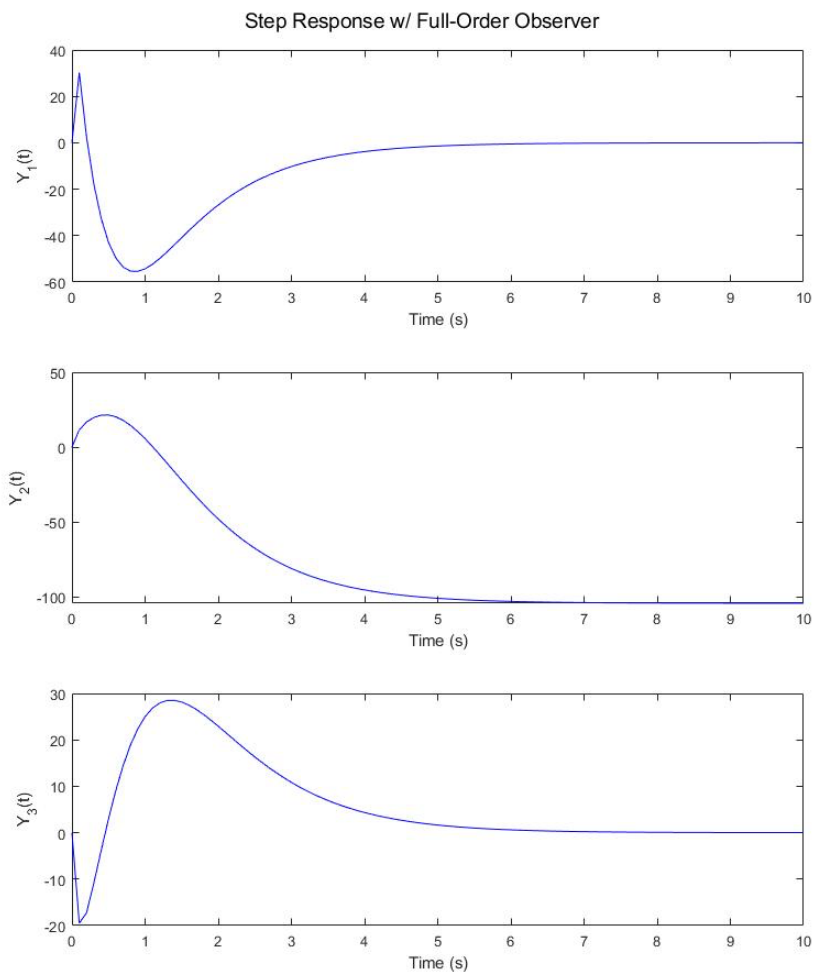
$$L = \begin{bmatrix} 0.5 & 0 & 0 \\ 5.75 & 0 & -4 \\ 0 & 2 & -0.2 \\ -0.05 & 0 & 0 \end{bmatrix}$$

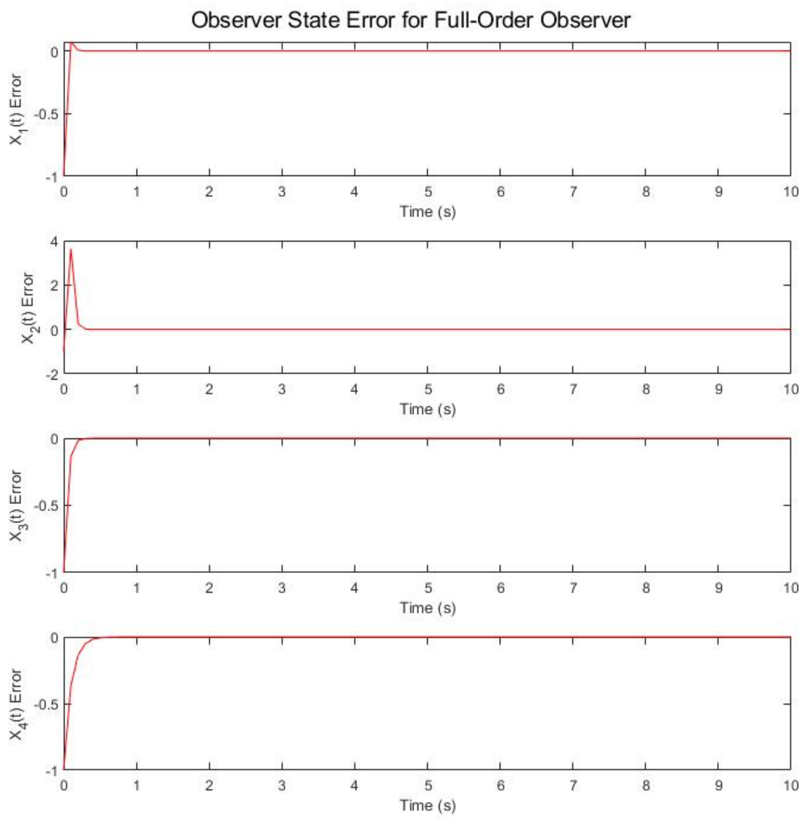
System Matrices for Combined State-Feedback Controller with Full-Order Observer:

$$A_f = \left[\begin{array}{c|c} (A-BK) & -BK \\ \hline 0_{4\times4} & (A-LC) \end{array}\right] ; \quad B_f = \left[\begin{array}{c} B \\ \hline 0_{4\times1} \end{array}\right]$$

$$C_f = [C, C] \; ; \; D_f = 0_{3\times1}$$

Linear Simulation Results:





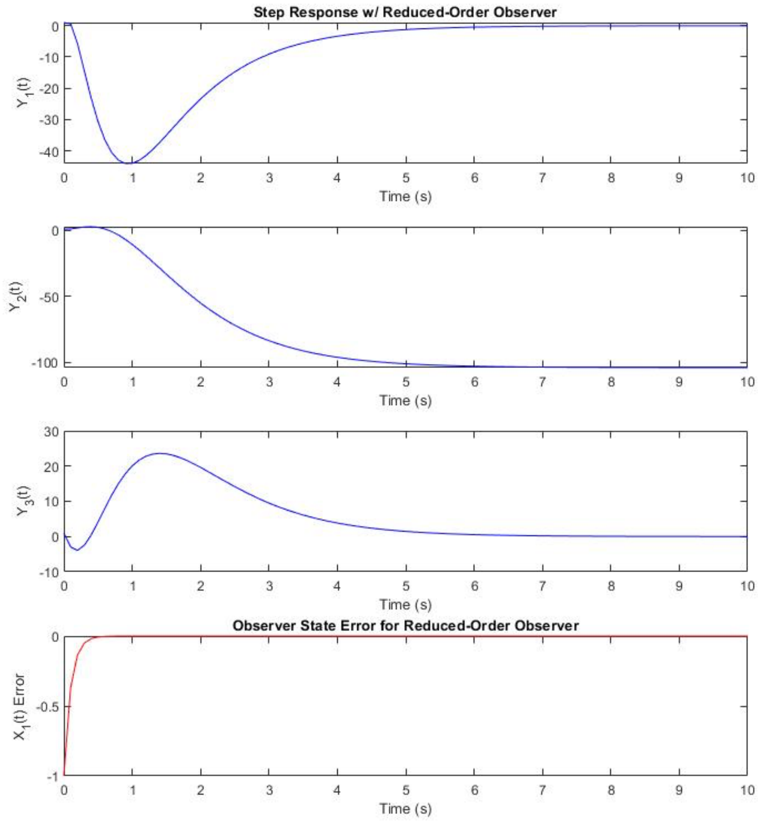
Output	Settling Time (s)
$Y_1(t)$	5.2236
$Y_2(t)$	5.2734
$Y_3(t)$	6.0537

System Matrices for Combined State-Feedback Controller with Reduced-Order Observer:

Since C isn't a matrix containing $I_{8\times8}$ and 0, the transformation $T = \begin{bmatrix} C \\ \hline R \end{bmatrix}$ must be applied to the system, where $R = [0 \ 1 \ 0 \ 0]$. The resulting system matrices are: $A_t = TAT^{-1}$, $B_t = TB$, $C_t = CT^{-1}$

Since the system originally has 3 outputs, the measurable and unmeasurable states are divided along the third row (and column, if applicable) for the A_t, B_t, x, and K matrices.

Linear Simulation Results:



Output	Settling Time (s)
$Y_1(t)$	5.3327
$Y_2(t)$	5.3134
$Y_3(t)$	6.1216

By comparing the lin.sim. results b/w the full-order and reduced-order observers, the reduced-order observer system has larger settling times for all three outputs.

b) To determine if the system can be observed through only one of the outputs, the rank test could be used for the A matrix and three C matrices, where the three C matrices represent the three different possible independent outputs:

$$C_1 = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}; \qquad Q_n = \begin{bmatrix} C_n \\ C_n A \\ C_n A^2 \\ C_n A^3 \end{bmatrix}$$

Rank Test

System is observable if $\text{rank}(Q_n) = n = 4$

Using MATLAB:

corresponds to $y = 10x_3$

$\text{rank}(Q_1) = 3$; $\underline{\text{rank}(Q_2) = 4}$; $\text{rank}(Q_3) = 3$

$\boxed{\text{Hence, the system can be observed through only } x_3\text{, which is the cart's position.}}$

MATLAB Code for Problem 3:

```
syms s;

% Setting up system matrices:
A = [0 1 0 0; 30 0 0 20; 0 0 0 1; -2 0 0 -10];
B = [0; -25; 0; 10];
C = [40 0 0 0; 0 0 10 0; 0 0 0 -5];
D = zeros(3, 1);

%%% Combined full-state feedback and full-order observer:

% Designing the controller:
Pd_U = [-1; -2; -3; -4];
K = place(A, B, Pd_U);

% Designing the observer:
Pd_O = 10 .* Pd_U;
L = place(A', C', Pd_O)';

% Setting up the system matrices of the combined full-state feedback and
% full-order observer system:
A_f = [(A - B*K), (-B*K); zeros(4, 4), (A - L*C)];
B_f = [B; zeros(4, 1)];
C_f = [C, C];
D_f = zeros(3, 1);

csys1 = ss(A_f, B_f, C_f, D_f); % Creating a continuous system using the
above system matrices

% Performing linear simulation:
t = [0:0.1:10];
u = ones(length(t),1);
x0 = [1 1 1 1 -1 -1 -1 -1];
[y1, t1, x1] = lsim(csys1,u,t,x0);
S1 = lsiminfo(y1,t1);

% Plotting results:
figure
subplot(3, 1, 1)
plot(t1, y1(:, 1), 'b')
xlabel('Time (s)')
ylabel('Y_1(t)')

subplot(3, 1, 2)
plot(t1, y1(:, 2), 'b')
xlabel('Time (s)')
ylabel('Y_2(t)')

subplot(3, 1, 3)
plot(t1, y1(:, 3), 'b')
xlabel('Time (s)')
ylabel('Y_3(t)')

sgtitle('Step Response w/ Full-Order Observer')
```

```
figure
subplot(4, 1, 1)
plot(t1, x1(:, 5), 'r')
xlabel('Time (s)')
ylabel('X_1(t) Error')

subplot(4, 1, 2)
plot(t1, x1(:, 6), 'r')
xlabel('Time (s)')
ylabel('X_2(t) Error')

subplot(4, 1, 3)
plot(t1, x1(:, 7), 'r')
xlabel('Time (s)')
ylabel('X_3(t) Error')

subplot(4, 1, 4)
plot(t1, x1(:, 8), 'r')
xlabel('Time (s)')
ylabel('X_4(t) Error')

sgtitle('Observer State Error for Full-Order Observer')

%%% Combined state feedback and reduced-order observer:

% Performing transformation on system matrices:
R = [0 1 0 0];
T = [C; R];
A_t = T * A * inv(T);
B_t = T * B;
C_t = C * inv(T);

K = place(A_t, B_t, Pd_U); % Recalculating the gains of the state-feedback
controller since the state matrices are transformed

% Dividing into measurable and unmeasurable states:
A11 = A_t(1:3, 1:3);
A12 = A_t(1:3, 4);
A21 = A_t(4, 1:3);
A22 = A_t(4, 4);
B1 = B_t(1:3);
B2 = B_t(4);

% Calculating the reduced-observer gain:
L_r = place(A22', A12', [-10]);
L_r = L_r';

% Splitting up the state-feedback controller gains:
K1 = K(1:3);
K2 = K(4);

% Calculating the matrices of the combined state feedback and reduced-order
% observer system:
A_r = A22 - (L_r * A12);
A_red = [(A_t - B_t*K) (-B_t*K2); zeros(1, 4) A_r];
```

```
B_red = [B_t; 0];
C_red = [C_t, zeros(3, 1)];
D_red = zeros(3, 1);

csys2 = ss(A_red, B_red, C_red, D_red); % Creating a continuous system using
the above system matrices

% Performing Linear Simulation:
t2 = [0:0.1:10];
u2 = ones(length(t2),1);
x0_2 = [1 1 1 1 -1];
[y2, t2, x2] = lsim(csys2, u2, t2, x0_2);
S2 = lsiminfo(y2, t2);

% Plotting results:
figure
subplot(4, 1, 1)
plot(t2, y2(:, 1), 'b')
xlabel('Time (s)')
ylabel('Y_1(t)')
title('Step Response w/ Reduced-Order Observer')

subplot(4, 1, 2)
plot(t2, y2(:, 2), 'b')
xlabel('Time (s)')
ylabel('Y_2(t)')

subplot(4, 1, 3)
plot(t2, y2(:, 3), 'b')
xlabel('Time (s)')
ylabel('Y_3(t)')

subplot(4, 1, 4)
plot(t2, x2(:, 5), 'r')
xlabel('Time (s)')
ylabel('X_1(t) Error')
title('Observer State Error for Reduced-Order Observer')

% Determining which output the system can be observed using rank test:
A = [0 1 0 0; 30 0 0 20; 0 0 0 1; -2 0 0 -10]; % State matrix

% First output:
C1 = [40 0 0 0; 0 0 0 0; 0 0 0 0];
Q1 = [C1; C1*A; C1*A*A; C1*A*A*A];
rQ1 = rank(Q1);

% Second output:
C2 = [0 0 0 0; 0 0 10 0; 0 0 0 0];
Q2 = [C2; C2*A; C2*A*A; C2*A*A*A];
rQ2 = rank(Q2);

% Third output:
C3 = [0 0 0 0; 0 0 0 0; 0 0 0 -5];
Q3 = [C3; C3*A; C3*A*A; C3*A*A*A];
rQ3 = rank(Q3);
```